An introduction to the Ricci flow neckpinch

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Singularity formation has long been one of the most intensively studied aspects of nonlinear PDE and in particular of geometric evolution equations. This article surveys some recent progress made in understanding singularity formation for the Ricci flow.

1. The neckpinch singularity

In the Ricci flow, one begins with a smooth Riemannian manifold (M^n, g_0) and evolves its metric by the equation

$$\frac{\partial}{\partial t} g = -2\,\mathrm{Rc}(g). \tag{1.1}$$

Standard long-time existence theorems for the flow imply that a singularity will occur at some finite time T if and only if there exists a sequence of points $x_i \in M$ and times $t_i \nearrow T$ such that

$$\lim_{i\to\infty} |\mathrm{Rm}\,(x_i, t_i)| = \infty.$$

Singularity formation is quite common. For example, whenever $R_{\min}(0) > 0$, the parabolic maximum principle allows one to compare solutions of the reaction-diffusion PDE

$$\frac{\partial}{\partial t} R = \Delta R + 2\,|\mathrm{Rc}|^2 \geq \Delta R + \frac{2}{n} R^2$$

satisfied by the scalar curvature with solutions of the ODE

$$\frac{dr}{dt} = \frac{2}{n} r^2$$

and thus conclude that a singularity must occur before $\frac{n}{2} R_{\min}^{-1}(0)$.

From one perspective, finite-time singularities of the Ricci flow have been intensively studied, especially in dimensions three and four. This study has been motivated by the fact that the flow can reveal geometric and topological information about the underlying manifold M^n in those dimensions. As is well known, Richard Hamilton is the architect of a well-developed program [**9**] to resolve the Geometrization Conjecture for closed 3-manifolds [**14**] by using the Ricci flow. At the time of this writing (July 2003) there has been recent remarkable progress [**11, 12**] made in this program by Grisha Perelman.

From the perspective of asymptotic analysis, however, remarkably little is known about singularity formation in the Ricci flow. For example, one cannot

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find in the literature examples of either formal or rigorous singularity analysis for the Ricci flow comparable to what has been done for the mean curvature flow [**3**] or for other reaction-diffusion equations like $u_t = \Delta u + u^p$. (See [**6, 7, 8**] and [**5**] for example.) In fact, there were until recently no rigorous constructions of finite-time singularities — except for trivial examples where the manifold is a product of constant-curvature factors, one of which vanishes all at once.

One says a solution $(M^n, g(t))$ of the Ricci flow encounters a *local singularity* at $T < \infty$ if there exists a proper compact subset $K \subset M^n$ such that

$$\sup_{K \times [0,T)} |\mathrm{Rm}| = \infty$$

but

$$\sup_{(M^n \setminus K) \times [0,T)} |\mathrm{Rm}| < \infty.$$

The first examples of local singularity formation for the Ricci flow were constructed by Miles Simon [**13**] on noncompact warped products $\mathbb{R} \times_f S^n$. In these examples, a supersolution of the PDE is used as an upper barrier to force a singularity to occur on a compact subset in finite time. The only other known examples were constructed in [**4**]. Here, the metric is a complete $\mathrm{U}(n)$-invariant shrinking gradient Kähler–Ricci soliton on the holomorphic line bundle L^{-k} over $\mathbb{CP}^{n-1}$ with twisting number $k \in \{1, \ldots, n-1\}$. As $t \nearrow T$, the $\mathbb{CP}^{n-1}$ which constitutes the zero-section of the bundle pinches off, while the metric remains nonsingular and indeed converges to a Kähler cone on the set $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_k$ which constitutes the rest of the bundle. Both of these families of examples live on noncompact manifolds.

A *neckpinch* is a special type of local singularity. There are quantitative measures of neck-like behavior for a solution of the Ricci flow [**10**], but the following qualitative characterization will suffice for our present purposes. One says a solution $\left(M^{n+1}, g(t)\right)$ of the Ricci flow undergoes a *neckpinch* at time $T < \infty$ if there exists a time-dependent open subset $N_t \subseteq M^{n+1}$ such that N_t is diffeomorphic to a quotient of $\mathbb{R} \times S^n$ by a finite group acting freely, and such that the pullback of the metric $g(t)|N_t$ to $\mathbb{R} \times S^n$ approaches the 'shrinking cylinder' soliton

$$ds^2 + 2(n-1)(T-t)\, g_{\mathrm{can}} \tag{1.2}$$

in a suitable sense as $t \nearrow T$. (Here g_{can} denotes the round metric of radius 1 on S^n.) Except for a sphere shrinking to a round point, the neckpinch is perhaps the simplest singularity which the Ricci flow can encounter. It is also one of the most important with regard to the goal of obtaining topological information from the Ricci flow: it is expected that one can perform a geometric-topological surgery on the underlying manifold M^n just prior to a neckpinch in such a way that the maximum curvature of the solution is reduced by an amount large enough to permit the flow to be continued on the piece or pieces that remain after the surgery.

The first rigorous examples of neckpinches for the Ricci flow are constructed by Sigurd Angenent and the author in [**2**]. In fact, these are the first examples of any sort of nontrivial pinching of the Ricci flow on compact manifolds. The remainder of this short note will discuss some of the results obtained there and outline key aspects of their proof. The main results of that paper are as follows.

THEOREM 1. *If $n \geq 2$, there exists an open subset of the family of $\mathrm{SO}(n+1)$-invariant metrics on S^{n+1} such that any solution of the Ricci flow starting at a metric in this set will develop a neckpinch at some time $T < \infty$. The singularity is*

Type-I (rapidly-forming). Any sequence of parabolic dilations formed at the developing singularity converges to a shrinking cylinder soliton (1.2) uniformly in any ball of radius $o\left(\sqrt{-(T-t)\log(T-t)}\right)$ *centered at the neck.*

Any $\mathrm{SO}(n+1)$-invariant metric on S^{n+1} can be written in the form

$$g = \varphi^2\, dx^2 + \psi^2\, g_{\mathrm{can}} \tag{1.3}$$

on the set $(-1,1)\times S^n$, which may be identified in the natural way with the sphere S^{n+1} with its north and south poles removed. The quantity $\psi(x,t) > 0$ may thus be regarded as the 'radius' of the totally geodesic hypersurface $\{x\}\times S^n$ at time t. It is natural to write geometric quantities related to g in terms of the distance

$$s\,(x) = \int_0^x \varphi\,(x)\ dx$$

from the equator. Then writing $\frac{\partial}{\partial s} = \frac{1}{\varphi}\frac{\partial}{\partial x}$ and $ds = \varphi\, dx$, one puts equation (1.3) into the nicer form

$$g = ds^2 + \psi^2\, g_{\mathrm{can}}.$$

Armed with this notation, one can make a precise statement about the asymptotics of the developing singularity.

THEOREM 2. *Let* $\bar{s}\,(t)$ *denote the location of the smallest neck. Then there are constants* $\delta > 0$ *and* $C < \infty$ *such that for* t *sufficiently close to* T *one has the estimate*

$$1 + o(1) \le \frac{\psi(x,t)}{\sqrt{2(n-1)(T-t)}} \le 1 + \frac{C}{(T-t)\,|\log(T-t)|}(s-\bar{s})^2 \tag{1.4}$$

in the inner layer $|s-\bar{s}| \le 2\sqrt{(T-t)\,|\log(T-t)|}$, *and the estimate*

$$\frac{\psi(x,t)}{\sqrt{T-t}} \le C\frac{s-\bar{s}}{\sqrt{(T-t)\,|\log(T-t)|}}\log\frac{s-\bar{s}}{\sqrt{(T-t)\,|\log(T-t)|}} \tag{1.5}$$

in the intermediate layer $2\sqrt{(T-t)\,|\log(T-t)|} \le s-\bar{s} \le (T-t)^{\frac{1}{2}-\delta}$.

The estimates in the theorem are exactly those one gets when one writes the evolution equation (2.2) satisfied by ψ with respect to the self-similar space coordinate $\sigma = (s-\bar{s})/\sqrt{T-t}$ and time coordinate $\tau = \log\left(1/(T-t)\right)$ and derives formal matched asymptotics. We expect therefore to show in a forthcoming paper that these bounds are sharp.

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2. How the solution evolves

The first task in studying neckpinches is to compute basic geometric quantities related to the metric (1.3). One begins by observing that g will solve the Ricci flow (1.1) if and only if φ and ψ evolve by

$$\varphi_t = n\frac{\psi_{ss}}{\psi}\varphi \tag{2.1}$$

and

$$\psi_t = \psi_{ss} - (n-1)\frac{1-\psi_s^2}{\psi} \tag{2.2}$$

respectively. In order that $g(t)$ extend to a smooth solution of the Ricci flow on S^{n+1}, one imposes the boundary conditions

$$\lim_{x\to\pm1}\psi_s = \mp 1. \tag{2.3}$$

REMARK 1. *The partial derivatives ∂_s and ∂_t do not commute, but instead satisfy*

$$[\partial_t, \partial_s] = -n\frac{\psi_{ss}}{\psi}\partial_s.$$

REMARK 2. *Equation (2.1) will disappear in what follows, because the evolution of φ is controlled by the quantity ψ_{ss}/ψ.*

The Riemann curvature tensor of (1.3) is determined by the sectional curvatures

$$K_0 = -\frac{\psi_{ss}}{\psi} \tag{2.4}$$

of the n 2-planes perpendicular to the spheres $\{x\}\times S^n$, and the sectional curvatures

$$K_1 = \frac{1-\psi_s^2}{\psi^2} \tag{2.5}$$

of the $\binom{n}{2} = n(n-1)/2$ 2-planes tangential to these spheres. The Ricci tensor of g is thus

$$\mathrm{Rc} = (nK_0)\,ds^2 + (K_0 + (n-1)\,K_1)\,g_{\mathrm{can}}, \tag{2.6}$$

and its scalar curvature is

$$R = 2nK_0 + n\,(n-1)\,K_1. \tag{2.7}$$

3. Bounds on the curvature and other derivatives

To begin the proof, we use the maximum principle obtain control of the first and second derivatives of ψ. We begin by studying the first spatial derivative $v = \psi_s$. The calculation

$$v_t = v_{ss} + \frac{n-2}{\psi}vv_s + \frac{n-1}{\psi^2}\left(1-v^2\right)v$$

implies that $|\psi_s|$ is nonincreasing whenever it exceeds $1 = \lim_{x\to\pm1}|\psi_s|$. We next compute that the sectional curvature K_1 evolves by

$$(K_1)_t = (K_1)_{ss} + (n+2)\,\frac{\psi_s}{\psi}\,(K_1)_s + 2\left[K_0^2 + (n-1)\,K_1^2\right],$$

hence conclude that its minimum is nondecreasing. To understand K_0, we consider the scale-invariant measure of the difference between the two sectional curvatures given by

$$a = \psi^2 (K_1 - K_0).$$

A key step in the proof is to notice that a satisfies an attractive evolution equation

$$a_t = a_{ss} + (n-4)\frac{\psi_s}{\psi}a_s - 4(n-1)\frac{\psi_s^2}{\psi^2}a,$$

which implies that a is uniformly bounded. To understand ψ_t, we make use of the observation that ψ_t, a, and the curvatures satisfy the relations

$$\psi_t = -\psi [K_0 + (n-1) K_1] = \psi \left(K_0 - \frac{1}{n}R\right) = \frac{a}{\psi} - n\psi K_1.$$

The main conclusions of this part of the proof may be organized as

Claim 1. *Let g be a solution of the Ricci flow such that $|\psi_s| \leq 1$ and $R \geq 0$ initially, and let $\alpha = \sup |a(\cdot, 0)|$. Then:*

1. *For as long as the solution exists, $|\psi_s| \leq 1$.*
2. *For as long as the solution exists,*
$$-\frac{\alpha}{\psi^2} \leq K_1 - K_0 \leq \frac{\alpha}{\psi^2}.$$
3. *$(K_1)_{\min}$ is nondecreasing.*
4. *There exists $C = C(n, \alpha)$ such that for as long as the solution exists,*
$$|\mathrm{Rm}| \leq \frac{C}{\psi^2}.$$
5. *For as long as the solution exists, $R \geq 0$ and $\psi_t < 0$.*
6. *ψ^2 is a uniformly Lipschitz-continuous function of time; in fact, one has $\left|(\psi^2)_t\right| \leq 2(\alpha + n)$.*
7. *If $g(t)$ exists for $0 \leq t < T$, then $\lim_{t \nearrow T} \psi$ exists for each $x \in [-1, 1]$.*

4. The solution keeps its profile

We call local minima of $x \mapsto \psi(x,t)$ *necks* and local maxima *bumps.* We are interested in solutions whose initial data has at least one neck. The second part of the proof is to establish the sense in which the profile of the initial data persists, in particular to show that the solution will become singular at its smallest neck and nowhere else.

We begin with the observation that the form of equation (2.2) allows one to apply the Sturmian theorem [**1**]. This says that the number of necks cannot increase with time, and further that all bumps/necks will be nondegenerate maxima/minima unless one or more necks and bumps come together and erase each other.

We next derive upper and lower bounds for the rate at which a neck shrinks. These estimates show that a singularity will develop at the smallest neck in finite time, unless the solution loses all its necks first.

The final step in this part of the proof is to show that no singularity occurs on the 'polar cap': the region between the last bump and the pole. To do this, we first use the tensor maximum principle to show that the Ricci curvature is positive there. Then we prove that when $\eta > 0$ is chosen sufficiently small, the quantity

$$b = \frac{|a|}{\psi^\eta} = \psi^{2-\eta} |K_1 - K_0|$$

remains bounded in a neighborhood $\mathcal{B}$ of the pole. Because the exponent η breaks scale invariance, b may be regarded as a pinching inequality for the curvatures on the polar cap. The bound on b thus lets us apply a blow-up argument which shows that singularity formation on a polar cap under our hypotheses would lead to a contradiction.

The main results of this part of the proof compose

CLAIM 2. *Let g be a solution of the Ricci flow such that $|\psi_s| \leq 1$ and $R \geq 0$. Assume that the solution keeps at least one neck.*

(1) *At any time, the derivative ψ_s has finitely many zeroes. The number of zeroes is nonincreasing in time. If ψ ever has a degenerate critical point (one where $\psi_s = \psi_{ss} = 0$ simultaneously) the number of zeroes of ψ_s drops.*
(2) *There exists a time T bounded above by $r_{\min}(0)^2/(n-1)$ such that the radius $r_{\min}(t)$ of the smallest neck satisfies*
$$(n-1)(T-t) \leq r_{\min}(t)^2 \leq 2(n-1)(T-t).$$
(3) *The solution is concave ($\psi_{ss} < 0$) on the polar caps.*
(4) *Let $x_*(t)$ denote the right-most bump. If that bump persists, then the limit $D = \lim_{t \nearrow T} \psi(x_*(t), t)$ exists. If $D > 0$, no singularity occurs on the polar caps.*

5. The solution converges to a shrinking cylinder

In the third part of the proof, we derive estimates which indicate that a neck-pinch asymptotically approaches the shrinking cylinder soliton (1.2). We start by considering the quantity

$$F = -\frac{K_0}{K_1} \log K_1.$$

Notice that F is positive in a neighborhood of a neck. Writing $K = -K_0$ and $L = K_1$ to simplify the notation, one computes that F evolves by

$$F_t = \Delta F + 2\left(\frac{\log L - 1}{L \log L}\right) L_s F_s + \left(\frac{2 - \log L}{\log L}\right) \frac{K L_s^2}{L^3} - 2P\left(\frac{\psi_s}{\psi}\right)^2 \frac{K + L}{L} + 2QK,$$

where

$$P = (n-1)\log L - 2\frac{K}{L}(\log L - 1)$$

and

$$Q = n - 1 - \frac{K^2}{L^2}(\log L - 1) - F.$$

Note that the Laplacian of a radially symmetric function is given by

$$\Delta = \frac{\partial^2}{\partial s^2} + n\frac{\psi_s}{\psi}\frac{\partial}{\partial s}.$$

An application of the maximum principle lets us bound F from above when it is positive and $K_1 = L$ is sufficiently large. The value of this estimate is that the factor $\log K_1$ breaks scale invariance. Our bound on F thus shows that K_0/K_1 becomes small near a forming neckpinch.

The main results obtained in this part of the proof are as follows.

CLAIM 3. *Let $g(t) : 0 \leq t < T$ be a maximal solution of the Ricci flow such that $|\psi_s| \leq 1$ and $R \geq 0$. Assume that the solution has at least one neck.*

(1) *A singularity occurs at the smallest neck at some time $T < \infty$. This singularity is of Type I; in particular, there exists $C = C(n, g_0)$ such that*
$$|\mathrm{Rm}| \leq \frac{C}{T-t}.$$
(2) *There exists $C = C(n, g_0)$ such that*
$$\frac{K}{L}\left[\log L + 2 - \log L_{\min}(0)\right] \leq C.$$
(3) *Let $\bar{s}(t)$ denote the location of the smallest neck. Then there are constants $\delta > 0$ and $C < \infty$ such that for t sufficiently close to T, one has the estimate*
$$1 \leq \frac{\psi(x,t)}{r_{\min}} \leq 1 + \frac{C}{-\log r_{\min}}\left(\frac{s-\bar{s}}{r_{\min}}\right)^2$$
in the inner layer $|s-\bar{s}| \leq 2r_{\min}\sqrt{-\log r_{\min}}$, and the estimate
$$\frac{\psi(x,t)}{r_{\min}} \leq C\frac{s-\bar{s}}{r_{\min}\sqrt{-\log r_{\min}}}\log\frac{s-\bar{s}}{r_{\min}\sqrt{-\log r_{\min}}}$$
in the intermediate layer $2r_{\min}\sqrt{-\log r_{\min}} \leq s-\bar{s} \leq r_{\min}^{1-2\delta}$.

6. Neckpinches happen

Finally, we show that there exist initial data $\Psi = \psi(0)$ meeting our hypotheses. In particular, we construct simple examples obtained by removing a neighborhood of the equator of a standard sphere and replacing it with a long thin neck. These examples satisfy $\Psi = \sqrt{A + Bs^2}$ near the equator (for appropriate constants A and B) and blend smoothly into the standard sphere metric on the polar caps. Our construction justifies

CLAIM 4. *There exist initial metrics*
$$g = ds^2 + \Psi^2 g_{\mathrm{can}}$$
for the Ricci flow on S^{n+1} which satisfy $|\Psi_s| \leq 1$, have positive scalar curvature, and possess a neck sufficiently small and a bump sufficiently large so that under the flow, the neck must disappear before the bump can vanish. Hence these solutions exhibit a neckpinch singularity in finite time.

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