

DYNAMIC INSTABILITY OF $\mathbb{C}\mathbb{P}^N$ UNDER RICCI FLOW

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ABSTRACT. The intent of this short note is to provide context for and an independent proof of the discovery of Klaus Kröncke [Kro13] that complex projective space with its canonical Fubini–Study metric is dynamically unstable under Ricci flow in all complex dimensions $N \geq 2$. The unstable perturbation is not Kähler. This provides a counterexample to a well known conjecture widely attributed to Hamilton. Moreover, it shows that the expected stability of the subspace of Kähler metrics under Ricci flow, another conjecture believed by several experts, needs to be interpreted in a more nuanced way than some may have expected.

CONTENTS

1. Introduction	1
2. Perelman’s entropy	3
3. Variational formulas	4
4. Proof of the main result	10
References	12

1. INTRODUCTION

Ricci solitons are stationary solutions of the Ricci flow dynamical system on the space of Riemannian metrics, modulo diffeomorphism and scaling. As such, it is natural to investigate their stability. Several distinct but related notions of stability appear in the literature. (I) One says a soliton metric g_0 is *dynamically stable* if for every neighborhood \mathcal{V} of g_0 there exists a neighborhood $\mathcal{U} \subseteq \mathcal{V}$ such that every (modified) Ricci flow solution originating in \mathcal{U} remains in \mathcal{V} and converges to a soliton in \mathcal{V} . (This is subtly different from the classical definition for continuous-time dynamical systems for two reasons. Unless a soliton is Ricci flat, one needs to modify the flow to make the soliton into a *bona fide* fixed point. Furthermore, stationary solutions frequently occur in families, so one typically has to deal with the presence of a center manifold.) (II) One says g_0 is *variationally stable* if the second variation of an appropriate choice of Perelman’s energy or entropy functionals is nonpositive at g_0 . (Note that the expander entropy was defined in [FIN05].) (III) One says that g_0 is *linearly stable* if the linearization of Ricci flow at g_0 has non-positive spectrum. Variational and linear stability are equivalent in the compact case, in the precise sense that the second-variation operator \mathcal{N} defined in (3) below

satisfies $2\mathcal{N} = \Delta_\ell - 1/\tau$ when acting on trace-free divergence-free tensors, where Δ_ℓ is the Lichnerowicz Laplacian.¹

Techniques to prove the implication (III) \Rightarrow (I) for flat metrics were first developed by one of the authors and collaborators. [GIK02]. Generalizing these, the other author proved for Ricci-flat metrics that (I) \Rightarrow (III), and that the converse holds if g_0 is integrable [Ses06]. Haslhofer–Müller removed the integrability assumption, replacing it with the slightly stronger hypothesis that g_0 is a local maximizer of Perelman’s λ -functional [HM14]. This must be verified directly and does not follow from (II), because these metrics are only weakly variationally stable. In any case, the implication (I) \Rightarrow (II) is clear for all compact solitons.

There are many other applications of stability theory to Ricci flow, too numerous to survey thoroughly here. An interested reader may consult the following (not complete!) list of examples, ordered by publication year: [Ye93], [DWW05], [DWW07], [SSS08], [Kno09], [LY10], [SSS11], [Has12], [Wu13], [Bam14], and [WW16].

Kröncke extended the techniques of Haslhofer–Müller [HM14] to study stability at general compact Einstein manifolds. Our interest here is narrower. In his investigation, Kröncke discovered (Corollary 1.8) the unexpected fact that complex projective space with its canonical Fubini–Study metric is dynamically unstable. We note that Kröncke’s work [Kro13] was accepted for publication after the original version of this paper was completed. Nonetheless, we believe his instability result deserves independent attention because of its considerable broad interest, for at least three reasons:

(A) The result negatively answers a well-known conjecture widely attributed to Hamilton. See, *e.g.*, the discussion of his conjecture in the introduction to [CZ12], as well as Hamilton’s own analysis of the behavior of $\mathbb{C}\mathbb{P}^2$ in Section 10 of [Ham95].

(B) The result shows that the conjecture of experts that the subspace of Kähler metrics is an attractor for Ricci flow involves more complicated dynamical behavior than some may have expected. It is well known that $(\mathbb{C}\mathbb{P}^N, g_{\text{FS}})$ is stable under Kähler perturbations [TZ13]. And it is easy to see that its unstable perturbation is not Kähler. However, if a flow originating at this perturbation asymptotically approaches the subspace of Kähler metrics, monotonicity of Perelman’s shrinker entropy ν implies that it cannot do so at the nearest Kähler candidate $(\mathbb{C}\mathbb{P}^N, g_{\text{FS}})$. Rather, if it converges to a Kähler singularity model, that metric must be sufficiently far away.

(C) Finally, in real dimension $n = 4$, dynamic instability of $\mathbb{C}\mathbb{P}^2$ shows that only the first four metrics in the list of conjectured stable singularity models, $\mathcal{S}^4 > \mathcal{S}^3 \times \mathbb{R} > \mathcal{S}^2 \times \mathbb{R}^2 > \mathcal{L}_{-1}^2 > \mathbb{C}\mathbb{P}^2$, ordered by the central density Θ introduced in [CHI04], are candidates to be generic singularity models. Here, \mathcal{L}_{-1}^2 denotes the “blowdown soliton” discovered in [FIK03]. At least in this dimension, it is reasonable to conjecture that solutions originating at the unstable perturbation

¹There is a subtlety in this equivalence and in proving (III) \Rightarrow (I) that should be noted. The linearization of Ricci flow is not strictly parabolic, because the flow is invariant under the action of the infinite-dimensional diffeomorphism group. So one either has to work orthogonally to the orbit of that group, or else fix a gauge, which converts the linearization into the Lichnerowicz Laplacian. Such a choice is necessary if one wishes to prove dynamic stability by standard semigroup methods, but can create a finite-dimensional unstable manifold within the orbit of the diffeomorphism group, especially in the presence of positive curvature. See, *e.g.*, Lemma 7 and the remarks that follow in [Kno09].

of $\mathbb{C}\mathbb{P}^2$ become singular in finite time by crushing the distinguished fiber $\mathbb{C}\mathbb{P}^1 \subset \mathbb{C}\mathbb{P}^2$, and converge after parabolic rescaling to \mathcal{L}_{-1}^2 . For further context, see the discussion and related results in [IKS17].

Because of these important consequences, we believe that it is valuable to produce an independent proof of Kröncke's discovery, using related but distinct methods. That is the purpose of this short note, in which we reprove the following:

Main Theorem. *For all complex dimensions $N \geq 2$, complex projective space with its canonical Fubini–Study metric, $(\mathbb{C}\mathbb{P}^N, g_{\text{FS}})$, is not a local maximum of Perelman's shrinker entropy ν . Consequently, it is dynamically unstable under Ricci flow. The unstable perturbation is conformal but not Kähler.*

2. PERELMAN'S ENTROPY

Perelman's entropy functional \mathcal{W} , introduced in [Per02], is defined on a closed Riemannian manifold (\mathcal{M}^n, g) by

$$\mathcal{W}(g, f, \tau) = \int_{\mathcal{M}^n} \left\{ \tau(R + |\nabla f|^2) + (f - n) \right\} (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV.$$

Suitably minimizing \mathcal{W} yields the shrinker entropy,

$$\nu[g] = \inf \left\{ \mathcal{W}(g, f, \tau) : f \in C^\infty(\mathcal{M}^n), \tau > 0, (4\pi\tau)^{-n/2} \int e^{-f} dV = 1 \right\}.$$

Along a smooth variation $g(s)$ such that $\frac{d}{ds}\Big|_{s=0} g(s) = h$, Perelman showed that

$$\frac{d}{ds}\Big|_{s=0} \nu[g(s)] = (4\pi\tau)^{-n/2} \int_{\mathcal{M}^n} \langle \frac{1}{2}g - \tau(\text{Rc} + \nabla^2 f), h \rangle e^{-f} dV.$$

From this, he obtained the beautiful result that ν is nondecreasing along any compact solution of Ricci flow — in fact, strictly increasing except on gradient shrinking solitons normalized so that

$$(1) \quad \text{Rc} + \nabla^2 f = \frac{1}{2\tau}g.$$

As originally observed in [CHI04] — see [CZ12] for details, along with a complete derivation of the strictly more complicated formula that holds at a nontrivial shrinking soliton — if $g(s)$ is a smooth family of metrics on \mathcal{M}^n such that $g = g(0)$ is Einstein, then the second variation of ν at $s = 0$ is given by

$$(2) \quad \frac{d^2}{ds^2}\Big|_{s=0} \nu[g(s)] = \frac{\tau}{V} \int_{\mathcal{M}^n} \langle \mathcal{N}(h), h \rangle dV,$$

where $V = \text{Vol}(\mathcal{M}^n, g)$, and

$$(3) \quad \mathcal{N}(h) = \frac{1}{2}\Delta h + \text{Rm}(h, *) + \text{div}^* \text{div} h + \frac{1}{2}\nabla^2 v_h - \frac{\bar{H}}{2n\tau}g.$$

Here $\bar{H} = (\int_{\mathcal{M}^n} H dV)/V$ is the mean of $H = \text{tr}_g h$, and v_h at $s = 0$ is the unique solution of

$$(4) \quad \left(\Delta + \frac{1}{2\tau}\right)v_h = \nabla^k \nabla^\ell h_{\ell k} \quad \text{satisfying} \quad \int_{\mathcal{M}^n} v_h dV = 0.$$

In components, we write (3) as

$$(5) \quad (\mathcal{N}(h))_{ij} = \frac{1}{2}(\Delta h)_{ij} + R_{kij}^\ell g^{km} h_{m\ell} - \frac{1}{2}g^{k\ell}(\nabla_i \nabla_\ell h_{kj} + \nabla_j \nabla_\ell h_{ki}) \\ + \frac{1}{2}\nabla_i \nabla_j v_h - \frac{\bar{H}}{2n\tau}g_{ij},$$

where our index convention is $R_{ijk}^\ell = \partial_i \Gamma_{jk}^\ell - \partial_j \Gamma_{ik}^\ell + \Gamma_{im}^\ell \Gamma_{jk}^m - \Gamma_{jm}^\ell \Gamma_{ik}^m$. For later use, we introduce the variant

$$\tilde{\mathcal{N}} = \frac{1}{2}\Delta h + \text{Rm}(h, *) + \text{div}^* \text{div} h + \frac{1}{2}\nabla^2 v_h \quad \left(= \mathcal{N} + \frac{\bar{H}}{2n\tau}g \right).$$

It is a classical fact [Bes87] that at any compact Einstein manifold other than the standard sphere, the space $C^\infty(S_2(T^*\mathcal{M}))$ of smooth sections of the bundle of symmetric covariant 2-tensors admits an orthogonal decomposition

$$C^\infty(S_2(T^*\mathcal{M})) = \text{im}(\text{div}^*) \oplus \mathcal{C} \oplus (\ker(\text{div}) \cap \ker(\text{tr})),$$

where \mathcal{C} is the space of infinitesimal conformal transformations, $C^\infty(\mathcal{M}^n) * g$. This decomposition is also orthogonal with respect to the second variation of ν (see Theorem 1.1 of [CH15]).

It is well known (see Example 2.3 of [CHI04] or Theorem 1.4 of [CH15]) that $(\mathbb{C}\mathbb{P}^N, g_{\text{FS}})$ is neutrally variationally stable. The neutral direction is attained by $h = \varphi g \in \mathcal{C}$, where φ belongs to the first nontrivial eigenspace of the Laplacian,

$$(6) \quad \left(\Delta_{\text{FS}} + \frac{1}{\tau} \right) \varphi = 0, \quad \int_{\mathbb{C}\mathbb{P}^N} \varphi \, dV_{\text{FS}} = 0,$$

where the subscripts indicate that the Laplacian and volume form are those of the Fubini-Study metric g_{FS} . Note that by (1), its Einstein constant is $1/(2\tau)$.

3. VARIATIONAL FORMULAS

In this section, we calculate the third variation of Perelman's shrinker entropy at the Fubini-Study metric. We begin by recalling some classical first-variation formulas [Bes87].

Proposition. *Let $g(s)$ be a smooth one-parameter variation of $g = g(0)$ such that $\frac{d}{ds}\Big|_{s=0} g(s) = h$, and set $H = \text{tr}_g h$. Then one has:*

$$(7) \quad \frac{d}{ds}\Big|_{s=0} g^{ij} = -h^{ij},$$

$$(8) \quad \frac{d}{ds}\Big|_{s=0} \Gamma_{ij}^k = \frac{1}{2}(\nabla_i h_j^k + \nabla_j h_i^k - \nabla^k h_{ij}),$$

$$(9) \quad \frac{d}{ds}\Big|_{s=0} R_{ijk}^\ell = \frac{1}{2} \left\{ \begin{array}{l} \nabla_i \nabla_k h_j^\ell - \nabla_i \nabla^\ell h_{jk} - \nabla_j \nabla_k h_i^\ell \\ + \nabla_j \nabla^\ell h_{ik} + R_{ijm}^\ell h_k^m - R_{ijk}^\ell h_m^\ell \end{array} \right\},$$

$$(10) \quad \frac{d}{ds}\Big|_{s=0} R = -\Delta H + \text{div}(\text{div} h) - \langle \text{Rc}, h \rangle,$$

$$(11) \quad \frac{d}{ds}\Big|_{s=0} dV = \frac{1}{2}H \, dV.$$

Assumptions and Notation. For simplicity, we adopt the following conventions. $(\mathbb{C}\mathbb{P}^N, g_{\text{FS}})$ denotes complex projective space with its canonical Fubini–Study metric. Then one has $\text{Rc}[g_{\text{FS}}] = (2\tau)^{-1}g_{\text{FS}}$. We denote its real dimension by $n = \dim_{\mathbb{R}} \mathbb{C}\mathbb{P}^N = 2 \dim_{\mathbb{C}} \mathbb{C}\mathbb{P}^N = 2N$. By compactness, we may take $g(s)$ to be the smooth family

$$(12) \quad g(s) = g_{\text{FS}} + sh, \quad s \in (-\varepsilon, \varepsilon),$$

where

$$h = \varphi g_{\text{FS}},$$

with φ the unique solution of (6). Note that in the variation (12) that we consider, φ is the fixed function defined in (6), and hence is independent of s . Below, to avoid notational prolixity, we write g for $g(s)$ at arbitrary $s \in (-\varepsilon, \varepsilon)$. Formulas should be assumed to hold at any s unless explicitly stated otherwise or decorated with $\big|_{s=0}$, in which case everything in sight is to be evaluated at $(\mathbb{C}\mathbb{P}^N, g(0) = g_{\text{FS}})$. If $A(s)$ is any smooth one-parameter family of tensor fields depending on $s \in (-\varepsilon, \varepsilon)$, we write

$$\frac{d}{ds}A$$

for the derivative evaluated at arbitrary s , and

$$A' = \left. \frac{d}{ds} \right|_{s=0} A$$

for the derivative evaluated at $s = 0$. We adopt similar notation for higher-order derivatives. Thus, as noted above, our selected variation has the property that $\varphi' = 0$, *i.e.*, $0 = h' = h'' = \dots$, a fact that we use frequently below. Finally, having fixed h , we simply write $\mathcal{N} = \mathcal{N}(h)$ and $v = v_h$ where no confusion will result.

Remark 1. We note that to prove dynamic instability of $\mathbb{C}\mathbb{P}^N$, it suffices to exhibit one unstable variation. Our choice with $\varphi' = 0$ matches that used by Cao–Zhu [CZ12], whose results we use below. Kröncke makes a different choice in the proof of Proposition 9.1 of [Kro13], taking $g(t) = (1 + tv(t))g_{\text{FS}}$, where $v(t) = \varphi/(1 + t\varphi)$. Other choices (differing to second order or above) are certainly possible.

Remark 2. Our simple choice of variation, with $\varphi' = 0$, means that we do not have the freedom to force equation (6) to hold for all s in an open set. Consequently, we do not differentiate that equation in what follows.

We now establish various identities needed to compute

$$\left. \frac{d^3}{ds^3} \right|_{s=0} \nu[g(s)].$$

Even though many of these identities are well known to experts, we derive them here to keep this note as transparent and self-contained as possible.

Lemma 1. *If g and $h = \varphi g_{FS}$ are as above, then using δ to denote the Kronecker delta function, one has:*

$$(13) \quad \frac{d}{ds} \Big|_{s=0} g^{ij} = -\varphi g^{ij},$$

$$(14) \quad \frac{d}{ds} \Big|_{s=0} \Gamma_{ij}^k = \frac{1}{2} (\nabla_i \varphi \delta_j^k + \nabla_j \varphi \delta_i^k - \nabla^k \varphi g_{ij}),$$

$$(15) \quad \frac{d}{ds} \Big|_{s=0} R_{ijk}^\ell = \frac{1}{2} (\nabla_i \nabla_k \varphi \delta_j^\ell - \nabla_j \nabla_k \varphi \delta_i^\ell + \nabla_j \nabla^\ell \varphi g_{ik} - \nabla_i \nabla^\ell \varphi g_{jk}),$$

$$(16) \quad \frac{d}{ds} \Big|_{s=0} R = -(n-1)\Delta\varphi - \frac{n}{2\tau}\varphi,$$

$$(17) \quad \frac{d}{ds} \Big|_{s=0} dV = \frac{n}{2}\varphi dV.$$

Proof. Formulas (13)–(17) follow from (7)–(11) by direct substitution. \square

Lemma 2. *If g and $h = \varphi g_{FS}$ are as above, then one has:*

$$(18) \quad \frac{d}{ds} \Big|_{s=0} \tau = 0,$$

$$(19) \quad \frac{d}{ds} \Big|_{s=0} V = 0,$$

$$(20) \quad \frac{d}{ds} \Big|_{s=0} \bar{H} = \frac{n(n-2)}{2V} \|\varphi\|^2,$$

where $\|\varphi\|^2 = \int_{\mathbb{C}P^N} \varphi^2 dV$.

Proof. To establish (18), we recall that by Lemma 2.4 of [CZ12], using the fact that our metric is Einstein with $Rc = (2\tau)^{-1}g_{FS}$ (and hence that we can take $f = 0$ in Lemma 2.4 of [CZ12]), we have

$$\tau' = \tau \frac{\int_{\mathcal{M}^n} \langle Rc, h \rangle dV}{\int_{\mathcal{M}^n} R dV}.$$

Furthermore, at $s = 0$, equation (6) implies that

$$\int_{\mathbb{C}P^N} \langle Rc, h \rangle dV = \frac{1}{2\tau} \int_{\mathbb{C}P^N} H dV = \frac{n}{2\tau} \int_{\mathbb{C}P^N} \varphi dV = 0.$$

Equation (18) follows.

Equation (19) follows from (17) and (6), because at $s = 0$,

$$V' = \int_{\mathbb{C}P^N} (dV)' = \frac{n}{2} \int_{\mathbb{C}P^N} \varphi dV = 0.$$

Finally, equation (20) follows from (19) and the computation

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} \int_{\mathbb{C}P^N} g^{ij} h_{ij} dV &= \int_{\mathbb{C}P^N} (g^{ij})' h_{ij} dV + \int_{\mathbb{C}P^N} g^{ij} h_{ij} (dV)' \\ &= \int_{\mathbb{C}P^N} \left(-\varphi H + H \frac{n}{2} \varphi \right) dV \\ &= \left(-n + \frac{n^2}{2} \right) \int_{\mathbb{C}P^N} \varphi^2 dV. \end{aligned}$$

\square

We next compute derivatives of the Laplacian.

Lemma 3. *If g and $h = \varphi g_{\text{FS}}$ are as above, then (at $s = 0$) we have*

$$(21) \quad \Delta' = -\varphi\Delta + \frac{n-2}{2} \nabla \nabla \varphi.$$

Proof. For any smooth (possibly s -dependent) function u , using formulas (13) and (14) and collecting terms yields

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} \{g^{ij}(\partial_i \partial_j u - \Gamma_{ij}^k \partial_k u)\} &= \Delta u' - \varphi \Delta u - g^{ij}(\Gamma_{ij}^k)' \nabla_k u \\ &= \Delta u' - \varphi \Delta u + \frac{n-2}{2} \langle \nabla \varphi, \nabla u \rangle, \end{aligned}$$

whence the result follows. (We note that for $u = \varphi$, we have $\varphi' = 0$.) \square

Lemma 4. *If g and $h = \varphi g_{\text{FS}}$ are as above, then (at $s = 0$) we have*

$$\Delta'' = 2\varphi^2 \Delta.$$

Proof. If u is a smooth function that is independent of s , then

$$(\Delta u)'' = (g^{ij})'' \nabla_i \nabla_j u + 2(g^{ij})' (\nabla_i \nabla_j u)' + g^{ij} (\nabla_i \nabla_j u)''.$$

Using the Neumann series for $(g_{\text{FS}} + s\varphi g_{\text{FS}})^{-1}$, one sees easily that $(g^{ij})'' = 2\varphi^2 g^{ij}$. Equation (13) gives $(g^{-1})' = -\varphi g^{-1}$, and as in the proof of Lemma 3, one has

$$g^{ij} (\nabla_i \nabla_j u)' = \frac{n-2}{2} \langle \nabla \varphi, \nabla u \rangle.$$

Using the fact that $u' = 0$, one has

$$(\nabla_i \nabla_j u)'' = -(\Gamma_{ij}^k)'' \nabla_k u,$$

where by part (1) of Lemma 28 of [AK07], we have

$$\begin{aligned} (\Gamma_{ij}^k)'' &= -h_\ell^k (\nabla_i h_j^\ell + \nabla_j h_i^\ell - \nabla^\ell h_{ij}) \\ &= -\varphi (\nabla_i \varphi \delta_j^k + \nabla_j \varphi \delta_i^k - \nabla^k \varphi g_{ij}). \end{aligned}$$

Putting these together, we obtain

$$(\Delta u)'' = 2\varphi^2 \Delta u - 2\varphi \frac{n-2}{2} \langle \nabla \varphi, \nabla u \rangle + \varphi(2-n) \langle \nabla \varphi, \nabla u \rangle = 2\varphi^2 \Delta u,$$

which finishes the proof. \square

Lemma 5. *If g and $h = \varphi g_{\text{FS}}$ are as above, then at $s = 0$, $v = 2\varphi$, and $\tilde{\mathcal{N}}$ vanishes pointwise.*

Proof. To prove the first claim, we note that $\nabla^k \nabla^\ell h_{\ell k} = \Delta \varphi = -\varphi/\tau$ by (6), and hence by (4) we have

$$\left(\Delta + \frac{1}{2\tau}\right) v = -\frac{\varphi}{\tau}, \quad \text{with} \quad \int_{\mathbb{C}\mathbb{P}^N} v dV = 0.$$

On the other hand, by (6), we have

$$\left(\Delta + \frac{1}{2\tau}\right) 2\varphi = -\frac{\varphi}{\tau}, \quad \text{with} \quad \int_{\mathbb{C}\mathbb{P}^N} 2\varphi dV = 0.$$

By uniqueness, we have $v = 2\varphi$.

To prove the second claim, one uses the identity $R_{ij}|_{s=0} = (2\tau)^{-1}g_{ij}$, without differentiating, along with (6) and the first claim to obtain

$$\begin{aligned}\tilde{N}_{ij} &= \frac{1}{2}\Delta\varphi g_{ij} + \varphi R_{ij} - \nabla_i \nabla_j \varphi + \frac{1}{2}\nabla_i \nabla_j v \\ &= \frac{1}{2}\left(\Delta\varphi + \frac{1}{\tau}\varphi\right)g_{ij} + \nabla_i \nabla_j \left(\frac{1}{2}v - \varphi\right) = 0.\end{aligned}$$

□

We now compute the third variation explicitly.

Lemma 6. *If g and $h = \varphi g_{\text{FS}}$ are as above, then*

$$\frac{d^3}{ds^3}\Big|_{s=0} \nu[g(s)] = -\frac{\tau}{(4\pi\tau)^{n/2}} \int_{\mathcal{M}^n} \left\langle h, \text{Rc}'' + (\nabla^2 f)'' - \left(\frac{1}{2\tau}g\right)'' \right\rangle dV,$$

where the three second derivatives on the right-hand side are given pointwise by (23), (25), and (26), respectively.

Proof. By Lemma 2.2 of [CZ12], at arbitrary $s \in (-\varepsilon, \varepsilon)$, one has

$$\begin{aligned}\frac{d}{ds} \nu(s) &= -\{\tau(4\pi\tau)^{-n/2}\} \int_{\mathbb{C}\mathbb{P}^N} \left\langle h, \text{Rc} + \nabla^2 f - \frac{1}{2\tau}g \right\rangle e^{-f} dV \\ &= -\{\tau(4\pi\tau)^{-n/2}\} \int_{\mathbb{C}\mathbb{P}^N} (g^{ip}g^{jq}h_{pq}) \left(R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau}g_{ij} \right) (e^{-f} dV),\end{aligned}$$

which we write schematically as $A \int_{\mathbb{C}\mathbb{P}^N} B * C * D$. A simple calculus exercise shows that

$$\begin{aligned}\nu'''(s) &= A'' \int_{\mathbb{C}\mathbb{P}^N} B * C * D + A \int_{\mathbb{C}\mathbb{P}^N} B'' * C * D + A \int_{\mathbb{C}\mathbb{P}^N} B * C'' * D + A \int_{\mathbb{C}\mathbb{P}^N} B * C * D'' \\ &\quad + 2A' \int_{\mathbb{C}\mathbb{P}^N} B' * C * D + 2A' \int_{\mathbb{C}\mathbb{P}^N} B * C' * D + 2A' \int_{\mathbb{C}\mathbb{P}^N} B * C * D' \\ &\quad + 2A \int_{\mathbb{C}\mathbb{P}^N} B' * C' * D + 2A \int_{\mathbb{C}\mathbb{P}^N} B' * C * D' + 2A \int_{\mathbb{C}\mathbb{P}^N} B * C' * D' .\end{aligned}$$

Because $(\mathbb{C}\mathbb{P}^N, g_{\text{FS}})$ is Einstein, we have $C|_{s=0} = 0$. By (18), $A' = 0$. Thus the formula above reduces to

$$\nu''' = A \int_{\mathbb{C}\mathbb{P}^N} B * C'' * D + 2A \int_{\mathbb{C}\mathbb{P}^N} B' * C' * D + 2A \int_{\mathbb{C}\mathbb{P}^N} B * C' * D'.$$

Using Lemma 2.3 of [CZ12], and (18), we obtain

$$\begin{aligned}C' &= (\text{Rc} + \nabla^2 f)' - \frac{1}{2\tau}h \\ &= -\frac{1}{2}\Delta h - \text{Rm}(h, \cdot) - \text{div}^* \text{div} h - \nabla^2 v \\ &= -\tilde{N}|_{s=0}.\end{aligned}$$

By Lemma 5, we have $C' = 0$, and hence

$$(22) \quad \nu'''(s) = A \int_{\mathbb{C}\mathbb{P}^N} B * C'' * D.$$

There are three terms in C'' . To compute the first, we apply part (3) of Lemma 28 of [AK07] to our conformal variation, obtaining

$$(23) \quad R''_{ij} = \left(\varphi \Delta \varphi - \frac{n-2}{2} |\nabla \varphi|^2 \right) g_{ij} + (n-2) \left(\varphi \nabla_i \nabla_j \varphi + \nabla_i \varphi \nabla_j \varphi \right),$$

where the right-hand side is computed with respect to the metric data of g_{FS} .

To compute the second term, we begin by observing that

$$(\nabla_i \nabla_j f)'' = \nabla_i \nabla_j f'' - 2(\Gamma_{ij}^k)' \nabla_k f' - (\Gamma_{ij}^k)'' \nabla_k f,$$

As noted above, the function $v(s)$ defined by

$$v := -2f' + H - 2\frac{\tau'}{\tau}(f - \nu)$$

in the discussion on page 9 of [CZ12] is given by (4) at $s = 0$. Moreover, by (18), we have

$$(24) \quad v|_{s=0} = -2f' + H.$$

Then equation (24) and Lemma 5 imply that

$$f' = \frac{n-2}{2} \varphi.$$

Now the fact that g_{FS} is Einstein implies that $f|_{s=0} = 0$, and so by (14) we get

$$(25) \quad \begin{aligned} (\nabla_i \nabla_j f)''|_{s=0} &= \nabla_i \nabla_j f'' + \langle \varphi, \nabla f' \rangle g_{ij} - \nabla_i \varphi \nabla_j f' - \nabla_j \varphi \nabla_i f' \\ &= \nabla_i \nabla_j f'' + \frac{n-2}{2} |\nabla \varphi|^2 g_{ij} - (n-2) \nabla_i \varphi \nabla_j \varphi. \end{aligned}$$

Similarly, using (18) and the fact that $g'' = 0$, we compute that the final term is

$$(26) \quad \left(\frac{1}{2\tau} g \right)'' = -\frac{\tau''}{2\tau^2} g.$$

The result follows, because $D|_{s=0} = (e^{-f} dV)|_{s=0} = dV$. \square

To calculate ν''' , we will use the following identity.

Lemma 7. *If g and $h = \varphi g_{\text{FS}}$ are as above, then*

$$\left(\Delta + \frac{1}{2\tau} \right) f'' = -n\varphi \Delta \varphi - \frac{n}{2\tau} \varphi^2 - \frac{n\tau''}{4\tau^2}.$$

Proof. Recall that since $f = f(s)$ is the minimizer of $\nu = \nu(s)$ for every s , it satisfies the following elliptic equation:

$$\tau(-2\Delta f + |\nabla f|^2 - R) - f + \nu = 0.$$

At $s = 0$, we have $\tau' = 0$ by (18) and $\nu'' = 0$ by our choice of variation. (Indeed, at $s = 0$, we have $\nu'' = A \int_{\mathbb{C}\mathbb{P}^N} B * C' * D = 0$, as seen in the proof of Lemma 6.) Because $f|_{s=0} = 0$, it follows that at $s = 0$,

$$(27) \quad 2\tau(\Delta f)'' + f'' = \tau \left\{ (|\nabla f|^2)'' - R'' \right\} - R\tau''.$$

Again using $f|_{s=0} = 0$, $(g^{-1})'' = 2\varphi^2 g^{-1}$ (see Lemma 4), and $f' = ((n-2)/2)\varphi$ (see Lemma 6), we get

$$(28) \quad \begin{aligned} (|\nabla f|^2)'' &= 2\varphi^2 |\nabla f|^2 + 4\langle \nabla f, \nabla f' \rangle + 2|\nabla f'|^2 \\ &= \frac{(n-2)^2}{2} |\nabla \varphi|^2. \end{aligned}$$

Again using $f|_{s=0} = 0$ and $(\nabla_i \nabla_j f)' = \nabla_i \nabla_j f' = ((n-2)/2) \nabla_i \nabla_j \varphi$ along with (25), we compute that

$$\begin{aligned} (\Delta f)'' &= (g^{ij})'' \nabla_i \nabla_j f + 2(g^{ij})' (\nabla_i \nabla_j f)' + g^{ij} (\nabla_i \nabla_j f)'' \\ (29) \quad &= \Delta f'' - (n-2) \varphi \Delta \varphi + \frac{(n-2)^2}{2} |\nabla \varphi|^2. \end{aligned}$$

Finally, contracting (15) yields

$$R'_{jk} = -\frac{n-2}{2} \nabla_j \nabla_k \varphi - \frac{1}{2} \Delta \varphi g_{jk}.$$

Using this with the identities $(g^{-1})' = 2\varphi^2 g^{-1}$, $R = g^{ij} R_{ij}$ and $R = n/(2\tau)$, along with equations (13) and (23), we compute that

$$\begin{aligned} R'' &= (g^{ij})'' R_{ij} + 2(g^{ij})' R'_{ij} + g^{ij} R''_{ij} \\ &= 2\varphi^2 R + 2\varphi \left\{ \frac{n-2}{2} \Delta \varphi + \frac{n}{2} \Delta \varphi \right\} + n(\varphi \Delta \varphi - \frac{n-2}{2} |\nabla \varphi|^2) + (n-2)(\varphi \Delta \varphi + |\nabla \varphi|^2) \\ &= 4(n-1) \varphi \Delta \varphi - \frac{(n-2)^2}{2} |\nabla \varphi|^2 + \frac{n}{\tau} \varphi^2. \end{aligned}$$

Now combining this identify with equations (27), (28), and (29), we obtain

$$\begin{aligned} 2\tau \Delta f'' + f'' &= \tau \{ 2(n-2) \varphi \Delta \varphi - (n-2)^2 |\nabla \varphi|^2 \} \\ &\quad + \tau \left\{ -4(n-1) \varphi \Delta \varphi + (n-2)^2 |\nabla \varphi|^2 - \frac{n}{\tau} \varphi^2 \right\} - R\tau'' \\ &= -2n\tau \varphi \Delta \varphi - n\varphi^2 - \frac{n\tau''}{2\tau}, \end{aligned}$$

which completes the proof. \square

4. PROOF OF THE MAIN RESULT

Theorem 8. *If g and $h = \varphi g_{\text{FS}}$ are as above, then*

$$\frac{d^3}{ds^3} \Big|_{s=0} \nu[g(s)] = \frac{n-2}{(4\pi\tau)^{n/2}} \int_{\mathbb{C}\mathbb{P}^N} \varphi^3 dV.$$

Proof. Using Lemma 6, equations (23), (25), and (26), and the fact that $h = \varphi g$, we compute at $s = 0$, where $e^{-f} = 1$, that

$$\begin{aligned} \frac{d^3}{ds^3} \Big|_{s=0} \nu[g(s)] &= -\frac{\tau}{(4\pi\tau)^{n/2}} \int_{\mathbb{C}\mathbb{P}^N} \left\{ n\varphi(\varphi \Delta \varphi - \frac{n-2}{2} |\nabla \varphi|^2) + (n-2)(\varphi^2 \Delta \varphi + |\nabla \varphi|^2) \right\} dV \\ &\quad - \frac{\tau}{(4\pi\tau)^{n/2}} \int_{\mathbb{C}\mathbb{P}^N} (\varphi \Delta f'' + \frac{n(n-2)}{2} \varphi |\nabla \varphi|^2 - (n-2) \varphi |\nabla \varphi|^2) dV \\ &\quad - \frac{\tau}{(4\pi\tau)^{n/2}} \left(-\frac{n\tau''}{2\tau^2} \right) \int_{\mathbb{C}\mathbb{P}^N} \varphi dV \\ &= -\frac{\tau}{(4\pi\tau)^{n/2}} \left\{ 2(n-1) \int_{\mathbb{C}\mathbb{P}^N} \varphi^2 \Delta \varphi dV + \int_{\mathbb{C}\mathbb{P}^N} \varphi \Delta f'' dV \right\}, \end{aligned}$$

where we used the consequence of (6) that $\int_{\mathbb{C}\mathbb{P}^N} \varphi \, dV = 0$. To calculate the second integral in the last line above, we use equation (6) and Lemma 7 to see that

$$\begin{aligned} \int_{\mathbb{C}\mathbb{P}^N} \varphi \Delta f'' \, dV &= \int_{\mathbb{C}\mathbb{P}^N} \varphi \left(\Delta + \frac{1}{2\tau} - \frac{1}{2\tau} \right) f'' \, dV \\ &= -n \int_{\mathbb{C}\mathbb{P}^N} \varphi^2 \Delta \varphi \, dV - \frac{n}{2\tau} \int_{\mathbb{C}\mathbb{P}^N} \varphi^3 \, dV - \frac{n\tau''}{4\tau^2} \int_{\mathbb{C}\mathbb{P}^N} \varphi \, dV - \frac{1}{2\tau} \int_{\mathbb{C}\mathbb{P}^N} \varphi f'' \, dV \\ &= -n \int_{\mathbb{C}\mathbb{P}^N} \varphi^2 \Delta \varphi \, dV - \frac{n}{2\tau} \int_{\mathbb{C}\mathbb{P}^N} \varphi^3 \, dV + \frac{1}{2} \int_{\mathbb{C}\mathbb{P}^N} \Delta \varphi f'' \, dV \\ &= -n \int_{\mathbb{C}\mathbb{P}^N} \varphi^2 \Delta \varphi \, dV - \frac{n}{2\tau} \int_{\mathbb{C}\mathbb{P}^N} \varphi^3 \, dV + \frac{1}{2} \int_{\mathbb{C}\mathbb{P}^N} \varphi \Delta f'' \, dV, \end{aligned}$$

where in the last step, we used the fact that the Laplacian is self-adjoint. Thus by using (6) to evaluate $\Delta \varphi$, we conclude that

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} \nu[g(s)] &= \frac{\tau}{(4\pi\tau)^{n/2}} \left\{ 2 \int_{\mathbb{C}\mathbb{P}^N} \varphi^2 \Delta \varphi \, dV + \frac{n}{\tau} \int_{\mathbb{C}\mathbb{P}^N} \varphi^3 \, dV \right\} \\ &= \frac{n-2}{(4\pi\tau)^{n/2}} \int_{\mathbb{C}\mathbb{P}^N} \varphi^3 \, dV. \end{aligned}$$

□

Our Main Theorem now follows from Theorem 8 and the following observation:

Lemma 9. *If $N > 1$, there is a solution φ of (6) such that*

$$\int_{\mathbb{C}\mathbb{P}^N} \varphi^3 \, dV \neq 0.$$

Proof. Our approach here is essentially the same as in [Kro13] and uses results from Part III-C of [BGM71]. We include it here merely for completeness. We denote by \mathcal{P}_k the vector space of polynomials on \mathbb{C}^{N+1} such that $f(cz) = c^k \bar{c}^k f(z)$ and by $\mathcal{H}_k \subset \mathcal{P}_k$ the subspace of harmonic polynomials. Then the eigenfunctions of the k^{th} eigenvalue of the Laplacian on $(\mathbb{C}\mathbb{P}^N, g_{\text{FS}})$ are the restrictions of functions in \mathcal{H}_k , and one has a decomposition

$$(30) \quad \mathcal{P}_k = \mathcal{H}_k \oplus r^2 \mathcal{P}_{k-1} \quad \text{for all integers } k \geq 1.$$

It follows that the dimension of the first nontrivial eigenspace of the the Laplacian on $(\mathbb{C}\mathbb{P}^N, g_{\text{FS}})$ is $(N+1)^2 - 1 = N(N+2)$.

Using the hypothesis $N > 1$, one can define real-valued functions

$$f_1(z) = z_1 \bar{z}_2 + \bar{z}_1 z_2, \quad f_2(z) = z_2 \bar{z}_3 + \bar{z}_2 z_3, \quad f_3(z) = z_3 \bar{z}_1 + \bar{z}_3 z_1.$$

We choose $f = f_1 + f_2 + f_3$ and denote by φ its restriction to $\mathbb{C}\mathbb{P}^N$. We will show that this choice (not unique in general) has the desired properties. By considering isometries $z_j \mapsto -z_j$, it is easy to verify that $\int_{\mathbb{S}^{2N-1}} f \, dV = 0$. One has

$$f^3 = \sum_{j=1}^3 f_j^3 + 3 \sum_{j=1}^3 \sum_{k \neq j} f_j f_k^2 + 6 f_1 f_2 f_3.$$

Consideration of the same isometries $z_j \mapsto -z_j$ similarly shows that the integrals of all terms above vanish, except possibly the last. We expand that term, obtaining

$$\begin{aligned} f_1 f_2 f_3 &= 2|z_1|^2 |z_2|^2 |z_3|^2 \\ &\quad + |z_1|^2 (z_2^2 \bar{z}_3^2 + \bar{z}_2^2 z_3^2) + |z_2|^2 (z_1^2 \bar{z}_3^2 + \bar{z}_1^2 z_3^2) + |z_3|^2 (z_1^2 \bar{z}_2^2 + \bar{z}_1^2 z_2^2). \end{aligned}$$

Consideration of isometries $z_j \mapsto iz_j$ shows that the integrals of all terms here except the first vanish, whereupon we obtain

$$\int_{\mathcal{S}^{2N-1}} f^3 dV = 12 \int_{\mathcal{S}^{2N-1}} |z_1|^2 |z_2|^2 |z_3|^2 dV = 12 \text{Vol}(\mathcal{S}^{2N-1}).$$

Now by (30), there are functions $F_k \in \mathcal{H}_k$ such that $f^3 = F_3 + r^2 F_2 + r^4 F_1 + r^6 F_0$, where the restriction of each F_k to \mathcal{S}^{2N+1} is an eigenfunction of its Laplacian, with $\int_{\mathcal{S}^{2N-1}} F_k dV = 0$ for $k = 3, 2, 1$. The fact that $\int_{\mathcal{S}^{2N-1}} f^3 dV > 0$ thus implies that $F_0 > 0$, and the same conclusion then holds for the corresponding decomposition $\varphi^3 = \Phi_3 + r^2 \Phi_2 + r^4 \Phi_1 + r^6 \Phi_0$ into eigenvalues of the Laplacian on $\mathbb{C}\mathbb{P}^N \approx \mathcal{S}^{2N+1}/\mathcal{S}^1$. The result follows. \square

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