An injectivity radius estimate for sequences of solutions to the Ricci flow having almost nonnegative curvature operators

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1 Background and motivation

Some of the most challenging problems in geometric evolution equations stem from efforts to understand the singularities they develop. For the Ricci flow in particular, improving our understanding of singularities is a goal of substantial interest, and is currently one of the main focal points of research in the subject. Indeed, Richard Hamilton has formulated a well-developed program to use the Ricci flow to resolve Thurston's Geometrization Conjecture [T-82] for a closed 3-manifold $\mathcal{M}^3$. (See [H-95a], [H-97], [H-99], and the survey article [CC-99].) Because one expects singularities to occur for Ricci flow evolutions starting from a large set of initial Riemannian 3-manifolds $(\mathcal{M}^3, g_0)$, the strategy for dealing with such singularities constitutes a fundamental part of Hamilton's program. One of the methods for understanding singularities is to take limits of parabolic dilations approaching a singularity and then to analyze the possible limits that result. Such limits are called *singularity models*. In particular, let $(\mathcal{M}^n, g(t))$ be a solution of the Ricci flow on a maximal time interval $0 \leq t < T \leq \infty$. If $(\mathcal{M}^n, g(t))$ becomes singular in the sense that $\limsup_{t \nearrow T} |\mathrm{Rm}(x,t)| = \infty$, one carefully chooses points $x_i \in \mathcal{M}$ and times $t_i \in (0,T)$ with $t_i \nearrow T$. (The precise criteria for making these choices are somewhat technical, and will not be discussed here.) One dilates space by rescaling the metric so as to normalize $|\mathrm{Rm}(x_i, t_i)|$. Then one translates and dilates time so as to obtain a new marked solution $(\mathcal{M}^n, g_i(t), x_i : \tau_i \leq t < T_i)$ such that the metric $g_i(0)$ is a scalar multiple of $g(t_i)$. In the final step, one wants to obtain a singularity model as a pointed limit $(\mathcal{M}_\infty, g_\infty(t), x_\infty)$ of the sequence of marked solutions

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$(\mathcal{M}^n, g_i(t), x_i)$, using Gromov-type compactness arguments such as those in [H-95b]. (An excellent survey of related compactness theory can be found in [P-97].)

This last step cannot be accomplished without an injectivity radius estimate for the sequence $(\mathcal{M}^n, g_i(t), x_i)$. Such an estimate would follow from a proof of Hamilton's Little Loop Lemma for solutions on compact 3-manifolds with arbitrary initial metrics. The validity of the Little Loop Lemma would also rule out the formation of the so-called cigar soliton as a singularity model. A statement of the Little Loop Lemma can be found in §15 of [H-95a]. The proof there is incomplete, but Hamilton has announced a complete (unpublished) proof valid in dimension three for solutions with nonnegative sectional curvature. The only reason the condition of nonnegative sectional curvature is needed is so that one can apply a suitable Li–Yau–Hamilton (LYH) estimate (also referred to as a differential Harnack estimate). Such estimates became prominent in geometric analysis through the pioneering work [LY-86] of Peter Li and Shing-Tung Yau for the heat equation on Riemannian manifolds, and work of Hamilton [H-88],[H-93],[H-95c] for geometric evolution equations.

Because LYH estimates for the Ricci flow with *arbitrary* initial metrics on 3-manifolds are presently unknown, one must resort to various *ad hoc* methods to establish the injectivity radius estimates that are critical for taking limits of parabolic dilations, hence for studying singularities, hence for understanding their geometric and topological consequences. Such methods are available in certain special cases. For instance, Hamilton has proved an isoperimetric inequality (§23 of [H-95a]) that implies an injectivity radius estimate for appropriately chosen sequences of dilations approaching a Type I singularity of the Ricci flow in dimension three. In dimension two, Hamilton has proved an injectivity radius estimate crucial for establishing convergence of the flow on $\mathcal{S}^2$. (See §12 of [H-95a] and [C-91].) Hamilton has also proved an injectivity radius estimate for odd-dimensional solitons that is useful for dimension reduction. (See §22 of [H-95a].) Wilhelm Klingenberg's injectivity radius estimate is independent of the Ricci flow but it is very useful in the study of the Ricci flow. For Kähler manifolds with positive bisectional curvature, similarly there is an injectivity radius estimate of Xiuxiong Chen and Gang Tian [CT-00] useful for the study of the Kähler-Ricci flow.

There is another important case in which an injectivity radius estimate is expected to hold — namely, for sequences with almost nonnegative curvature operators. (The precise definition of such sequences will be given below.) Hamilton has given a proof that an injectivity radius estimate should hold in this case also, and has in fact used this result in an essential way in his classification [H-97] of 4-manifolds with positive isotropic curvature and no essential incompressible space form. However, his proof of this estimate (§25 of [H-95a]) appears incomplete, because of a gap in an essential step of the argument. The purpose of this note is to announce a new, complete proof of an injectivity radius estimate for such sequences.

2 The result

In order to state our result properly, it is necessary to make some definitions. Let

$$\{\mathcal{M}_i^n, g_i(t), O_i, F_i : i \in \mathbb{N}\}$$

be a sequence of complete marked solutions of the Ricci flow

$$\frac{\partial}{\partial t} g_i(t) = -2\,\mathrm{Rc}(g_i(t))$$

defined on a common time interval (α, ω), where $-\infty \leq \alpha < 0 < \omega \leq \infty$. Each solution is marked by an origin O_i and a frame $F_i = \{e_1^i, \ldots, e_n^i\}$ at O_i which is orthonormal with respect to $g_i(0)$. We say such a sequence has *uniformly bounded geometry* if there exists a family $\{C_k : k \in \mathbb{N} \cup \{0\}\}$ of constants such that

$$\sup_{i \in \mathbb{N}} \sup_{\mathcal{M}_i \times (\alpha,\omega)} \left| \nabla^k \mathrm{Rm}(g_i) \right|_{g_i} \leq C_k.$$

We denote the smallest eigenvalue of the curvature operator

$$\mathrm{Rm}_i(x,t) \doteqdot \mathrm{Rm}(g_i(t))|_x : \Lambda^2 T_x \mathcal{M}_i \to \Lambda^2 T_x \mathcal{M}_i$$

by $\lambda_1(\mathrm{Rm}_i)(x,t)$. We say a sequence

$$\{\mathcal{M}_i^n, g_i(t), O_i, F_i : i \in \mathbb{N}\}$$

with uniformly bounded geometry is a *sequence with almost nonnegative curvature operators* if the following three conditions hold:

Assumption 1 There exists a sequence $\delta_i \searrow 0$ such that

$$-1 \leq -\delta_i \leq \lambda_1(\mathrm{Rm}_i)(x,t)$$

for all $x \in \mathcal{M}_i$ and $t \in (\alpha, \omega)$.

Assumption 2 The diameters are tending to infinity, namely

$$\lim_{i \to \infty} [\mathrm{diam}(\mathcal{M}_i^n, g_i(0))] = \infty.$$

Assumption 3 Each origin O_i is a bump-like point of positive curvature at time $t = 0$; in other words, there exists $\varepsilon > 0$ such that for all i,

$$\lambda_1(\mathrm{Rm}_i)(O_i, 0) \geq \varepsilon.$$

Having established the necessary notation, we now announce the following:

Theorem 1 *For any sequence with almost nonnegative curvature operators and* $\mathrm{sect}(g_i)(x,0) \leq 1$ *for all* $x \in \mathcal{M}_i$ *and* $i \in \mathbb{N}$*, there exists a subsequence*

$$\{\mathcal{M}_i^n, g_i(t), O_i, F_i\}$$

such that for all i,

$$\mathrm{inj}_{g_i(0)}(O_i) \geq 1.$$

Remark 2 *This result is equivalent to Theorem 25.1 of [H-95a]. But as will be explained below, there appears to be a gap in an essential step of the proof given there.*

Remark 3 *Assumption 1 is automatically satisfied in dimension three if the sequence* $\{\mathcal{M}_i^n, g_i(t), O_i, F_i\}$ *arises from dilations about a singularity. This follows from §24 of [H-95a], [I-93], or §4 of [H-99].*

Remark 4 *If Assumption 2 does not hold (namely, if the diameters are uniformly bounded) then Theorem 1 is not needed for the purpose of determining geometrizability in the special case that all* $\mathcal{M}_i^3$ *are topologically the same closed manifold. Indeed, if one has an injectivity radius estimate for a subsequence, then one can take a limit using the techniques in [H-95b]. On the other hand, if no subsequence satisfies an injectivity radius estimate, then the original sequence collapses. In the (typical) case that all* $\mathcal{M}_i^3$ *are topologically a fixed smooth* 3*-manifold* $\mathcal{M}^3$*, this collapse implies by Cheeger–Gromov theory [CG-86, CG-90] that* $\mathcal{M}^3$ *is a graph manifold (all of which are known to be geometrizable).*

Remark 5 *In future work, we hope to weaken Assumption 3 by addressing the split case that some but not all of the eigenvalues of the curvature operator* Rm_i *can be uniformly bounded from below by* $\varepsilon > 0$ *at* O_i *at time* 0.

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3 The method of proof

In this section, we outline our proof of Theorem 1. The details of the proof appear in [CKL-01].

Our method follows essentially the same four steps as does the argument given in §25 of [H-95a], namely:

Step1 Use Assumptions 1 and 2 to find arbitrarily long minimizing geodesics along which the curvature is arbitrarily close to nonnegative.

Step 2 Use Assumptions 1 and 3 and the strong maximum principle to show that curvature is uniformly positive on large balls centered at the origins.

Step 3 Construct sets which mimic the sublevel sets of a Busemann function. Show that these sets are uniformly bounded in an appropriate sense.

Step 4 Use a second-variation argument along the long geodesics found in Step 1 to rule out short geodesic 1-gons in these sets.

However, Step 3 differs in quite significant ways from the approach found in [H-95a]. In order to motivate our departure from the methods employed there, let $\mathcal{S}_1^{n-1}$ denote the unit $(n-1)$-sphere in $\mathbb{R}^n$. Because the frame F_i induces a canonical isometry $\mathbb{R}^n \to T_{O_i}\mathcal{M}_i$, there is a well-defined function $\sigma_i : \mathcal{S}_1^{n-1} \to (0,\infty]$ denoting the distance to the cut locus in unit directions from the origin in $(\mathcal{M}_i^n, g_i(0))$. Before he has ruled out collapse of the sequence $(\mathcal{M}_i^n, g_i(0))$, Hamilton claims that the σ_i converge to a continuous function $\ell_\infty : \mathcal{S}_1^{n-1} \to [0,\infty]$ defined by

$$\ell_\infty(V) \doteqdot \lim_{V_i \to V} \sigma_i(V_i),$$

in particular asserting that this limit is independent of the sequence $V_i \to V$. Using the claimed continuity of ℓ_∞ in an essential way, Hamilton constructs sets which mimic the sublevel sets of the Busemann function.

An important step of Hamilton's argument that ℓ_∞ is well-defined and continuous is the construction of a Jacobi field in a geodesic tube for the case that $\exp_{O_i}(\ell_i V_i) = \exp_{O_i}(\ell_i W_i)$ for a sequence of distinct vectors such that $|V_i - W_i| \to 0$. The following example indicates why this approach encounters difficulties. It is *not* a counterexample to Hamilton's claim, because it does not include bump-like points of positive curvature. But it does illustrate the difficulty in proving that claim before one has ruled out collapse of the sequence. In particular, we do not see why the expression Hamilton writes out using a degenerating sequence of geodesic 2-gons yields a nontrivial Jacobi field which vanishes at its endpoints.

Example 7 *Consider a sequence* $\{\mathcal{T}_i^2 : i = 1, 2, \dots\}$ *of collapsing flat tori with fundamental domains*

$$[-i, i] \times \left[-\frac{1}{i}, \frac{1}{i}\right] \subset \mathbb{R}^2.$$

Take $O_i = (0,0)$, *and define constant-speed geodesics*

$$\alpha_i, \beta_i : \left[0, \frac{\sqrt{i^2-1}}{i}\right] \to \mathcal{T}_i^2$$

by

$$\alpha_i(s) = \left(s, \frac{s}{\sqrt{i^2-1}}\right) \qquad \text{and} \qquad \beta_i(s) = \left(s, -\frac{s}{\sqrt{i^2-1}}\right).$$

Then $\operatorname{length}\alpha_i = \operatorname{length}\beta_i = 1$ *for all* i. *But* α_i *and* β_i *converge in the universal cover* $\mathbb{R}^2$ *to the segment* $s \mapsto (s, 0)$ *defined for* $s \in [0,1]$. *Since* $\mathbb{R}^2$ *is flat, there is no nontrivial Jacobi field which vanishes at its endpoints.*

One can also construct 'local counterexamples' with constant positive curvature by removing small neighborhoods of the cone points from $\mathcal{S}^2/\mathbb{Z}_i$ *for* $i \in \mathbb{N}$, *and letting* $i \to \infty$. *This construction does not produce global counterexamples, since gluing thin infinite cylinders* $\mathcal{S}^1_{1/(2i)} \times (0,\infty)$ *to both ends and smoothing the metric will not result in metrics of almost nonnegative curvature.*

In the remainder of this section, we shall describe our implementation of the four main steps in the proof, giving particular attention to the innovations in Step 3 that are essential to our strategy.

3.1 Step 1 — finding ray-like directions

The goal of this step of the proof is to find arbitrarily long minimizing geodesics along which the curvature is arbitrarily close to nonnegative.

For each marked manifold $\{\mathcal{M}_i^n, g_i(t), O_i, F_i\}$, the frame F_i defines a canonical isometry

$$I_i : (\mathbb{R}^n, g_{\text{can}}) \to (T_{O_i}\mathcal{M}_i, g_i(O_i, 0)).$$

Thus there is for each i a well-defined map $\sigma_i : \mathcal{S}_1^{n-1} \to (0, \infty]$ such that $\sigma_i(V)$ is the distance to the cut locus of $g_i(0)$ in the direction identified with $V \in \mathcal{S}_1^{n-1}$. Let $\mathfrak{S}(V)$ denote the set of all sequences $\{V_i\} \subset \mathcal{S}_1^{n-1}$ such that $\lim_{i\to\infty} |V_i - V|_{g_{\text{can}}} = 0$. In contrast to the function ℓ_∞ introduced by Hamilton, we define $\sigma_\infty : \mathcal{S}_1^{n-1} \to [0, \infty]$ by

$$\sigma_\infty(V) \doteqdot \sup_{\mathfrak{S}(V)} \left(\limsup_{i\to\infty} \sigma_i(V_i) \right).$$

Hamilton defined a set $\mathcal{D}$ of *distinguished directions* by $\mathcal{D} \doteqdot \ell_\infty^{-1}(\infty)$. Instead, we define a set $\mathcal{R}_\infty$ of *ray-like directions* for the sequence $\{\mathcal{M}_i^n, g_i(0), O_i, F_i\}$ by

$$\mathcal{R}_\infty \doteqdot \sigma_\infty^{-1}(\infty).$$

Remark 8 *Along the lines of Hamilton's reasoning, if one could show that ℓ_∞ were well-defined and continuous, then for each $V \in \ell_\infty^{-1}(\infty)$, one would have $\lim_{i\to\infty} \sigma_i(V) = \infty$. In contrast, we have $V \in \mathcal{R}_\infty$ if and only if there is a subsequence $\{V_{i_j}\}$ from $\mathcal{S}_1^{n-1}$ such that $V_{i_j} \to V$ and $\sigma_{i_j}(V_{i_j}) \to \infty$. This necessitates the introduction of the parameter L in our definition of the replacements for the sublevel sets $N_i(L, K)$ of a Busemann function in Step 3, below. This parameter complicates the proof there that the sets $N_i(L, K)$ are eventually bounded uniformly.*

Remark 9 *With our definition, $\mathcal{R}_\infty$ may become smaller each time we pass to a subsequence. Steps 3 and 4 of our proof address this issue carefully.*

The main observation of Step 1 of our construction is that $\mathcal{R}_\infty$ is compact and nonempty. If $V \in \mathcal{R}_\infty$, then there exists $\{V_{i_j}\} \subset \mathcal{S}_1^{n-1}$ such that $V_{i_j} \to V$ and $\sigma_{i_j}(V_{i_j}) \to \infty$. Note that the curvature of $\left(\mathcal{M}_{i_j}^n, g_{i_j}(t)\right)$ is bounded from below by $-\delta_{i_j} \nearrow 0$. In particular, by taking j large enough, there will be an arbitrarily long minimizing geodesic in a direction V_{i_j} along which the curvature is arbitrarily close to nonnegative.

3.2 Step 2 — finding large balls of positive curvature

The goal of this step of the proof is to show that we can make the curvature positive in arbitrarily large neighborhoods of the origin by going sufficiently far out in the sequence. The strategy is to take limits in geodesic tubes, where an injectivity radius estimate is not needed. The curvature estimate itself then follows from the strong maximum principle, which lets us extend local conditions to arbitrarily large sets.

Part 1 In the first part of Step 2, Hamilton shows that any sequence having uniformly bounded geometry contains a subsequence that is *preconvergent in geodesic tubes,* meaning that for every direction $V \in \mathcal{S}_1^{n-1}$ and every length $L > 0$, the pullbacks of $g_i(t)$ converge uniformly in each C^k norm to a solution of the Ricci flow that exists for $t \in (\alpha, \omega)$ in a geodesic tube of length L in the direction V. This part of the proof uses familiar covering-space arguments.

Part 2 In the second part of the proof, we find a subsequence that is *preconverging to positive curvature,* meaning that for each $\rho > 0$, there are $\eta(\rho) > 0$ and $\iota(\rho)$ such that $\lambda_1(\mathrm{Rm}_i)(x, 0) \geq \eta(\rho)$ for all $i \geq \iota(\rho)$ and all x whose distance from O_i measured with respect to $g_i(0)$ is no greater than ρ. This is where the strong maximum principle is used.

3.3 Step 3 — mimicking the sublevel sets of a Busemann function

In this step, we depart significantly from the strategy suggested by Hamilton in §25 of [H-95a], and consequently must introduce new ideas.

Part 1 The first part of Step 3 of our proof is an essential modification of Hamilton's construction, which itself is a modification of the construction of the sublevel sets of a Busemann function. Recall that Gromoll and Meyer proved that if $(\mathcal{M}^n, g)$ is a complete noncompact manifold of positive sectional curvature bounded above by κ, then its injectivity radius can be bounded from below by $\pi/\sqrt{\kappa}$. A familiar way to prove this is to fix an origin $O \in \mathcal{M}^n$, use the rays emanating from O to construct a Busemann function, use that Busemann function to construct a totally convex neighborhood N of O, and then use a second variation argument along those rays to rule out short geodesics in the neighborhood N.

Our modified construction still follows this basic model. Given length scales K (like 1) and L (large), we construct sets $N_i(L, K)$ which act as substitutes for the sublevel sets $b^{-1}(-\infty, K]$ of a Busemann function b. (The corresponding sets in §25 of [H-95a] are constructed using those V for which $\ell_\infty(V) = \infty$, but our proof avoids using the function ℓ_∞.) We introduce the parameter L in the definition of $N_i(L, K)$ as a requirement that the geodesic segments used in its definition be of length at least L. These segments act as substitutes for the rays used in the construction of a Busemann function. Each set $N_i(L, K)$ is compact and weakly star shaped with respect to O_i. Moreover, $N_i(L,K)$ contains the closed ball of radius $\min\{\pi, K\}$, and is contained in the closed ball of radius $\max\{L,K\}$. However it is essential for the remainder of the proof to show that the sets $N_i(L,K)$ are uniformly bounded independently of L, at least for all i larger than some $I(L)$.

Part 2 As mentioned above, the second and most critical part of Step 3 is to bound the sets $N_i(L,K)$ appropriately. This step is the most difficult part of our entire proof, and contains the main innovations of our method.

We begin by showing that for any $\varepsilon > 0$, we can (after passing to a suitable subsequence) find an ε-*net* in $\mathcal{R}_\infty$, namely a finite subset $\{V_\alpha\}$ such that no member of $\mathcal{R}_\infty$ lies more than distance ε away from some V_α, and such that the lim sup in the definition of $\sigma_\infty(V_\alpha)$ is actually attained as a limit for each V_α. Construction of this ε-net is essential to ensure robustness under the action of passing to further subsequences.

Then we establish the sense in which the sets $N_i(L,K)$ are bounded. Note that we do not quite show that the $N_i(L,K)$ are uniformly bounded. Instead, we prove the following critical:

> *Boundedness Property.* Any sequence preconverging to positive curvature contains a subsequence for which there exists a constant $C < \infty$ depending on K such that for each $L \in (0, \infty)$, there exists $I(L)$ such that for all $i \geq I(L)$, we have
>
> $$N_i(L,K) \subseteq B_i(O_i, C).$$

This result is absolutely crucial to the remainder of the proof. Note that if $\mathcal{M}_i \subseteq B(O_i, L)$, then $N_i(L,K) = \mathcal{M}_i$. In particular, if each $\mathcal{M}_i$ is compact, then $N_i(\operatorname{diam}(\mathcal{M}_i), K) = \mathcal{M}_i$. This illustrates the need for the restriction in the Boundedness Property that i be large enough, depending on L. The lack of knowledge that ℓ_∞ (or σ_∞) is continuous requires us to introduce the parameter L in the definition of $N_i(L,K)$. However, this also makes it much more difficult to prove that the sets $N_i(L,K)$ are uniformly bounded. By contrast, this fact would have been obvious in the corresponding part of §25 of [H-95a] — if one knew that ℓ_∞ were well-defined and continuous. We refer the reader to [CKL-01] for the details of the proof of this result, in which we use the ε-net construction in a fundamental way.

3.4 Step 4 — ruling out short geodesics

The final step in our proof is essentially the same as the analogous argument in §25 of [H-95a]. Roughly speaking, the argument is as follows: to obtain a contradiction, one may suppose that for all i large enough there exists a geodesic 1-gon (unit-speed closed geodesic which is smooth everywhere except possibly at its base point) β_i of length $L(\beta_i) < 2$ and such that β_i is the shortest element of the set of all nondegenerate geodesic 1-gons contained in $N_i(L,1)$. In the easier case that there exists a subsequence along which each β_i is smooth at its base point, one obtains a contradiction by applying a second-variation argument along minimal geodesics from β_i to sufficiently distant points. In the (typical) case that β_i fails to be smooth, one perturbs β_i and applies a more delicate but essentially similar argument, again obtaining a contradiction. These arguments complete the proof of Theorem 1.

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