RICCI FLOW NECKPINCHES WITHOUT ROTATIONAL SYMMETRY

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Abstract. We study "warped Berger" solutions $(\mathcal{S}^1 \times \mathcal{S}^3, G(t))$ of Ricci flow: generalized warped products with the metric induced on each fiber $\{s\} \times \mathrm{SU}(2)$ a left-invariant Berger metric. We prove that this structure is preserved by the flow, that these solutions develop finite-time neckpinch singularities, and that they asymptotically approach round product metrics in space-time neighborhoods of their singular sets, in precise senses. These are the first examples of Ricci flow solutions without rotational symmetry that become asymptotically rotationally symmetric locally as they develop local finite-time singularities.

Contents

1. Introduction

There are many examples of solutions of parabolic geometric PDE that become round as they develop global singularities: for instance, this phenomenon has been observed, in chronological order, for 3-manifolds of positive Ricci curvature evolving by Ricci flow [11], for convex hypersurfaces evolving by mean curvature flow [12], for compact embedded solutions of curve-shortening flow [6, 10], and for 1/4-pinched solutions of Ricci flow [3]. Although these examples are usually viewed as special cases of the propensity of geometric flows to asymptotically approach constant-curvature geometries, it can be informative instead to interpret them in the spirit of Klein's *Erlangen Program* as examples of symmetry enhancement along geometric

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flows, with these solutions asymptotically acquiring larger symmetry groups than the symmetry groups of their initial data.

There is growing evidence that the same phenomenon holds locally in space-time neighborhoods of local singularities. For example, rotationally-symmetric solutions of Ricci flow that develop neckpinch singularities asymptotically acquire the additional translational symmetry of the cylinder soliton [1, 2]. More recently, it has been shown that any complete noncompact 2-dimensional solution of mean curvature flow that is sufficiently C^3-close to a standard round neck at some time will develop a finite-time singularity and become asymptotically rotationally symmetric in a space-time neighborhood of that singularity [7, 8]. As well, numerical experiments support the expectation that broader classes of mean curvature flow solutions asymptotically develop additional local symmetries as they become singular [9]. All of these results contribute to the developing heuristic principle that singularities of parabolic geometric evolution equations are nicer than one might naively expect.

In this paper, we obtain an analogous result for 4-dimensional solutions of Ricci flow, but with comparatively weaker hypotheses on the initial data than those used in [7, 8]. We replace the relatively strong hypothesis of C^3-closeness with the less restrictive structural assumption that the metrics under consideration are Riemannian submersions of a certain form. Specifically, we consider cohomogeneity-one (generalized warped product) solutions $\big(\mathcal{S}^1 \times \mathcal{S}^3, G(t)\big)$, where

$$G = (\mathrm{d}s)^2 + \Big\{ f^2\, \omega^1 \otimes \omega^1 + g^2 \big(\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3\big) \Big\}. \tag{1}$$

For each $s \in \mathcal{S}^1$, the quantity in braces is a left-invariant metric on the fiber $\mathrm{SU}(2) \approx \mathcal{S}^3$ over s, written with respect to a coframe $(\omega^1, \omega^2, \omega^3)$ that is algebraically dual to a fixed Milnor frame. The quantities f and g that control these geometries depend only on $s \in \mathcal{S}^1$ and $t \geq 0$; and the $\mathcal{S}^3$ fibers are round if and only if $f \equiv g$. We provide a detailed description of these geometries, which we call *warped Berger metrics,* in Appendix A. As shown there, this *Ansatz* is preserved by Ricci flow. We prove that if a metric of this form develops a local "neckpinch" singularity, then the fibers become asymptotically round in space-time neighborhoods of its singular sets. This asymptotic roundness is manifest in the quotient $f(s,t)\big/g(s,t)$ approaching 1 locally in C^2 as t approaches the singularity time. More precisely, we prove progressively stronger results under these progressively stronger assumptions:

Assumption 1. $\big(\mathcal{S}^1 \times \mathcal{S}^3, G(t)\big)$ *is a warped Berger solution of Ricci flow such that*

(1) $f \leq g$ *at* $t = 0$*;*
(2) $\big\{\min_{\mathcal{S}^1\times\mathcal{S}^3} R\big\}\big\{\max_{\mathcal{S}^1\times\mathcal{S}^3} g^2\big\} > -3$ *at* $t = 0$*; and*
(3) *there exists* $T < \infty$ *such that* $\limsup_{t\nearrow T} \max_{s\in\mathcal{S}^1} |\mathrm{Rc}(s,t)| = \infty$.

Assumption 2. $\big(\mathcal{S}^1 \times \mathcal{S}^3, G(t)\big)$ *is a warped Berger solution of Ricci flow that satisfies Assumption 1 and has the additional properties that at* $t = 0$*,*

(1) $f \geq (1-\varepsilon) g$ *for some small* $\varepsilon > 0$*;*[1] *and*
(2) $|f_s| \leq 1$.

Assumption 3. $\big(\mathcal{S}^1 \times \mathcal{S}^3, G(t)\big)$ *is a warped Berger solution of Ricci flow that satisfies Assumption 2 and is reflection symmetric at* $t = 0$*, with its smallest neck located at a fixed point* $\xi_* \in \mathcal{S}^1$.

[1] Note that $\varepsilon = 1/20$ is sufficiently small.

As we observe below, it follows easily from our construction in Appendix B that these assumptions are not vacuous. Moreover, one readily verifies that these assumptions allow for initial geometries that exhibit substantial initial deformations away from rotational symmetry in a neighborhood of the developing neck. Our main results are as follows:

Main Theorem. *The eccentricity of every warped Berger solution of Ricci flow is uniformly bounded: there exists C_0 depending only on the initial data such that the estimate*

$$|f-g| \le C_0 \min\{f,g\} \tag{2}$$

holds pointwise for as long as the solution exists, without additional assumptions.

(I) *There exist open sets of warped Berger metrics satisfying Assumption 1 such that all solutions originating in these sets develop local neckpinch singularities at some $T<\infty$. Each such solution has the properties that*

(A) *the ordering $f \le g$ is preserved; and*

(B) *the singularity is Type-I, with $|\operatorname{Rc}| \le C\big(\min f(\cdot,t)\big)^{-2}$, and*

$$\frac{1}{C}\sqrt{T-t} \le \min f(\cdot,t) \le C\sqrt{T-t}.$$

(II) *There exist open sets of warped Berger metrics satisfying Assumption 2 such that as solutions originating in these sets become singular, they become asymptotically round at rates that break scale invariance. Specifically, in addition to the properties above, they satisfy the following*[2] *C^0, C^1, and C^2 bounds at the neck:*[3] [4]

$$(T-t)^{-1/2}|f-g| \le C\sqrt{T-t}, \tag{3}$$

$$(T-t)|\kappa_{12}-\kappa_{23}| \le C\sqrt{T-t}, \tag{4}$$

$$(T-t)|\kappa_{01}-\kappa_{02}| \le C\sqrt{T-t}. \tag{5}$$

In a neighborhood of each smallest neck, where $\kappa_{01}<0$, there is the further bound

$$(T-t)\left(|\kappa_{01}|+|\kappa_{02}|\right) \le \frac{C}{|\log(T-t)|}. \tag{6}$$

The radius of a smallest neck is $(1+o(1))\,2\sqrt{T-t}$. Type-I blowups $\tilde{G}=(T-t)^{-1}G$ of the solution converge near each neck to the shrinking cylinder soliton. If S is the arclength from a smallest neck, and $\sigma := S/\sqrt{T-t}$, then there exist constants c, $C<\infty$ independent of time, such that as $t \nearrow T$, the estimates

$$1+o(1) \le \frac{f}{2\sqrt{T-t}} \le 1 + C\,\frac{\sigma^2}{|\log(T-t)|} \tag{7}$$

and

$$1+o(1) \le \frac{g}{2\sqrt{T-t}} \le \big(1+o(1)\big)\left(1 + C\,\frac{\sigma^2}{|\log(T-t)|}\right) \tag{8}$$

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[2]The first three estimates, which are proved in Section 5, hold under the weaker assumption that $|f_s| \le 2/\sqrt{3}$ initially.

[3]The κ_{ij} here are the sectional curvatures defined in equations (10)–(13) below. Note that for simple warped-product metrics with $f=g$, one has $\kappa_{12}=\kappa_{23}$ and $\kappa_{01}=\kappa_{02}$.

[4]We arrange these estimates to emphasize the scale invariance of the quantities on the LHS.

hold for $|\sigma| \leq c\sqrt{|\log(T-t)|}$*, and the estimate*

$$\frac{f}{\sqrt{T-t}} + \frac{g}{\sqrt{T-t}} \leq C\,\frac{|\sigma|}{\sqrt{|\log(T-t)|}}\sqrt{\log\left(\frac{|\sigma|}{\sqrt{|\log(T-t)|}}\right)} \tag{9}$$

holds for $c\sqrt{|\log(T-t)|} \leq |\sigma| \leq (T-t)^{-\varepsilon/2}$*, for* $\varepsilon \in (0,1)$*.*

(III) *There exist open sets of reflection-symmetric warped Berger metrics satisfying Assumption 3 such that any solution originating in these sets has the following property: for any small* δ *and large* Σ*, there exist* $T_* < T$ *and* C *such that the stronger estimate*

$$(T-t)^{-1/2}|f-g| \leq C(T-t)^{1+\delta}$$

holds for all $|\sigma| \leq \Sigma$ *and* $T_* < t < T$*.*

We note that warped Berger solutions may also develop global singularities in finite time; see Remarks 2–3 below. We further note that the assumption $f \leq g$ is geometrically natural for initial data giving rise to neckpinch singularities, in the following sense. Manifolds with $f \gg g$ locally resemble a product of a small $\mathcal{S}^2$ with a large surface and can have substantially negative scalar curvature. So it is not unreasonable to expect qualitatively different behavior for solutions originating from such initial data.

Our results in this paper are obtained in a series of Lemmas that prove more than we have summarized in the Main Theorem. The paper is organized as follows. In Appendix A, we review basic geometric calculations that show in particular that the metric *Ansatz* (1) and the inequality $f \leq g$ are preserved under Ricci flow. In Section 2, we summarize the conclusions of Appendix A that are needed in the remainder of the paper. In Section 3, we first prove estimate (2), which requires no assumptions beyond the form (1) of the metric. The results in Part (I) of the Main Theorem, which rely only on Assumption 1, are proved in the remainder of Section 3 and Section 4. The results in Part (II) of the Main Theorem, which rely on Assumption 2, are proved in Sections 5–6. The results in Part (III), which rely on Assumption 3, are proved in Section 7. In Appendix B, we demonstrate the existence of sets (open in the subspace of metrics with prescribed symmetries) of initial data that satisfy our various Assumptions. Then in Appendix C, we study parabolic dilations that motivate the calculations in Section 7 and lead one to expect that the precise asymptotics proved in [2] for rotationally symmetric neckpinches should be satisfied by the the non-rotationally symmetric solutions analyzed here.

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2. Ricci flow equations for warped Berger metrics

It follows from the calculations in Appendix A that for metrics of the form (1), the curvatures of the metric induced on each fiber $\{s\} \times \mathcal{S}^3$ are $\hat{\kappa}_{12} = \hat{\kappa}_{31} = f^2/g^4$ and $\hat{\kappa}_{23} = (4g^2 - 3f^2)/g^4$. The curvatures of the corresponding vertical planes in the total space are

$$\kappa_{12} = \kappa_{31} = \frac{f^2}{g^4} - \frac{f_s g_s}{fg} \tag{10}$$

and

$$\kappa_{23} = \frac{4g^2 - 3f^2}{g^4} - \frac{g_s^2}{g^2}. \tag{11}$$

The curvatures of mixed vertical-horizontal planes in the total space are

$$\kappa_{01} = -\frac{f_{ss}}{f} \tag{12}$$

and

$$\kappa_{02} = \kappa_{03} = -\frac{g_{ss}}{g}. \tag{13}$$

Using (10)–(13) together with (55), one determines that the Ricci flow equations for these geometries take the form

$$f_t = f_{ss} + 2\frac{g_s}{g} f_s - 2\frac{f^3}{g^4}, \tag{14a}$$

$$g_t = g_{ss} + \left(\frac{f_s}{f} + \frac{g_s}{g}\right) g_s + 2\frac{f^2 - 2g^2}{g^3}. \tag{14b}$$

If $f = g$, this system reduces to equation (10) in [1], with $n = 3$ and $\psi = f$.

To obtain this strictly parabolic form (14) for the Ricci flow equations, we have fixed a gauge,[5] replacing the non-geometric coordinate $\xi \in \mathcal{S}^1$ with a coordinate $s(\xi, t)$ representing arclength from a fixed but arbitrary point $\xi_0 \in \mathcal{S}^1$. By a variant of Calabi's trick, we may always assume that s is a smooth coordinate at any spatial point where we apply the maximum principle. We note that this choice of gauge results in the commutator formula

$$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = -(\log \rho)_t \frac{\partial}{\partial s} = -\left(\frac{f_{ss}}{f} + 2\frac{g_{ss}}{g}\right)\frac{\partial}{\partial s}. \tag{15}$$

Remark 1. The system (14) can be re-expressed in the more geometric form

$$\begin{aligned}(\log f)_t &= -\kappa_{01} - 2\kappa_{12},\\ (\log g)_t &= -\kappa_{02} - \kappa_{23} - \kappa_{31}.\end{aligned}$$

3. Controlling the evolving geometries

To proceed, we derive various evolution equations implied by the Ricci flow system (14). In doing so, we use the fact that for any C^2 function $\phi(s)$, one has

$$\Delta\phi = \phi_{ss} + \left\{\frac{f_s}{f} + 2\frac{g_s}{g}\right\}\phi_s. \tag{16}$$

3.1. **The shape of the metric.** Because $\mathcal{S}^1 \times \mathcal{S}^3$ is compact, there are well-defined functions M and $\check{M}$ given by

$$M(s,t) := \min\{f(s,t),\, g(s,t)\} \qquad \text{and} \qquad \check{M}(t) := \min_{s \in \mathcal{S}^1} M(s,t). \tag{17}$$

One readily verifies that $\check{M}$ is a Lipschitz continuous function of time.[6]

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[5]By equation (56) in Appendix A, the gauge function $\rho := \frac{\partial s}{\partial \xi}$ evolves by $(\log\rho)_t = \frac{f_{ss}}{f} + 2\frac{g_{ss}}{g}$. If $f = g$, this evolution equation reduces to equation (11) in [1], with $\varphi = \rho$.

[6]Comments related to this verification appear in the proof of Lemma 2.

We begin by considering scale-invariant quantities $(f-g)/g$ and $(g-f)/f$ that measure eccentricity: how far each fiber $\{s\}\times\mathcal{S}^3$ is from being round. The evolution of these quantities is governed by the equations

$$\left(\frac{f-g}{g}\right)_t = \Delta\left(\frac{f-g}{g}\right) + \left(\frac{g_s}{g} - \frac{f_s}{f}\right)\left(\frac{f-g}{g}\right)_s - 4\,\frac{f}{g^3}\left(\frac{f+g}{g}\right)\left(\frac{f-g}{g}\right) \tag{18}$$

and

$$\left(\frac{g-f}{f}\right)_t = \Delta\left(\frac{g-f}{f}\right) + \left(\frac{f_s}{f} - \frac{g_s}{g}\right)\left(\frac{g-f}{f}\right)_s - 4\,\frac{f}{g^3}\left(\frac{g+f}{f}\right)\left(\frac{g-f}{f}\right), \tag{19}$$

respectively. Using these equations, we show that the fibers must become round near any points where f or g become zero, as expressed in the following Lemma.

Lemma 1. *There exists C_0 depending only on the initial data such that the estimate*

$$|f-g| \le C_0 M$$

holds for as long as a given solution exists.

Proof. Applying the parabolic maximum principle to the evolution equations (18) and (19) for $(f-g)/g$ and $(g-f)/f$, one obtains C_0 depending only on the initial data such that

$$\left|\frac{f-g}{g}\right| \le C_0 \qquad \text{and} \qquad \left|\frac{g-f}{f}\right| \le C_0$$

for as long as a solution exists. The result immediately follows. □

We next derive a two-sided time-dependent bound for $\check{M}$, starting with an upper bound.

Lemma 2. *If there exists $T<\infty$ such that $\check{M}(T)=0$, then there exists a uniform constant C such that*

$$\check{M}^2 \le C(T-t).$$

Proof. We recall the standard fact that $\check{f}(t) := \min_{s\in\mathcal{S}^1} f(s,t)$ is a Lipschitz continuous function of time, so that its derivative $\check{f}'(t)$ exists for almost every $t\in(0,T)$. We here slightly abuse notation by regarding f as $f(\xi,t)$, where the spatial coordinate ξ is independent of time — i.e., we here ignore the arclength coordinate $s(\xi,t)$.

If t is such that $\check{f}'(t)$ exists, then it follows from the implicit function theorem that there exists a function $\bar{\xi}(\bar{t})$ defined for all $\bar{t}$ in a sufficiently small neighborhood of t such that $f(\bar{\xi}(\bar{t}),\bar{t}) = \check{f}(\bar{t}) = \min_{s\in\mathcal{S}^1} f(s,t)$. Therefore, $f_{\bar{\xi}}(\bar{\xi}(\bar{t}),\bar{t})=0$, and one has

$$\begin{aligned}
\frac{d}{dt}\check{f}(t) &= \frac{\partial}{\partial t} f(\bar{\xi}(t),t) + \frac{\partial}{\partial\bar{\xi}} f(\bar{\xi}(t),t)\frac{d\bar{\xi}}{dt}\\
&= f_t(\bar{\xi}(t),t)\\
&= f_{ss}(\bar{\xi}(t),t) - 2\frac{f^3(\bar{\xi}(t),t)}{g^4(\bar{\xi}(t),t)}\\
&\ge -\frac{C}{\check{f}(t)},
\end{aligned}$$

since f attains its global minimum in space at $\bar{\xi}(t)$, and since it follows from Lemma 1 that f and g are comparable. Thus there exists a uniform constant C such that $\frac{d}{dt}\{(\check{f})^2\} \ge -C$.

An entirely analogous argument applies to $\check{g}(t) := \min_{s\in\mathbb{S}^1} g(s,t)$. It follows easily that $\frac{d}{dt}(\check{M}^2) \geq -C$ holds almost everywhere in time,[7] whereupon integration yields

$$-\check{M}^2(t) = \check{M}^2(T) - \check{M}^2(t) \geq -C(T-t).$$

□

Lemma 3. *Suppose that at time $t=0$, the metric satisfies $f \leq g$, and the scalar curvature satisfies $\left\{\min_{\mathbb{S}^1\times\mathbb{S}^3} R\right\}\left\{\max_{\mathbb{S}^1\times\mathbb{S}^3} g^2\right\} > -3$. If there exists $T<\infty$ such that $\check{M}(T)=0$, then there exists a uniform constant c such that*

$$\check{M}^2 \geq c(T-t).$$

Proof. The positive function $m(t) := \min_{\mathbb{S}^1\times\mathbb{S}^3}(fg^2)$ is Lipschitz continuous. It follows from (14) that

$$\frac{\partial}{\partial t}\log(fg^2) = \left(\frac{f_{ss}}{f} + 2\frac{g_{ss}}{g}\right) + 4\frac{f_s g_s}{fg} + 2\frac{g_s^2}{g^2} + 2\frac{f^2}{g^4} - \frac{8}{g^2}.$$

Since R is a supersolution of the heat equation (in the sense that $(\partial_t - \Delta)R \geq 0$), there exists a constant r_0 depending only on the initial data such that for as long as the flow exists, one has

$$r_0 \leq R = \kappa_{01} + \kappa_{02} + \kappa_{03} + \kappa_{12} + \kappa_{23} + \kappa_{31}.$$

Substituting in expressions (10)–(13) for the curvatures and simplifying, one obtains

$$g f_{ss} + 2 f g_{ss} \leq (4 - r_0 g^2)\frac{f}{g} - \frac{f^3}{g^3} - 2 f_s g_s - \frac{f}{g} g_s^2. \tag{20}$$

Using this estimate and the consequence of Lemma 25 (in Appendix A) that the ordering $f \leq g$ is preserved along the flow, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\log(fg^2) &\leq \left(\frac{4 - r_0 g^2}{g^2} - \frac{f^2}{g^4} - 2\frac{f_s g_s}{fg} - \frac{g_s^2}{g^2}\right) + 4\frac{f_s g_s}{fg} + 2\frac{g_s^2}{g^2} + 2\frac{f^2}{g^4} - \frac{8}{g^2} \\
&= \frac{4 - r_0 g^2}{g^2} + \frac{f^2}{g^4} + 2\,\frac{f_s g_s}{fg} + \frac{g_s^2}{g^2} - \frac{8}{g^2} \\
&\leq -\frac{3 + r_0 g^2}{g^2} + 2\frac{g_s}{g}(\log(fg^2))_s - \frac{3g_s^2}{g^2}.
\end{aligned}$$

This implies that almost everywhere in time, one has

$$\frac{d}{dt}(\log m) \leq -r_0 - \frac{3}{g^2}.$$

It is easy to see from (14) that if $f \leq g$, then $g_{\max}(\cdot,t)$ is a non-increasing function of time. So it follows from our assumptions on f, g, and R that there exists $c_0 > 0$ such that $r_0 \geq -(3-c_0)/g^2$, which implies that

$$\frac{dm}{dt} \leq -c_0 f \leq -c(fg^2)^{\frac{1}{3}},$$

where $c>0$ is another uniform constant whose existence follows from Lemma 1. Because there exists $T<\infty$ with $\check{M}(T)=0$, it is clear that $m(T)=0$. Integrating the a.e. inequality

$$\frac{dm}{dt} \leq -cm^{\frac{1}{3}}$$

[7] This differential inequality may be interpreted as the lim sup of forward difference quotients.

over the time interval $[t, T]$, we thus obtain

$$m(t)^{\frac{2}{3}} \geq c\,(T-t).$$

Now by Lemma 1, the inequality

$$f^3 \geq \frac{fg^2}{C} \geq \frac{\min_{\mathbb{S}^1\times\mathbb{S}^3}(fg^2)}{C} = \frac{m(t)}{C}$$

holds everywhere in space and time, which implies in particular that [8]

$$\min f(\cdot,t)^2 \geq \frac{m(t)^{\frac{2}{3}}}{C} \geq \frac{c}{C}(T-t).$$

The same reasoning applies to $\min g(\cdot,t)^2$, whence the result follows. □

3.2. **Evolution of first derivatives.** Using (15), it is straightforward to compute that

$$(f_s)_t = \Delta(f_s) - 2\frac{f_s}{f}(f_s)_s - \left\{6\frac{f^2}{g^4} + 2\frac{g_s^2}{g^2}\right\} f_s + 8\frac{f^3}{g^5}\,g_s \tag{21}$$

and

$$(g_s)_t = \Delta(g_s) - 2\frac{g_s}{g}(g_s)_s + \left\{\frac{4}{g^2} - \frac{g_s^2}{g^2} - \frac{f_s^2}{f^2} - 6\frac{f^2}{g^4}\right\} g_s + 4\frac{f}{g^3}f_s. \tag{22}$$

If $f = g$, these reduce to equation (16) in [1].

Lemma 4. *Suppose that $f \leq g$ at time $t = 0$, and define*

$$C_f := \max\left\{\frac{2}{\sqrt{3}},\ \max|f_s(\cdot,0)|\right\},$$

$$C_g := \max\left\{2\sqrt{2},\ \max|g_s(\cdot,0)|\right\}.$$

Then for as long as a solution exists, one has

$$|f_s| \leq C_f \qquad \textit{and} \qquad |g_s| \leq C_g.$$

Proof. Consider $(f_s)_{\max}$, and assume that $(f_s)_{\max} \geq C > 0$ where C is sufficiently large. By Lemmas 1 and 25 (in Appendix A), we have

$$f(\cdot,t) \leq g(\cdot,t) \leq (1+C_0)f(\cdot,t) \tag{23}$$

for as long as a solution exists, where C_0 is the uniform constant in Lemma 1. Recalling the evolution equation (21) for f_s and applying weighted Cauchy–Schwarz,[9] we obtain

$$\begin{aligned}
\frac{d}{dt}(f_s)_{\max} &\leq -(f_s)_{\max}\left(6\frac{f^2}{g^4} + 2\frac{g_s^2}{g^2}\right) + 8\frac{f^3}{g^5}g_s \\
&\leq -C\left(6\frac{f^2}{g^4} + 2\frac{g_s^2}{g^2}\right) + \frac{4}{\sqrt{3}}\frac{g_s^2}{g^2} + \frac{12}{\sqrt{3}}\frac{f^6}{g^8} \\
&\leq -C\left(6\frac{f^2}{g^4} + 2\frac{g_s^2}{g^2}\right) + \frac{4}{\sqrt{3}}\frac{g_s^2}{g^2} + \frac{12}{\sqrt{3}}\frac{f^2}{g^4} \\
&\leq 0,
\end{aligned}$$

[8] Here as elsewhere in this paper, we follow the convention in analysis that uniform constants are allowed to change from line to line without relabeling.

[9] To wit, we estimate $ab \leq \epsilon a^2 + \frac{1}{4\epsilon}b^2$, with $a = \frac{g_s}{g}$, $b = \frac{f^3}{g^4}$, and $\epsilon = \frac{1}{2\sqrt{3}}$.

if we choose $C \geq 2/\sqrt{3}$. This implies that there is a sufficiently large constant C such that $(f_s)_{\max} \leq C$ uniformly, as long as the flow exists. Similarly we also get a uniform bound $(f_s)_{\min} \geq -C$.

We now consider $(g_s)_{\max}$. Suppose $(g_s)_{\max} \geq C_g \geq \sqrt{8}$. Then

$$\frac{4}{g^2} - \frac{g_s^2}{g^2} \leq -\frac{1}{2}\frac{g_s^2}{g^2} \leq -\frac{4}{g^2}.$$

So it follows from the evolution equation (22) for g_s that

$$\frac{d}{dt}(g_s)_{\max} \leq -\left\{\frac{4}{g^2} + \frac{f_s^2}{f^2}\right\}(g_s)_{\max} + \frac{4ff_s}{g^3}.$$

Using weighted Cauchy–Schwarz and the fact that the inequality $f \leq g$ is preserved, we obtain

$$\frac{4ff_s}{g^3} \leq \frac{4f^4}{g^6} + \frac{f_s^2}{f^2} \leq \frac{4}{g^2} + \frac{f_s^2}{f^2}.$$

Hence at any sufficiently large value of $(g_s)_{\max}$, one has

$$\frac{d}{dt}(g_s)_{\max} \leq 0.$$

A similar argument shows that $(g_s)_{\min} \geq -C_g$. □

Corollary 5. *If $f \leq g$ initially, then there exists C depending only on the initial data such that the estimate*

$$|\kappa_{12}| + |\kappa_{31}| + |\kappa_{23}| \leq \frac{C}{M^2}$$

holds for as long as a solution exists.

Proof. Because

$$\kappa_{12} = \kappa_{31} = \frac{f^2}{g^4} - \frac{f_s g_s}{fg} \qquad \text{and} \qquad \kappa_{23} = \frac{4}{g^2} - \frac{3f^2}{g^4} - \frac{g_s^2}{g^2},$$

the stated bound follows immediately from Lemma 1 and Lemma 4. □

3.3. **Evolution of second derivatives.** After further tedious but straightforward computations, one finds that κ_{01} and κ_{02} evolve by

$$(\kappa_{01})_t = \Delta(\kappa_{01}) + 2\kappa_{01}^2 - 4\left\{\frac{g_s^2}{g^2} + \frac{f^2}{g^4}\right\}\kappa_{01} + 4\left\{\kappa_{12} + \frac{f^2}{g^4}\right\}\kappa_{02} + 12\frac{f_s^2}{g^4} + 40\frac{f^2g_s^2}{g^6} - 48\frac{ff_sg_s}{g^5} - 4\frac{f_sg_s^3}{fg^3} \tag{24}$$

and

$$(\kappa_{02})_t = \Delta(\kappa_{02}) + 2\kappa_{02}^2 + \left(\frac{4f^2}{g^4} - \frac{2f_sg_s}{fg}\right)\kappa_{01} + \left(\frac{8}{g^2} - \frac{8f^2}{g^4} - \frac{2f_s^2}{f^2} - \frac{4g_s^2}{g^2}\right)\kappa_{02} - \frac{4f_s^2}{g^4} + \frac{24ff_sg_s}{g^5} - \frac{2f_s^3g_s}{f^3g} - \frac{24f^2g_s^2}{g^6} + \frac{8g_s^2}{g^4} - \frac{2g_s^4}{g^4}, \tag{25}$$

respectively. If $f = g$, these reduce to equation (22) in [1], using the identifications $K = -\kappa_{01} = -\kappa_{02}$ and $L = \kappa_{12} = \kappa_{23}$.

It follows from [14] that a singularity occurs at $T<\infty$ only if

$$\limsup_{t\nearrow T}\max_{s\in\mathbb{S}^1}|\operatorname{Rc}(s,t)|=\infty.$$

We now show that all remaining curvatures are controlled by $\check{M}$ at a finite-time singularity.

Lemma 6. *Suppose that at time $t=0$, the metric satisfies $f\leq g$, and the scalar curvature satisfies $\{\min_{\mathbb{S}^1\times\mathbb{S}^3}R\}\{\max_{\mathbb{S}^1\times\mathbb{S}^3}g^2\}>-3$. If the norm of Rc becomes unbounded as $t\nearrow T<\infty$, then $\check{M}(T)=0$, and there exists a uniform constant C such that*

$$|\kappa_{01}|+|\kappa_{02}|+|\kappa_{03}|\leq\frac{C}{\check{M}^2}.$$

Proof. Corollary 5 bounds the sectional curvatures of vertical planes by C/M^2. So it remains only to consider the mixed curvatures κ_{01} and $\kappa_{02}=\kappa_{03}$.

To control κ_{01} from above, we work with $K^*:=\kappa_{01}+a\,f_s^2/f^2+b\,g_s^2/g^2$, where a and b are positive constants to be chosen. Clearly, it follows from this definition that K^* is an upper bound for κ_{01}. To derive an estimate for the time derivative of K^*, we begin by estimating the time derivative of κ_{01}. Applying Lemma 1, Lemma 4, and the Cauchy–Schwarz inequality to equation (24), we obtain

$$\begin{aligned}(\kappa_{01})_t&\leq\Delta(\kappa_{01})+2\kappa_{01}^2+C\left\{\frac{\kappa_{01}}{M^2}+\frac{\kappa_{02}}{M^2}+\frac{1}{M^4}\right\}\\&\leq\Delta(\kappa_{01})+C\left\{\kappa_{01}^2+\kappa_{02}^2+\frac{1}{M^4}\right\}.\end{aligned}$$

We next calculate the time derivatives of the quadratic terms in K^*, obtaining

$$\begin{aligned}\left(\frac{f_s^2}{f^2}\right)_t&=\Delta\left(\frac{f_s^2}{f^2}\right)-\frac{8f_s^2}{g^4}-\frac{4f_s^2}{f^4}+\frac{16ff_sg_s}{g^5}-\frac{4f_s^2g_s^2}{f^2g^2}-\frac{4f_s^2}{f^2}\kappa_{01}-2\kappa_{01}^2,\\\left(\frac{g_s^2}{g^2}\right)_t&=\Delta\left(\frac{g_s^2}{g^2}\right)+\frac{8ff_sg_s}{g^5}-\frac{16f^2g_s^2}{g^6}+\frac{16g_s^2}{g^4}-\frac{2f_s^2g_s^2}{f^2g^2}-\frac{6g_s^4}{g^4}-\frac{4g_s^2}{g^2}\kappa_{02}-2\kappa_{02}^2.\end{aligned}$$

Again applying Lemma 1, Lemma 4, and weighted Cauchy–Schwarz, we get

$$\begin{aligned}\left(\frac{f_s^2}{f^2}\right)_t&\leq\Delta\left(\frac{f_s^2}{f^2}\right)-\kappa_{01}^2+\frac{C}{M^4},\\\left(\frac{g_s^2}{g^2}\right)_t&\leq\Delta\left(\frac{g_s^2}{g^2}\right)-\kappa_{02}^2+\frac{C}{M^4}.\end{aligned}$$

It immediately follows that for a,b chosen large enough, one has

$$K_t^*\leq\Delta K^*+\frac{C}{M^4},$$

and hence

$$\frac{d}{dt}(K^*)_{\max}\leq\frac{C}{\check{M}^4}.\tag{26}$$

This inequality, together with the mean value theorem, imply that K^* and hence κ_{01} cannot approach $+\infty$ on any time interval on which $\check{M}$ is bounded away from zero.

To control κ_{01} from below, we work with $K_* := \kappa_{01} - c\, g_s^2/g^2$, where c is a positive constant to be chosen. This quantity clearly serves as a lower bound for κ_{01}. Calculating as above, we obtain the estimate

$$(\kappa_{01})_t \geq \Delta(\kappa_{01}) - C\left\{\kappa_{02}^2 + \frac{1}{M^4}\right\}.$$

Combining this with the inequality derived above for g_s^2/g^2, we see that for c chosen large enough, one has $(K_*)_t \geq \Delta K_* - C/M^4$, and hence

$$\frac{d}{dt}(K_*)_{\min} \geq -\frac{C}{\check{M}^4}. \tag{27}$$

It follows that K_*, and hence κ_{01}, cannot approach $-\infty$ on any time interval on which $\check{M}$ is bounded away from zero. Combining this result with that obtained above, we see that κ_{01} becomes singular at a finite time T only if $\check{M} = 0$.

To determine the specific relation between κ_{01} and $\check{M}$, we combine the estimates for $\check{M}$ obtained in Lemma 2 and Lemma 3 with estimates (26) and (27), thereby obtaining

$$\frac{d}{dt}K^*_{\max} \leq \frac{C}{(T-t)^2} \qquad \text{and} \qquad \frac{d}{dt}(K_*)_{\min} \geq -\frac{C}{(T-t)^2}.$$

Integrating these inequalities leads to the estimate $|\kappa_{01}| \leq C_1 + C/(T-t)$. Then applying Lemma 2 again, we get the desired control on κ_{01}, which is

$$|\kappa_{01}| \leq \frac{C}{\check{M}^2}.$$

The estimate for $|\kappa_{02}|$ is obtained similarly, using $\kappa_{02} + a\, f_s^2/f^2 + b\, g_s^2/g^2$ for an upper bound, and $\kappa_{02} - c\, f_s^2/f^2$ for a lower bound. □

4. Analysis of singularities

In this section, we study solutions of Ricci flow satisfying Assumption 1, as stated in the introduction.

Remark 2. To see that Assumption 1 is not vacuous, it suffices to observe that initial data with $f \leq g$ both constant have strictly positive constant scalar curvature $R = (4g^2 - f^2)/g^4$. So there is a neighborhood of these products in the space of metrics $\mathfrak{Met}(\mathcal{S}^1 \times \mathcal{S}^3)$ such that all warped Berger solutions originating from initial data in this neighborhood satisfy the first two hypotheses of the Assumption and also become singular in finite time. The last fact follows from the standard estimate $R_{\max}(t) \geq \big([R_{\max}(0)]^{-1} - t/2\big)^{-1}$. By Lemma 1, these solutions will develop finite-time (global or local) singularities and will satisfy $f = g$ (hence "become round") at all points where $M = \check{M} = 0$.

Remark 3. It is expected that open sets of warped Berger solutions will encounter global singularities in which the geometry shrinks uniformly around the $\mathcal{S}^1$ factor; for example, this is expected for solutions originating from initial data sufficiently near the products described in Remark 2. On the other hand, we show in Appendix B below that there exist open sets of warped Berger solutions that develop local neckpinch singularities. Unless otherwise stated, the results in this paper apply to both cases.

It is clear from Corollary 5 and the proof of Lemma 6 that the singular set Σ, i.e. the set of points $\{\xi\} \times \mathcal{S}^3 \subseteq \mathcal{S}^1 \times \mathcal{S}^3$ such that $\limsup_{t\nearrow T} |\operatorname{Rc}(\xi,t)| = \infty$, coincides with the set Σ_0 of points such that $M(\xi,t) \searrow 0$ as $t \nearrow T$. Moreover, Lemma 3 shows that the singularity is Type-I. It therefore follows from [5] that $\Sigma = \Sigma_R$, where Σ_R denotes the set of points at which the scalar curvature blows up at the Type-I rate as $t \nearrow T$.

Our first observation is that the solution has a well-defined profile at the singular time.

Lemma 7. *If a solution $\big(\mathcal{S}^1 \times \mathcal{S}^3, G(t)\big)$ of Ricci flow satisfies Assumption 1 and becomes singular at $T < \infty$, then the limits $\lim_{t\nearrow T} f(\xi,t)$ and $\lim_{t\nearrow T} g(\xi,t)$ both exist for all $\xi \in \mathcal{S}^1$.*

Proof. We observe that

$$(f^2)_t = 2ff_{ss} + 4\frac{f}{g}f_s g_s - 4\frac{f^4}{g^4}$$

and

$$(g^2)_t = 2gg_{ss} + 2\left(\frac{g}{f}f_s g_s + g_s^2\right) + 4\frac{f^2}{g^2} - 8.$$

It thus follows from Lemmas 1, 4, and 6 that there is a uniform constant C such that

$$|(f^2)_t| \le C \qquad \text{and} \qquad |(g^2)_t| \le C.$$

Consequently both f^2 and g^2 are uniformly Lipschitz-continuous functions of time. □

Corollary 8. *If a solution $\big(\mathcal{S}^1 \times \mathcal{S}^3, G(t)\big)$ of Ricci flow satisfies Assumption 1 and becomes singular at $T < \infty$, then there exists a uniform constant C such that*

$$f^2(s,t) \ge f^2(s,0) - Ct \qquad \textit{and} \qquad g^2(s,t) \ge g^2(s,0) - Ct.$$

5. Sharper estimates

In this section, we obtain stronger results under the more restrictive hypotheses on the initial data detailed in Assumption 2 in the introduction, with the goal of breaking scaling invariance. It follows easily from Remark 2 above that Assumption 2 is not vacuous. Furthermore, Remark 6 below shows that the assumption is satisfied by an open set of warped Berger solutions that develop local singularities.

Our first result shows that solutions originating from original data that are not too far from round become asymptotically round near their singular sets at a rate that breaks scale invariance, hence that improves upon the scale-invariant C^0 estimate of Lemma 1.

Lemma 9. *If a solution $\big(\mathcal{S}^1 \times \mathcal{S}^3, G(t)\big)$ satisfies Assumption 2, then there exists a uniform constant C such that for as long as the flow exists, one has $0 < \frac{1}{f} - \frac{1}{g} \le C$ and hence*

$$0 < g - f \le CM^2. \tag{28}$$

Proof. Define $h := \frac{1}{f} - \frac{1}{g}$. It easily follows from (14) that h evolves by

$$h_t = \Delta h + \left(\frac{g_s}{g^2} + \frac{f_s}{f^2}\right) g h_s + (f-g)\left(\frac{2f+4g}{g^5} - \frac{f_s^2}{f^4}\right). \tag{29}$$

By Assumption 2 and Lemma 1, the inequality $g \le (1+C_0)f$ is preserved, where $1 + C_0 = (1-\varepsilon)^{-1}$. By Assumption 2 and Lemma 4, the inequality $|f_s| \le 2/\sqrt{3}$ persists as well. If ε is small enough[10] that $2(1-\varepsilon)^5 + 4(1-\varepsilon)^4 > 4/3$, which follows from Assumption 2, then we have

$$\frac{2f^5 + 4f^4 g - f_s^2 g^5}{f^4 g^5} \ge \frac{2(1-\varepsilon)^5 + 4(1-\varepsilon^4) - \frac{4}{3}}{f^4} > 0.$$

Combining this inequality and Lemma 25 (which guarantees that $f \le g$ for as long as the flow exists) with evolution equation (29), we obtain

$$\frac{d}{dt} h_{\max} \le 0.$$

It follows that $0 < g - f \le Cfg$. This inequality and Lemma 1 together imply (28). $\square$

We next obtain a C^1 estimate for solutions satisfying Assumption 2. This estimate improves upon Lemma 4 and shows that solutions become round in spatial neighborhoods of their singular sets.

Lemma 10. *If a solution $\left(\mathcal{S}^1 \times \mathcal{S}^3, G(t)\right)$ satisfies Assumption 2, then there exists a uniform constant C such that for as long as the flow exists, one has*

$$|(f-g)_s| \le CM. \tag{30}$$

Proof. Consider the quantity

$$Q := \left(\frac{f_s}{f} - \frac{g_s}{g}\right)^2. \tag{31}$$

We claim that if one can show that $Q \le C$ for some uniform constant C, then estimate (30) follows. To verify this claim, we observe that if $Q \le C$, then one has

$$\begin{aligned}
\left(\frac{f_s - g_s}{f}\right)^2 &= \left\{\frac{f_s}{f} - \frac{g_s}{g} + g_s\left(\frac{1}{g} - \frac{1}{f}\right)\right\}^2 \\
&\le 2Q + 2|g_s|^2\left(\frac{1}{f} - \frac{1}{g}\right)^2 \\
&\le C.
\end{aligned}$$

Here we have used Lemmas 4 and 9 to bound the second term on the second line. This implies the result we want in the form $|f_s - g_s|^2 \le Cf^2 = CM^2$.

We proceed to prove $Q \le C$. We readily verify that Q evolves by

$$\frac{\partial}{\partial t} Q = \Delta Q - \frac{Q_s^2}{2Q} - 2Q\left(\frac{2g_s^2}{g^2} + \frac{f_s^2}{f^2} + \frac{8f^2}{g^4}\right) + \frac{16 g_s\left(f^2 - g^2\right)}{g^5}\left(\frac{f_s}{f} - \frac{g_s}{g}\right).$$

[10] $\varepsilon = 1/4$ is sufficiently small here.

We obtain

$$\begin{aligned}\frac{d}{dt}\,Q_{\max} &\le -\frac{16(1-\varepsilon)^2}{f^2}\,Q_{\max} + 16\,\frac{|g_s|(g-f)(g+f)\sqrt{Q_{\max}}}{g^5}\\ &\le \frac{-16(1-\varepsilon)^2\,Q_{\max} + C\sqrt{Q_{\max}}}{f^2}\end{aligned}$$

by using Lemmas 4 and 9. Because the numerator is negative if $Q_{\max} > \frac{C^2}{256(1-\varepsilon)^2}$, we conclude that $Q_{\max} \le C'$. Estimate (30) follows. ☐

The results obtained thus far imply that the curvatures of vertical planes $\kappa_{12} = \kappa_{31}$ and κ_{23} become close near a singularity at a rate that breaks scale invariance.

Corollary 11. *If a solution $\big(\mathcal{S}^1 \times \mathcal{S}^3, G(t)\big)$ satisfies Assumption 2, then there is a uniform constant C such that for as long as the flow exists, one has*

$$|\kappa_{12} - \kappa_{23}| \le \frac{C}{M}, \tag{32}$$

and hence

$$(T-t)|\kappa_{12} - \kappa_{23}| \le C\sqrt{T-t}. \tag{33}$$

Proof. From the curvature formulas (10) and (11), one readily verifies that

$$\kappa_{12} - \kappa_{23} = 4\frac{(f+g)(f-g)}{g^4} + \frac{g_s(fg_s - gf_s)}{fg^2}. \tag{34}$$

Estimate (32) then follows from (34), together with Lemmas 4, 9, and 10. Finally, applying Lemmas 2 and 3 to (32), we obtain estimate (33). ☐

The mixed sectional curvatures also become close at a rate that breaks scaling.

Lemma 12. *If a solution $\big(\mathcal{S}^1 \times \mathcal{S}^3, G(t)\big)$ satisfies Assumption 2, then there exists a uniform constant C such that for as long as the flow exists, one has*

$$|\kappa_{01} - \kappa_{02}| \le \frac{C}{\check{M}}.$$

Proof. We define the quantity $k := \frac{f_s}{f} - \frac{g_s}{g}$, observing that it follows from Lemma 10 that $k = \sqrt{Q}$ satisfies $|k| \le C$. Using equations (14) and (21)–(22), one readily calculates that k evolves by

$$k_t = \Delta k - \left(\frac{2g_s^2}{g^2} + \frac{f_s^2}{f^2} + \frac{8f^2}{g^4}\right)k + \frac{8g_s\,(f^2-g^2)}{g^5}.$$

If we differentiate both sides of this equation with respect to s, use the commutator (15), and recall formula (16) for the Laplacian, we obtain

$$\frac{\partial}{\partial t}(k_s) = \Delta(k_s) - A\,k_s - B\,k + D, \tag{35}$$

where A, B, and D are functions of (s,t) defined by

$$A := \frac{8f^2}{g^4}, \quad B := \left\{\frac{2g_s^2}{g^2} + \frac{f_s^2}{f^2} + \frac{8f^2}{g^4}\right\}_s, \quad D := \left\{\frac{8g_s\,(f^2-g^2)}{g^5}\right\}_s.$$

Lemmas 2, 3, 4, 6, 9, and 10 imply that A and $|B| + |D|$ may be estimated by

$$\frac{c}{T-t} \le A \le \frac{C}{T-t} \qquad \text{and} \qquad |B| + |D| \le \frac{C}{(T-t)^{3/2}}. \tag{36}$$

Using (35), we readily calculate

$$(k_s^2)_t = \Delta(k_s^2) - 2k_{ss}^2 - 2A\,k_s^2 - 2B\,kk_s + 2D\,k_s.$$

Then using the maximum principle, weighted Cauchy–Schwarz, and estimate (36), and recalling that $|k|$ is uniformly bounded, we find that

$$\begin{aligned}\frac{d}{dt}(k_s^2)_{\max} &\leq -2Ak_s^2 - 2Bkk_s + 2Dk_s \\ &\leq -2Ak_s^2 + \left(Ak_s^2 + \frac{B^2}{A}k^2\right) + \left(Ak_s^2 + \frac{D^2}{A}\right) \\ &\leq \frac{C}{(T-t)^2}.\end{aligned}$$

Integrating this in time, using Lemma 2, and enlarging C if necessary, we get

$$|(k_s)_{\max}| \leq \frac{C}{\sqrt{T-t}} \leq \frac{C}{\check{M}}. \tag{37}$$

Recalling the definition of k, we see that (37) implies that

$$|k_s| = \left|\frac{f_{ss}}{f} - \frac{g_{ss}}{g} + \frac{g_s^2}{g^2} - \frac{f_s^2}{f^2}\right| \leq \frac{C}{\check{M}}.$$

This estimate, together with Lemmas 4 and 10, implies that

$$|\kappa_{01} - \kappa_{02}| \leq \frac{C}{\check{M}} + |k|\left|\frac{f_s}{f} + \frac{g_s}{g}\right| \leq \frac{C}{\check{M}},$$

as desired. □

6. Local convergence to the shrinking cylinder soliton

In this section, we demonstrate that solutions originating from initial data that satisfy Assumption 2 converge locally, after parabolic rescaling, to the rotation- and translation-invariant shrinking cylinder soliton.

We begin by deriving an improved C^1 bound for the metric component f.

Lemma 13. *If a solution $\left(\mathcal{S}^1 \times \mathcal{S}^3, G(t)\right)$ satisfies Assumption 2, then there exists a uniform constant C such that for as long as the flow exists, one has*

$$f_s^2 \leq 1 + Cf.$$

Proof. Based on the evolution equation (21) for f_s, one easily determines that the evolution equation for the quantity $L := (f_s^2 - 1)/f$ is given by

$$L_t = \Delta L - \frac{2f}{g^4} + \frac{f_s^2}{f^3} - \frac{10ff_s^2}{g^4} - \frac{f_s^4}{f^3} + \frac{16f^2f_sg_s}{g^5} - \frac{4f_s^2g_s^2}{fg^2} - \frac{2f_{ss}^2}{f}.$$

Consider a point and time where $L_{\max}$ is attained. We may assume that at the maximum of L, we have $f_s \neq 0$, since otherwise $f_s^2 \leq 1 + Cf$ as desired. Then because $L_s = 0$ at $L_{\max}$, we can write

$$f_{ss} = \frac{-f_s + f_s^3}{2ff_s}.$$

By expanding $-2f_{ss}^2/f$, this implies that

$$\frac{d}{dt}L_{\max} \leq -\frac{1}{2f^3} - \frac{2f}{g^4} + \frac{2f_s^2}{f^3} - \frac{10ff_s^2}{g^4} - \frac{3f_s^4}{2f^3} + \frac{16f^2f_sg_s}{g^5} - \frac{4f_s^2g_s^2}{fg^2}.$$

Furthermore, using Lemmas 4, 9, and 10, we see that

$$\frac{f^2 f_s g_s}{g^5} = \frac{f_s^2 f^2}{g^5} + A \qquad \text{and} \qquad \frac{f_s^2 g_s^2}{f g^2} = \frac{f_s^4}{f g^2} + B,$$

where

$$|A| = \left| \frac{f^2 g_s (g_s - f_s)}{f g^2} \right| \le \frac{C}{f^2} \quad \text{and} \quad |B| = \left| \frac{2 f_s^3 \,(g_s - f_s)}{f g^2} + \frac{f_s^2 \,(g_s - f_s)^2}{f g^2} \right| \le \frac{C}{f^2},$$

for a uniform constant C. Thus we obtain

$$\begin{aligned}
\frac{d}{dt} L_{\max} &\le \left(\frac{16 f^2}{g^5} - \frac{12 f}{g^4} - \frac{4}{f g^2} \right) - \left(\frac{3}{2f} - \frac{4f}{g^2} \right) L^2 - \left(\frac{10 f^2}{g^4} + \frac{8}{g^2} + \frac{1}{f^2} - \frac{16 f^3}{g^5} \right) L + \frac{C}{f^2} \\
&\le - \left(\frac{10 f^2}{g^4} + \frac{8}{g^2} + \frac{1}{f^2} - \frac{16 f^3}{g^5} \right) L + \frac{C}{f^2},
\end{aligned}$$

because

$$\frac{16 f^2}{g^5} - \frac{12 f}{g^4} - \frac{4}{f g^2} = \frac{4 (f-g)\,(4 f^2 + f g + g^2)}{f g^5} \le 0.$$

Therefore,

$$\frac{d}{dt} L_{\max} \le -\frac{L}{f^2 g^5}\,(g^5 + 8 f^2 g^3 + 10 f^4 g - 16 f^5) + \frac{C}{f^2}.$$

If ε in $(1-\varepsilon) g \le f \le g$ is small enough[11] so that $10(1-\varepsilon)^4 + 8(1-\varepsilon)^2 - \varepsilon - 15 > 0$, then we have $g^5 + 8 f^2 g^3 + 10 f^4 g - 16 f^5 > \varepsilon f^5$, snd hence obtain

$$\frac{d}{dt} L_{\max} \le -\frac{\tilde{C} \varepsilon \, L}{f^2} + \frac{C}{f^2},$$

where $\tilde{C}$ is a uniform constant. So either $L_{\max} \le C/(\tilde{C}\varepsilon)$, or else L is decreasing at $L_{\max}$. This implies that for as long as the flow exists,

$$L_{\max}(t) \le \max \left\{ L_{\max}(0),\ \frac{C}{\tilde{C}\varepsilon} \right\}.$$

□

As a tool for controlling the second spatial derivative of f, we next consider the quantity [12]

$$F := f f_{ss} \log f, \tag{38}$$

and show that it is bounded from below in certain space-time neighborhoods of a local singularity. We define the neighborhoods of interest as follows. For fixed $0 < \delta \ll 1$, there exists by Lemma 2 a time $t_\delta \in [0, T)$ such that the radius of each neck that becomes singular satisfies $f \le \delta$ for all $t_0 \le t < T$. Because $f_{ss} > 0$ at each local minimum of f, the set

$$\Omega = \left\{ f_{ss} \log \left(\frac{f}{\delta} \right) < 0 \right\}$$

describes an open interval around that neck (or those necks) for all $t \in (t_\delta, T)$.[13]

[11] $\varepsilon = 1/20$ is sufficiently small here.

[12] This quantity may be compared to F defined in (25) of [1]. In that paper, one has $f = g = \psi$. So the quantity F in [1] simplifies to $F = 2\psi\psi_{ss} |\log \psi\,|$ at a neck, and is bounded from above.

[13] If there are several equally small necks, Ω may have several connected components in space. This does not pose a problem for the argument that follows.

Lemma 14. *If a solution $\big(\mathbb{S}^1 \times \mathbb{S}^3, G(t)\big)$ satisfies Assumption 2, then there exists a constant C such that for as long as the flow exists, one has*

$$F \geq -C$$

in the neck-like region Ω.

Proof. It follows from Lemmas 3 and 6 that $F \geq C \log(T - t_\delta)/\sqrt{T - t_\delta}$ at $t = t_\delta$. Moreover, the definition of Ω guarantees that $F \geq -C(\delta)$ at the endpoints of each component of Ω for all times $t_\delta < t < T$. Hence F is uniformly bounded on the parabolic boundary of Ω. To complete the proof, we show that F is bounded from below at all interior points. To do so, in the following argument, we use the facts that $f_{ss} > 0$ and $f < \delta$ inside Ω.

Differentiating (14) using the commutator (15), we compute that F evolves by

$$F_t = \Delta F - 4\,\frac{f_s}{f}\,F_s + N, \tag{39}$$

where the reaction term N is given by

$$\begin{aligned} N := -f\log f\left(\frac{12 f f_s^2}{g^4} - \frac{48 f^2 f_s g_s}{g^5} + \frac{40 f^3 g_s^2}{g^6} - \frac{4 f_s g_s^3}{g^3}\right) - \frac{8 f^4 \log f}{g^4}\left(\frac{f_{ss}}{f} - \frac{g_{ss}}{g}\right) \\ - 2 f_{ss}^2 \log f + \frac{4 f_s^2 f_{ss}}{f} - 4 f \log f\left(\frac{g_s^2 f_{ss}}{g^2} + \frac{f_s g_s g_{ss}}{g^2} - \frac{f_s^2 f_{ss}}{f^2}\right). \end{aligned}$$

To proceed, we estimate the various terms in N one-by-one:

Beginning with the coefficient of $-f \log f$, we observe that there exists a uniform constant C such that

$$\begin{aligned} \frac{12 f f_s^2}{g^4} - \frac{48 f^2 f_s g_s}{g^5} + \frac{40 f^3 g_s^2}{g^6} - \frac{4 f_s g_s^3}{g^3} &\geq \frac{12 f_s^2}{f^3 (1 + Cf)^4} - \frac{48 f_s\,(f_s + Cf)}{f^3} \\ &\quad + \frac{40 \big\{f_s + (g_s - f_s)\big\}^2}{f^3 (1 + Cf)^6} - \frac{4 f_s \left\{f_s + (g_s - f_s)\right\}^3}{f^3} \\ &\geq \frac{4 f_s^2 (1 - f_s^2)}{f^3} - \frac{C}{f^2} \\ &\geq -\frac{C}{f^2}. \end{aligned}$$

To obtain this estimate, we use Lemmas 4, 9, and 10, and then, in the last step, Lemmas 3 and 13. It follows from this estimate, using Lemma 3 again, that [14]

$$-f\log f\left(\frac{12 f f_s^2}{g^4} - \frac{48 f^2 f_s g_s}{g^5} + \frac{40 f^3 g_s^2}{g^6} - \frac{4 f_s g_s^3}{g^3}\right) \geq C\,\frac{\log(T-t)}{\sqrt{T-t}}. \tag{40}$$

Similarly, relying on Lemma 12, we obtain

$$-\frac{8 f^4 \log f}{g^4}\left(\frac{f_{ss}}{f} - \frac{g_{ss}}{g}\right) \geq C\,\frac{\log(T-t)}{\sqrt{T-t}}. \tag{41}$$

Further we observe that at any interior point of Ω, one has

$$\frac{4 f_s^2 f_{ss}}{f} > 0. \tag{42}$$

[14] Here we use the fact that $|\log x|/x$ is monotone decreasing for $0 < x < 1$.

Finally, using the positivity of f_{ss} in Ω and applying Lemmas 3, 4, 9, 10, and 12, we observe that

$$\begin{aligned}\frac{g_s^2 f_{ss}}{g^2}+\frac{f_s g_s g_{ss}}{g^2}-\frac{f_s^2 f_{ss}}{f^2} &\geq \frac{f_s g_s g_{ss}}{g^2}-\frac{f_s^2 f_{ss}}{f^2}\\ &= \frac{f_s g_s}{g}\left(\frac{g_{ss}}{g}-\frac{f_{ss}}{f}\right)+\frac{f_s\big\{f(g_s-f_s)+f_s(f-g)\big\}}{fg}\,\frac{f_{ss}}{f}\\ &\geq -\frac{C}{f\sqrt{T-t}}-C\,\frac{f_{ss}}{f}.\end{aligned}$$

Combining this estimate with Lemma 3, we obtain

$$-4f\log f\left(\frac{g_s^2 f_{ss}}{g^2}+\frac{f_s g_s g_{ss}}{g^2}-\frac{f_s^2 f_{ss}}{f^2}\right)\geq C\,\frac{\log(T-t)+F}{\sqrt{T-t}}. \tag{43}$$

To proceed, we assume that $F_{\min}(t)$ is attained at an interior point of Ω. Using inequalities (40)–(43) to estimate the right-hand side of equation (39), one obtains

$$\frac{d}{dt}F_{\min}\geq C\,\frac{\log(T-t)+F}{\sqrt{T-t}}-2f_{ss}^2\log f.$$

Since $\log f\leq 0$ in Ω, we have

$$\frac{d}{dt}F_{\min}\geq C\,\frac{F_{\min}+\log(T-t)}{\sqrt{T-t}}.$$

This inequality, together with the maximum principle, implies that

$$\begin{aligned}F_{\min}(t)&\geq e^{-C'\sqrt{T-t}}\left\{e^{C'\sqrt{T-t_\delta}}F_{\min}(t_\delta)+C\int_{t_\delta}^{t}\frac{\log(T-\tau)\,e^{C'\sqrt{T-\tau}}}{\sqrt{T-\tau}}\mathrm{d}\tau\right\}\\ &\geq e^{C'(\sqrt{T-t_\delta}-\sqrt{T-t})}F_{\min}(t_\delta)-C''.\end{aligned}$$

Since, as noted above, $F_{\min}(t_\delta)\geq C\log(T-t_\delta)/\sqrt{T-t_\delta}$, the proof is complete. □

Using Lemmas 2 and 3, we obtain the following consequence of Lemma 14.

Corollary 15. *In the neighborhood Ω of the smallest neck(s), where the sectional curvature κ_{01} is negative, the scale-invariant quantities $(T-t)|\kappa_{01}|$ and $(T-t)|\kappa_{02}|$ satisfy*

$$(T-t)|\kappa_{01}|\leq\frac{C}{|\log(T-t)|}\qquad\text{and}\qquad(T-t)|\kappa_{02}|\leq\frac{C}{|\log(T-t)|}.$$

Proof. By Lemmas 2 and 3, the estimate for κ_{01} is a straightforward consequence of Lemma 14. Then combining Lemma 12 with this estimate for κ_{01}, we obtain

$$\begin{aligned}(T-t)|\kappa_{02}|&\leq(T-t)|\kappa_{01}|+(T-t)|\kappa_{01}-\kappa_{02}|\\ &\leq\frac{C}{|\log(T-t)|}+C\sqrt{T-t}\\ &\leq\frac{C}{|\log(T-t)|}.\end{aligned}$$

□

We now prove cylindricality at a singularity. Without loss of generality, we confine our considerations to a single component of Ω. We choose $\xi_1(t)$ such that $f(s(\xi_1(t),t),t) = \check{M}(t)$ in that component for all times t sufficiently close to T, and we define arclength from the neck by

$$S(\xi,t) := s(\xi,t) - s(\xi_1(t),t).$$

Lemma 16. *There exist uniform constants* $0<\varepsilon<1$ *and* c, $C<\infty$ *such that for all times* t *sufficiently close to* T*, one has*

$$1 \le \frac{f}{\check{M}} \le 1 + C\,\frac{(S/\check{M})^2}{\big|\log \check{M}\big|} \qquad \textit{and} \qquad 1 \le \frac{g}{\check{M}} \le \big(1+o(1)\big)\left(1 + C\,\frac{(S/\check{M})^2}{\big|\log \check{M}\big|}\right)$$

for $|S| \le c\check{M}\sqrt{|\log \check{M}|}$*, and*

$$\frac{f}{\check{M}} + \frac{g}{\check{M}} \le C\,\frac{|S/\check{M}|}{\sqrt{|\log \check{M}|}}\sqrt{\log\left(\frac{|S/\check{M}|}{\sqrt{|\log \check{M}|}}\right)}$$

for $c\check{M}\sqrt{|\log \check{M}|} \le |S| \le \check{M}^{1-\varepsilon}$.

Proof. We carry out the argument for the side of the neck on which $S \ge 0$; the other side is treated analogously. Because $f_s > 0$ where $S > 0$, we can use f as a coordinate there. More precisely, we use $\ell := \log f$. Then because $\frac{ds}{d\ell} = f/f_s$, we can state the conclusion of Lemma 14 as

$$\frac{\partial}{\partial \ell}(f_s^2) = 2ff_{ss} \le -\frac{C}{\ell}.$$

Integrating this inequality and using the calculus fact that $\log x \le x-1$ for $x \ge 1$, we obtain

$$f_s^2 \le C\log\left(\frac{\log \check{M}}{\log f}\right) \le C\left(\frac{\log \check{M}}{\log f} - 1\right).$$

This estimate implies that

$$\sqrt{C}\,\frac{dS}{d\ell} \ge \frac{f}{\sqrt{\frac{\log \check{M}}{\log f} - 1}} = \frac{\frac{df}{d\ell}}{\sqrt{\frac{\log \check{M}}{\ell} - 1}},$$

which upon another integration yields

$$\sqrt{C}\,S \ge \int_{\check{M}}^{f} \frac{d\tilde{f}}{\sqrt{\frac{\log \check{M}}{\log \tilde{f}} - 1}}.$$

We change the variable of integration to $\varphi = \tilde{f}/\check{M}$, obtaining

$$\sqrt{C}\,S \ge \check{M}\int_1^{f/\check{M}} \sqrt{\frac{-\log \check{M} - \log\varphi}{\log\varphi}}\,\mathrm{d}\varphi.$$

Restricting to a smaller neighborhood of the neck if necessary so that $f \le \check{M}^{-3/4}$, we ensure that $\sqrt{-\log \check{M} - \log\varphi} \ge \frac{1}{2}\sqrt{-\log \check{M}}$ and so obtain the simpler estimate

$$2\sqrt{C}\,\frac{S}{\check{M}\sqrt{-\log \check{M}}} \ge \int_1^{f/\check{M}} \frac{\mathrm{d}\varphi}{\sqrt{\log\varphi}}.$$

As observed in Proposition 9.3 of [1], this inequality implies that

$$\frac{f}{\check{M}} \leq 1 + C' \,\frac{(S/\check{M})^2}{|\log \check{M}|}$$

for $S \leq c\check{M}\sqrt{|\log \check{M}|}$, and

$$\frac{f}{\check{M}} \leq C'' \,\frac{S/\check{M}}{\sqrt{|\log \check{M}|}}\sqrt{\log\left(\frac{S/\check{M}}{\sqrt{|\log \check{M}|}}\right)} \tag{44}$$

for larger values of S.

To obtain the estimates for g, we argue as follows. Because f is monotone increasing moving away from the neck in Ω, we may use estimate (44) to see that if $S \leq \check{M}^{1-\varepsilon}$ for $\varepsilon \in (0,1)$, then $f = o(1)$ as $\check{M} \searrow 0$. Hence by Lemma 9, we obtain $g \leq (1 + Cf)f \leq \big(1 + o(1)\big)f$ as $\check{M} \searrow 0$. □

Lemma 3, Corollary 15, and Lemma 16 imply that a Type-I blowup of the metric,

$$\tilde{G} := (T-t)^{-1}G,$$

must converge near the singularity to the shrinking cylinder soliton. It follows that

$$\check{M} = \big(1 + o(1)\big)\, 2\sqrt{T-t}.$$

If we now denote the parabolically-rescaled distance from the neck by

$$\sigma := \frac{S}{\sqrt{T-t}},$$

then the conclusion of Lemma 16 may be recast as follows.

Corollary 17. *There exist uniform constants $0 < \varepsilon < 1$ and c, $C < \infty$ such that as $t \nearrow T$, the estimates*

$$1 + o(1) \leq \frac{f}{2\sqrt{T-t}} \leq 1 + C\,\frac{\sigma^2}{|\log(T-t)|}$$

and

$$1 + o(1) \leq \frac{g}{2\sqrt{T-t}} \leq \big(1 + o(1)\big)\left(1 + C\,\frac{\sigma^2}{|\log(T-t)|}\right)$$

hold for $|\sigma| \leq c\sqrt{|\log(T-t)|}$, and the estimate

$$\frac{f}{\sqrt{T-t}} + \frac{g}{\sqrt{T-t}} \leq C\,\frac{|\sigma|}{\sqrt{|\log(T-t)|}}\sqrt{\log\left(\frac{|\sigma|}{\sqrt{|\log(T-t)|}}\right)}$$

holds for $c\sqrt{|\log(T-t)|} \leq |\sigma| \leq (T-t)^{-\varepsilon/2}$.

7. Estimates for reflection-symmetric solutions

In this section, we derive our sharpest estimates for the eccentricity of a Ricci flow solution near a developing neckpinch, more than doubling the decay rate for the scale-invariant quantity $|f-g|/\sqrt{T-t}$ that we have obtained above. To accomplish this, we use ideas motivated by the formal asymptotics outlined in Appendix C, following the approach carried out rigorously in [2]. To make the arguments rigorous here, we impose Assumption 3 from Section 1, adding a technical hypothesis that guarantees that each solution under consideration is reflection symmetric, with its smallest neck occurring at $s = 0$. In this approach, we find that the evolution of the quantity we study below, which controls $|f - g|$, is governed by a favorable linear term and by a "forcing function" that represents the nonlinear terms involved. As in [2], we do not quite achieve the optimal decay predicted by the linear term, but we are able to prove decay at the rate of the forcing function.

Our first step, which does not need reflection symmetry, is a mild improvement to Lemma 9, to be used below.

Lemma 18. *If a solution $\big(\mathcal{S}^1 \times \mathcal{S}^3, G(t)\big)$ satisfies Assumption 2, then there exists a uniform constant C such that for as long as the flow exists, one has*

$$g - f \leq C f^3 |\log(T - t)|.$$

Proof. We define $P := f^{-2} - g^{-2} > 0$ and compute that

$$P_t = \Delta P + g^2 \left(\frac{4f_s^2}{f^3} + P_s\right) P_s + \frac{4(g^2 - f^2)(g^6 f_s^2 - f^6)}{f^6 g^6}.$$

Therefore, using the fact that $f \leq g$, one has

$$\frac{d}{dt} P_{\max} \leq \frac{4(g^2 - f^2)}{f^6 g^6} \Big\{ g^6 - f^6 + g^6 (f_s^2 - 1) \Big\}.$$

Now by Lemma 9, one has $g^6 - f^6 = (g^3 + f^3)(g^2 + fg + f^2)(g - f) \leq C f^7$. Then using Lemma 13, Lemma 9 again, and finally Lemma 3, one obtains

$$\begin{aligned} \frac{d}{dt} P_{\max} &\leq C \frac{g^2 - f^2}{f^6 g^6} f^7 \\ &\leq C \frac{1}{f^2} \leq \frac{C}{T - t}. \end{aligned}$$

Integrating this yields $P_{\max} \leq C\big\{1 - \log(T - t)\big\} \leq C' |\log(T - t)|$, whereupon unwrapping the definition of P and using Lemma 1 gives the result in the form

$$g - f = \frac{f^2 g^2}{f + g} P \leq C f^3 P_{\max} \leq C' f^3 |\log(T - t)|.$$

□

Next we perform a parabolic dilation as outlined in Appendix C, the purpose of which is to facilitate analysis of the solution very near the developing singularity, following the approach of [2]. We introduce new time $\tau := -\log(T - t)$ and space $\sigma := e^{\tau/2} s$ variables. Then we consider the quantity $x(\sigma, \tau)$ defined in equation (58) (found in Appendix C), which is

$$x = \frac{1}{2} e^{\tau/2} (f - g) = \frac{f - g}{2\sqrt{T - t}}.$$

As computed in Appendix C, the evolution of x is governed by

$$x_\tau = (\mathcal{A} - 3)x + N(x),$$

where the familiar linear operator

$$\mathcal{A} := \frac{\partial^2}{\partial \sigma^2} - \frac{\sigma}{2} \frac{\partial}{\partial \sigma} + 1$$

generates the quantum harmonic oscillator. The nonlinear quantity $N(x)$ is

$$N(x) := \frac{\varphi_\sigma \psi_\sigma}{(1+\varphi)(1+\psi)} x + \mathcal{I} x_\sigma - \frac{\varphi^2 + 2\varphi(2+\psi) - \psi\big\{14 + \psi[28 + 5\psi(4+\psi)]\big\}}{2(1+\psi)^4} x, \tag{45}$$

where

$$\varphi := u - 1 := \frac{e^{\tau/2} f}{2} - 1, \qquad \psi := v - 1 := \frac{e^{\tau/2} g}{2} - 1,$$

and $\mathcal{I}(\sigma,\tau)$ is the nonlocal term

$$\mathcal{I} := \int_0^\sigma \left(\frac{u_{\bar\sigma\bar\sigma}}{u} + 2\frac{v_{\bar\sigma\bar\sigma}}{v} \right) \mathrm{d}\bar\sigma. \tag{46}$$

In order to estimate the nonlinear terms above, we need the following analog of Lemma 4 from [2]. Note that this is the first time we use our strongest assumption on the initial data, that of reflection symmetry.

Lemma 19. *If a solution $\big(\mathcal{S}^1 \times \mathcal{S}^3, G(t)\big)$ satisfies Assumption 3, then there exist $\varepsilon \in (0,1)$ and c, C such that one has C^0 estimates*

$$1 - \frac{C}{\tau} \le u \le 1 + C\frac{\sigma^2}{\tau}, \qquad |\sigma| \le c\sqrt{\tau},$$
$$1 - \frac{C}{\tau} \le u \le C\frac{|\sigma|}{\sqrt{\tau}}\sqrt{\log\frac{|\sigma|}{\sqrt{\tau}}}, \qquad c\sqrt{\tau} \le |\sigma| \le e^{\varepsilon\tau},$$

$$1 - \frac{C}{\tau} \le v \le 1 + C\frac{1+\sigma^2}{\tau}, \qquad |\sigma| \le c\sqrt{\tau},$$
$$1 - \frac{C}{\tau} \le v \le C\frac{|\sigma|}{\sqrt{\tau}}\sqrt{\log\frac{|\sigma|}{\sqrt{\tau}}}, \qquad c\sqrt{\tau} \le |\sigma| \le e^{\varepsilon\tau},$$

and C^1 estimates

$$|u_\sigma| + |v_\sigma| \le C\frac{1+|\sigma|}{\tau}, \qquad |\sigma| \le c\sqrt{\tau},$$
$$|u_\sigma| + |v_\sigma| \le \frac{C}{\sqrt{\tau}}\sqrt{\log\frac{|\sigma|}{\sqrt{\tau}}}, \qquad c\sqrt{\tau|} \le |\sigma| \le e^{\varepsilon\tau}.$$

Proof. The upper bound for u follows immediately from Corollary 17.

To get the lower bound for u, we note that at the center of the neck, Lemma 14 and Lemma 3 imply that

$$f f_{ss} \le \frac{C}{|\log f|} \le \frac{C'}{\tau}.$$

Then using the implication of Lemma 2 that

$$\frac{f^4}{g^4} = 1 + \frac{f^4 - g^4}{g^4} = 1 + \frac{(f-g)(f^3 + f^2 g + f g^2 + g^3)}{g^4} \ge 1 - Cf \ge 1 - C' e^{-\tau/2},$$

we observe that at the center of the neck, where f achieves its minimum, one has

$$\frac{1}{2}(f^2)_t = f f_t = f f_{ss} - 2\frac{f^4}{g^4} \le -2 + \frac{C}{\tau},$$

which implies after integration that $f^2 \geq 4(T-t)\big(1 - C\tau^{-1}\big)$, hence that

$$u \geq \sqrt{1 - \frac{C}{\tau}} \geq 1 - \frac{C'}{\tau}.$$

The upper bound for v at large $|\sigma|$ is implied by Corollary 17. To get the upper bound at small $|\sigma|$, we note that by Lemma 9, one has $v \leq (1 + Cf)u$. But for $|\sigma| \leq c\sqrt{\tau}$, the upper bound for u implies that $f \leq Ce^{-\tau/2}$, which in turn implies the estimate.

To get the lower bound for v, we note that Lemma 12 implies that

$$g_{ss} \leq \frac{g}{f}\big(f_{ss} + C\big).$$

Therefore at the center of the neck, where g also achieves its minimum, we apply Lemmas 2, 9, and 14 to obtain

$$\begin{aligned}
\frac{1}{2}(g^2)_t &= g g_{ss} + 2\frac{f^2 - g^2}{g^2} - 2 \\
&\leq \frac{g^2}{f^2}(f f_{ss}) + Cf - 2 \\
&\leq -2 + \frac{C}{\tau}.
\end{aligned}$$

In the final step of the derivation of this estimate, we have used the fact that $f = \check{M} \leq Ce^{-\tau/2}$ at the center of the neck. Working with this estimate, we derive the lower bound for v using an argument very similar to that used to obtain the lower bound for u.

The derivative bounds in the statement of the Lemma follow readily from the C^0 and C^2 bounds we have obtained above, together with the fact that the definition of Ω ensures that u and v are convex there. For example, unwrapping definitions and using the estimate $f f_{ss}|\log f| \leq C$ from Lemma 14, one sees that

$$u u_{\sigma\sigma}|\log(2e^{-\tau/2}u)| = \frac{1}{4} f f_{ss}|\log f| \leq C. \tag{47}$$

Then for $|\sigma| \leq c\sqrt{\tau}$, a region in which Corollary 17 implies that $u = \mathcal{O}(1)$, our estimate (47) implies that $u_{\sigma\sigma} \leq C/\tau$, which after antidifferentiation yields

$$|u_\sigma| \leq C\frac{|\sigma|}{\tau}, \qquad \big(|\sigma| \leq c\sqrt{\tau}\big).$$

The remaining bounds are proved similarly. □

The operator $-\mathcal{A}$ is self-adjoint in the Hilbert space $\mathcal{G} := L^2(\mathbb{R};\, e^{-\sigma^2/4}\mathrm{d}\sigma)$, with discrete spectrum bounded below by -1. We denote the inner product in $\mathcal{G}$ by $\big(\cdot,\cdot\big)_{\mathcal{G}}$ and the norm by $\|\cdot\|_{\mathcal{G}}$.

The quantity x does not belong to $\mathcal{G}$ because it is not defined for all σ. We remedy this difficulty as follows. Let β be a smooth, even, bump function with $\beta(z) = 1$ for $|z| \leq 1$ and $\beta(z) = 0$ for $|z| \geq 2$. We define

$$X(\sigma,\tau) := \begin{cases} \beta(e^{-\varepsilon\tau/2}\sigma)\, x(\sigma,\tau), & \text{for } |\sigma| \leq 2e^{\varepsilon\tau/2}, \\ 0 & \text{for } |\sigma| > 2e^{\varepsilon\tau/2}, \end{cases}$$

where ε is the constant from Corollary 17. A computation shows that

$$X_\tau = (\mathcal{A} - 3)X + \beta N(x) + E,$$

where $E := \big(\beta_\tau - \beta_{\sigma\sigma} + \frac{\sigma}{2}\beta_\sigma\big)x - 2\beta_\sigma x_\sigma$ denotes the "error" induced by β.

Lemma 20. *The quantity E vanishes except for $e^{\varepsilon\tau/2} < |\sigma| < 2e^{\varepsilon\tau/2}$, and there exists a uniform constant C such that if $E \neq 0$, then*

$$|E(\sigma,\tau)| \le C|\sigma| \qquad \text{and} \qquad \|E(\cdot,\tau)\|_{\mathcal{G}} \le C\exp\left(-e^{\varepsilon\tau/2}\right).$$

As a consequence of Lemma 19, the proof of this result is identical to that of Lemma 7 in [2].

We are now ready to estimate the evolution of $\|X\|_{\mathcal{G}}^2$. In doing this, we use the fact that for any function W in the domain of the operator $\mathcal{A}$, the divergence form of that operator shows that $\mathcal{A}W = (e^{-\sigma^2/4}W_\sigma)_\sigma\, e^{-\sigma^2/4}$. Consequently one has

$$-\big(W, \mathcal{A}W\big)_{\mathcal{G}} = \int_{\mathbb{R}} (W_\sigma^2 - W^2)\, e^{-\sigma^2/4}\,\mathrm{d}\sigma,$$

and hence

$$\|W_\sigma\|_{\mathcal{G}}^2 = \|W\|_{\mathcal{G}}^2 - \big(W, \mathcal{A}W\big)_{\mathcal{G}}.$$

Thus we obtain

$$\begin{aligned} \frac{d}{d\tau}\|X\|_{\mathcal{G}}^2 &= 2\big(X, X_\tau\big)_{\mathcal{G}} \\ &= 2\big(X,\, \mathcal{A}X - 3X + \beta N(x) + E\big)_{\mathcal{G}} \\ &= -4\|X\|_{\mathcal{G}}^2 - 2\|X_\sigma\|_{\mathcal{G}}^2 + 2\big(X,\, \beta N(x) + E\big)_{\mathcal{G}}. \end{aligned}$$

We now define $N_0(x) := N(x) - \mathcal{I}x_\sigma$, where $N(x)$ is defined in (45) and $\mathcal{I}$ is defined in (46). We also define $E_0 := \beta_\sigma x$. It then follows from Cauchy–Schwarz that we have

$$\begin{aligned} \frac{d}{d\tau}\|X\|_{\mathcal{G}}^2 &= -4\|X\|_{\mathcal{G}}^2 - 2\|X_\sigma\|_{\mathcal{G}}^2 + 2\big(X,\, \mathcal{I}X_\sigma + N_0(x)\big)_{\mathcal{G}} + 2\big(X,\, E - \mathcal{I}E_0\big)_{\mathcal{G}} \\ &\le -4\|X\|_{\mathcal{G}}^2 + \|\mathcal{I}X\|_{\mathcal{G}}^2 + 2\big(X,\, N_0(x)\big)_{\mathcal{G}} + 2\big(X,\, E - \mathcal{I}E_0\big)_{\mathcal{G}}. \end{aligned} \tag{48}$$

To control $\frac{d}{d\tau}\|X\|_{\mathcal{G}}^2$, we start by deriving pointwise bounds for the nonlinear factors on the right hand side of (48).

Lemma 21. *If $\big(\mathcal{S}^1 \times \mathcal{S}^3, G(t)\big)$ is a Ricci flow solution satisfying Assumption 3, then for $|\sigma| \le c\sqrt{\tau}$, one has*

$$|\mathcal{I}| \le C\left(\frac{1}{\tau} + \frac{|\sigma|^3}{\tau^2}\right) \qquad \text{and} \qquad |N_0| \le C\left(\frac{1}{\tau} + \frac{\sigma^8}{\tau^4}\right)|x|;$$

while for $c\sqrt{\tau} \le |\sigma| \le e^{\varepsilon\tau}$, one has

$$|\mathcal{I}| \le C\frac{|\sigma|}{\tau}\log\frac{|\sigma|}{\sqrt{\tau}} \qquad \text{and} \qquad |N_0| \le C\frac{\sigma^4}{\tau^2}\left(\log\frac{|\sigma|}{\sqrt{\tau}}\right)^2|x|.$$

Proof. Our assumption of reflection symmetry allows us to integrate by parts and thus write the nonlocal term $\mathcal{I}$ defined in (46) as

$$\mathcal{I} = \frac{u_\sigma}{u} + 2\frac{v_\sigma}{v} \;+ \int_0^\sigma \frac{u_{\bar\sigma}^2}{u^2}\,\mathrm{d}\bar\sigma + 2\int_0^\sigma \frac{v_{\bar\sigma}^2}{v^2}\,\mathrm{d}\bar\sigma.$$

We now use $\varphi = u - 1$ and $\varphi_\sigma = u_\sigma$, together with $\psi = v - 1$ and $\psi_\sigma = v_\sigma$, and proceed to estimate the terms above using Lemma 19. This yields the stated bounds for $|\mathcal{I}|$.

The bounds for $|N_0|$ also follow easily from Lemma 19. For $|\sigma| \le c\sqrt{\tau}$, one has

$$|N_0| \le C\left\{\frac{1+\sigma^2}{\tau^2} + \frac{1+\sigma^2}{\tau} + \left(\frac{1+\sigma^2}{\tau}\right)^4\right\}|x| \le C'\left(\frac{1}{\tau} + \frac{\sigma^8}{\tau^4}\right)|x|.$$

The bound on $|N_0|$ for $c\sqrt{\tau} \le |\sigma| \le e^{\varepsilon\tau}$ is obtained similarly. □

Lemma 22. *If $\big(\mathcal{S}^1 \times \mathcal{S}^3, G(t)\big)$ is a Ricci flow solution satisfying Assumption 3, then for any δ sufficiently small, there exist C and τ^* depending on δ such that for all times $\tau \ge \tau^*$, one has*

$$\frac{d}{d\tau}\|X\|_{\mathcal{G}}^2 \le -(4-\delta)\|X\|_{\mathcal{G}}^2 + Ce^{-2(1+\delta)\tau}.$$

Proof. We estimate the terms on the RHS of (48), starting with $\big(X, E - \mathcal{I}E_0\big)_{\mathcal{G}}$. Given any $\delta_1 > 0$, we find by using weighted Cauchy–Schwarz and Lemma 20 that

$$\begin{aligned}\left|\big(X,E\big)_{\mathcal{G}}\right| &\le \frac{\delta_1}{4}\|X\|_{\mathcal{G}}^2 + \frac{1}{\delta_1}\|E\|_{\mathcal{G}}^2\\ &\le \frac{\delta_1}{4}\|X\|_{\mathcal{G}}^2 + \frac{C}{\delta_1}\exp(-2e^{\varepsilon\tau/2})\\ &\le \frac{\delta_1}{4}\|X\|_{\mathcal{G}}^2 + Ce^{-2\tau}\end{aligned}$$

for all $\tau \ge \tau_1$, where τ_1 is chosen sufficiently large, depending only on δ_1 and ε. Then using the facts that $E_0 = \beta_\sigma x$ is supported in $e^{\varepsilon\tau/2} < |\sigma| < 2e^{\varepsilon\tau/2}$, that $|\beta_\sigma|$ is bounded, and that Lemma 19 provides bounds for $|x|$ in that region, we apply Lemma 21 and thereby obtain the estimate

$$\|\mathcal{I}E_0\|_{\mathcal{G}} \le C\int_{e^{\varepsilon\tau/2}}^{\infty}\frac{\sigma^2}{\tau}\left(\log\frac{|\sigma|}{\sqrt{\tau}}\right)e^{-\sigma^2/4}\,\mathrm{d}\sigma \le C'\exp(-e^{\varepsilon\tau/2}),$$

exactly as in the proof of Lemma 20. Consequently, arguing as above, we obtain

$$\left|\big(X,\mathcal{I}E_0\big)_{\mathcal{G}}\right| \le \frac{\delta_1}{4}\|X\|_{\mathcal{G}}^2 + Ce^{-2\tau}.$$

Next we decompose $\|\mathcal{I}X\|_{\mathcal{G}}^2 + 2\big(X, N_0(x)\big)_{\mathcal{G}}$ into the sum of two quantities $\mathcal{J}_1$ and $\mathcal{J}_2$, defined as

$$\begin{aligned}\mathcal{J}_1 &:= \int_{|\sigma|\le\varepsilon_1\sqrt{\tau}}\left\{\mathcal{I}X^2 + 2N_0(x)X\right\}e^{-\sigma^2/4}\,\mathrm{d}\sigma,\\ \mathcal{J}_2 &:= \int_{\varepsilon_1\sqrt{\tau}\le|\sigma|\le 2e^{\varepsilon\tau/2}}\left\{\mathcal{I}X^2 + 2N_0(x)X\right\}e^{-\sigma^2/4}\,\mathrm{d}\sigma,\end{aligned}$$

for $\varepsilon_1 \le c$ to be chosen. Since, as a consequence of Lemma 21, we have $|\mathcal{I}| \le C$ and $|N_0(x)| \le CX$ for $|\sigma| \le c$, we can enforce the inequality $|\mathcal{J}_1| \le \frac{\delta_1}{4}\|X\|_{\mathcal{G}}^2$ by choosing ε_1 sufficiently small, depending only on δ_1 and C. Thus in what follows, we focus on $\mathcal{J}_2$.

Using Lemma 18 along with the weaker growth estimates in Lemma 19, we get

$$|x| = \frac{1}{2}e^{\tau/2}(g-f) \le C\frac{|\log(T-t)|}{\sqrt{T-t}}f^3 = C\tau e^{-\tau}u^3 \le C\tau e^{-\tau}\left(1 + \frac{\sigma^6}{\tau^3}\right).$$

So by Lemma 21, we have

$$|\mathcal{I}|X^2 \leq Ce^{-2\tau}\left(\tau + \frac{|\sigma|^{15}}{\tau^6}\right) \qquad \text{and} \qquad |N_0(x)X| \leq Ce^{-2\tau}\left(\tau + \frac{\sigma^{20}}{\tau^8}\right).$$

We now fix $0 < \delta_2 \leq 1/2$. Then there exist constants C depending on δ_2 such that

$$\begin{aligned}
|\mathcal{J}_2| &\leq Ce^{-2\tau}\int_{\varepsilon_1\sqrt{\tau}\leq|\sigma|\leq 2e^{\varepsilon\tau/2}}\left(\tau + \frac{\sigma^{20}}{\tau^8}\right)e^{-\delta_2\sigma^2/4}\,e^{-(1-\delta_2)\sigma^2/4}\mathrm{d}\sigma \\
&\leq Ce^{-2\tau}\int_{\varepsilon_1\sqrt{\tau}\leq|\sigma|\leq 2e^{\varepsilon\tau/2}} e^{-(1-\delta_2)\sigma^2/4}\mathrm{d}\sigma \\
&\leq Ce^{-2\tau}\,e^{-\frac{1-\delta_2}{4}\varepsilon_1^2\tau}.
\end{aligned}$$

Thus if we set $\delta_3 := (1-\delta_2)\varepsilon_1^2/8$, we get $|\mathcal{J}_2| \leq Ce^{-2(1+\delta_3)\tau}$, whence the result follows. □

If $\delta \leq 2/3$, then the ODE for $\|X\|_{\mathcal{G}}^2$ implies that for all $\tau \geq \tau^*$, one has $\|X\|_{\mathcal{G}} \leq Ce^{-(1+\delta)\tau}$. The regularizing effect of the heat equation lets us bootstrap this estimate by one spatial derivative. The proof is nearly identical to that of Lemma 9 in [2] but is even simpler, because the operator $\mathcal{A}-3$ has no unstable eigenmodes. Consequently, we omit the details, and state our result as follows:

Corollary 23. *If $\left(\mathcal{S}^1\times\mathcal{S}^3, G(t)\right)$ is a Ricci flow solution satisfying Assumption 3, then for any δ sufficiently small, there exist C and τ^* depending on δ such that for all $\tau \geq \tau^*$, one has*

$$\|X\|_{\mathcal{G}} + \|X_\sigma\|_{\mathcal{G}} \leq Ce^{-(1+\delta)\tau}.$$

Using Sobolev embedding, we find that this result implies that on bounded $|\sigma|$ intervals, one has a pointwise estimate $|x| = |X| \leq Ce^{\sigma^2/8}\big(\|X\|_{\mathcal{G}} + \|X_\sigma\|_{\mathcal{G}}\big)$. We therefore reach the following conclusion.

Corollary 24. *If $\left(\mathcal{S}^1\times\mathcal{S}^3, G(t)\right)$ is a Ricci flow solution satisfying Assumption 3, then for any δ sufficiently small and Σ large, there exist C and τ^* depending on δ and Σ such that for all $|\sigma| \leq \Sigma$ and $\tau \geq \tau^*$, one has*

$$\frac{g-f}{\sqrt{T-t}} = 2|x| \leq Ce^{-(1+\delta)\tau} = C(T-t)^{1+\delta}.$$

Thus as promised above, we obtain an improved rate of decay for the scale-invariant quantity $|f-g|/\sqrt{T-t}$. This decay indicates that the solution is rapidly approaching roundness in a spatial neighborhood of the center of the neck.

Appendix A. Warped Berger metrics

A.1. **The metrics we study.** To begin, we identify $\mathcal{S}^3$ with the Lie group SU(2), and consider general left-invariant metrics of the form

$$\hat{G} = f^2\omega^1\otimes\omega^1 + g^2\omega^2\otimes\omega^2 + h^2\omega^3\otimes\omega^3,$$

where the coframe $(\omega^1, \omega^2, \omega^3)$ is algebraically dual to a fixed Milnor frame (F_1, F_2, F_3) (see [4, Chapter 1]). The sectional curvatures of $\hat{G}$ are then

$$\begin{aligned}\hat{\kappa}_{12} &= \frac{(f^2-g^2)^2}{(fgh)^2} - 3\frac{h^2}{(fg)^2} + \frac{2}{f^2} + \frac{2}{g^2},\\ \hat{\kappa}_{23} &= \frac{(g^2-h^2)^2}{(fgh)^2} - 3\frac{f^2}{(gh)^2} + \frac{2}{g^2} + \frac{2}{h^2},\\ \hat{\kappa}_{31} &= \frac{(f^2-h^2)^2}{(fgh)^2} - 3\frac{g^2}{(fh)^2} + \frac{2}{f^2} + \frac{2}{h^2}.\end{aligned}$$

Notice that setting $f^2 = \varepsilon$ and $g^2 = h^2 = 1$ recovers the classic Berger collapsed sphere, shrinking the fibers of the Hopf vibration $\mathcal{S}^1 \hookrightarrow \mathcal{S}^3 \twoheadrightarrow \mathcal{S}^2$ with sectional curvatures $\kappa_{12} = \kappa_{31} = \varepsilon$ and $\kappa_{23} = 4 - 3\varepsilon$.

Now we consider Riemannian manifolds $(\mathcal{M}^4, G)$ having metrics of the form

$$\begin{aligned}G &= \rho^2(\mathrm{d}\xi)^2 + \hat{G}(\xi)\\ &= (\mathrm{d}s)^2 + f(s)^2\omega^1\otimes\omega^1 + g(s)^2\omega^2\otimes\omega^2 + h(s)^2\omega^3\otimes\omega^3,\end{aligned}$$

where $\mathrm{d}s := \rho\,\mathrm{d}\xi$. We assume for now that f, g, h depend only on $s(\xi)$, where $-\infty \le \xi_- < \xi < \xi_+ \le \infty$. We suppress dependence on ξ when possible, regarding f, g, h as functions of $s \in \mathcal{B} := (s_-, s_+)$, where $s_- = s(\xi_-)$ and $s_+ = s(\xi_+)$. We leave the boundary conditions at $s_\pm$, hence the topology of $\mathcal{M}^4$, open for now. We call these geometries *warped Berger metrics.* They are generalized warped products; for example, setting $f = g = h$ recovers the warped products studied in [1].

Note that $\pi : (\mathcal{M}^4, G) \to (\mathcal{B}, \check{G})$ is a Riemannian submersion, where $\check{G} = (\mathrm{d}s)^2$. As in [13], for each $x \in \mathcal{M}^4$ with $s = \pi(x)$, we define fibers $\mathcal{F}_x = \pi^{-1}(s) = (\mathcal{S}^3, \hat{G}(s))$. We identify $\mathcal{V}_x = \ker \pi_* : T_x\mathcal{M}^4 \to T_s\mathcal{B}$ with $T_x\mathcal{F}_x \subset T_x\mathcal{M}^4$. We set $\mathcal{H}_x = \mathcal{V}_x^\perp \subset T_x\mathcal{M}^4$. One calls $\mathcal{V}$ and $\mathcal{H}$ the vertical and horizontal distributions, respectively.

We define $F_0 := \frac{\partial}{\partial s}$, and let F_1, F_2, F_3 be the Milnor frame introduced above. Hereafter, we let Greek indices range in $0, \ldots, 3$ and Roman indices in $1, \ldots, 3$. We denote the components of the curvature tensor Rm of G by $R_{\alpha\beta\lambda\mu}$, and those of the curvature tensor $\hat{\mathrm{Rm}}$ of $\hat{G}$ by $\hat{R}_{ijk\ell}$. Then $\hat{R}_{1221} = (fg)^2\hat{\kappa}_{12}$, $\hat{R}_{2332} = (gh)^2\hat{\kappa}_{23}$, and $\hat{R}_{3113} = (fh)^2\hat{\kappa}_{31}$.

A.2. **Curvatures of the total space.** Recall that O'Neill [13] introduces $(2,1)$ tensor fields A and T that act on vector fields M, N by

$$A_M N = \mathcal{H}\big\{\nabla_{(\mathcal{H}M)}(\mathcal{V}N)\big\} + \mathcal{V}\big\{\nabla_{(\mathcal{H}M)}(\mathcal{H}N)\big\}, \tag{49}$$

and

$$T_M N = \mathcal{H}\big\{\nabla_{(\mathcal{V}M)}(\mathcal{V}N)\big\} + \mathcal{V}\big\{\nabla_{(\mathcal{V}M)}(\mathcal{H}N)\big\}, \tag{50}$$

respectively. Using these, one computes the tensor Rm from $\hat{\mathrm{Rm}}$, as follows.

We denote the connection 1-forms by Υ, so that $\nabla_{F_\alpha}F_\beta = \Upsilon_{\alpha\beta}^\gamma F_\gamma$. Note that these are not Christoffel symbols with respect to a chart; in particular, it is not true in general that $\Upsilon_{\alpha\beta}^\gamma = \Upsilon_{\beta\alpha}^\gamma$. However, we do have $\Upsilon_{0i}^\gamma = \Upsilon_{i0}^\gamma$, because $[F_0, F_i] = 0$, a fact that we use below.

The only forms Υ for G that differ from those $\hat{\Upsilon}$ for $\hat{G}$ are

$$\Upsilon_{0i}^i = \frac{1}{2}G^{ii}\partial_s(G_{ii}) \qquad \text{and} \qquad \Upsilon_{ii}^0 = -\frac{1}{2}\partial_s(G_{ii}). \tag{51}$$

One obtains these from the calculations

$$\partial_s G_{ii} = 2\langle \nabla_{F_0} F_i, F_i\rangle = 2\Upsilon^i_{0i} G_{ii}$$

and

$$\Upsilon^0_{ii} = \langle \nabla_{F_i} F_i, F_0\rangle = -\langle F_i, \nabla_{F_i} F_0\rangle = -\langle F_i, \nabla_{F_0} F_i\rangle = -\Upsilon^i_{0i} G_{ii}.$$

Hereafter, we only consider vector fields N that satisfy our warped Berger *Ansatz* $N = N^\alpha(s) F_\alpha$, so that all $F_i(N^0) = 0$. It follows that for any such vector fields M, N, one has

$$\nabla_M N - \hat{\nabla}_{(\mathcal{V}M)}(\mathcal{V}N) = M^0\Big\{\partial_s N^\beta F_\beta + \Upsilon^i_{0i} N^i F_i\Big\} + M^i\Big\{\Upsilon^0_{ii} N^i F_0 + \Upsilon^i_{i0} N^0 F_i\Big\}. \tag{52}$$

A.2.1. *Curvatures of vertical planes.* O'Neill's tensor T encodes the second fundamental form of the fibers $\mathcal{F}_x$ — that is to say, it encodes $\nabla - \hat{\nabla}$. We write $T_M N = M^\alpha N^\beta T^\gamma_{\alpha\beta} F_\gamma$. Formulas (50)–(52) show that for vertical vector fields U, V, one has $\nabla_U V - \hat{\nabla}_U V = U^i V^j T^\alpha_{ij} F_\alpha$, where all components T^α_{ij} vanish except $T^0_{ii} = \Upsilon^0_{ii}$, which have the values

$$T^0_{11} = -f f_s, \qquad T^0_{22} = -g g_s, \quad \text{and} \quad T^0_{33} = -h h_s.$$

O'Neill's formula for Rm applied to vertical vector fields U, V, W, P,

$$\langle R(U,V)W, P\rangle = \langle \hat{R}(U,V)W, P\rangle + \langle T_U W, T_V P\rangle - \langle T_V W, T_U P\rangle,$$

thus implies that the sectional curvatures of the vertical planes are

$$\kappa_{12} = \hat{\kappa}_{12} - \frac{f_s g_s}{fg},$$
$$\kappa_{23} = \hat{\kappa}_{23} - \frac{g_s h_s}{gh},$$
$$\kappa_{31} = \hat{\kappa}_{31} - \frac{f_s h_s}{fh}.$$

A.2.2. *Curvature of mixed planes.* Because $\mathcal{B}$ is one-dimensional, the only other curvatures we need consider are those involving planes $F_0 \wedge F_i$. For the same reason, O'Neill's tensor A, which measures the obstruction to integrability of the distribution $\mathcal{H}$, vanishes. These observations reduce the remaining curvature formulas to

$$\langle R(F_0, U)V, F_0\rangle = \langle (\nabla_{F_0} T)_U V, F_0\rangle - \langle T_U F_0, T_V F_0\rangle, \tag{53}$$

$$\langle R(U,V)W, F_0\rangle = \langle (\nabla_U T)_V W, F_0\rangle - \langle (\nabla_V T)_U W, F_0\rangle, \tag{54}$$

where U, V, W are again vertical vector fields.

To compute the curvatures given by (53), we first use (50) and (52) to see that

$$T_U F_0 = \mathcal{V}\Big(\nabla_U F_0\Big) = U^i \Upsilon^i_{i0} F_i.$$

Next we observe that $\nabla_0 T^k_{ij}{=}0$ for all i, j, k, and that $\nabla_0 T^0_{ij} = 0$ for $i \neq j$, while

$$\nabla_0 T^0_{ii} = \partial_s T^0_{ii} - 2\Upsilon^i_{0i} T^0_{ii}.$$

It thus follows from (53) that $R_{0ij0} = 0$ for all $i \neq j$, while the nonvanishing sectional curvatures $G^{ii} R_{0ii0}$ are

$$\kappa_{01} = -\frac{f_{ss}}{f}, \qquad \kappa_{02} = -\frac{g_{ss}}{g}, \quad \text{and} \quad \kappa_{03} = -\frac{h_{ss}}{h}.$$

Rather than use (54) to compute the remaining curvatures, it is easier to proceed as follows. To study this *Ansatz* under Ricci flow, it suffices to compute Rc, and the only remaining curvatures one needs to accomplish this are all elements of the form R_{0jjk} with $j \neq k$. By definition of Rm, one has

$$\begin{aligned} R(F_\alpha, F_\beta)F_\gamma &= \nabla_{F_\alpha}(\nabla_{F_\beta}F_\gamma) - \nabla_{F_\beta}(\nabla_{F_\alpha}F_\gamma) - \nabla_{[F_\alpha,F_\beta]}F_\gamma \\ &= F_\alpha\big(\Upsilon^\lambda_{\beta\gamma}\big)F_\lambda + \Upsilon^\mu_{\beta\gamma}\Upsilon^\lambda_{\alpha\mu}F_\lambda - F_\beta(\Upsilon^\lambda_{\alpha\gamma})F_\lambda - \Upsilon^\mu_{\alpha\gamma}\Upsilon^\lambda_{\beta\mu}F_\lambda - \nabla_{[F_\alpha,F_\beta]}F_\gamma. \end{aligned}$$

Therefore, because $[F_0, F_j] = 0$ and $j \neq k$, we obtain

$$R^k_{0jj} = \Upsilon^\mu_{jj}\Upsilon^k_{0\mu} - \Upsilon^\mu_{0j}\Upsilon^k_{j\mu} = \hat{\Upsilon}^k_{jj}\big(\Upsilon^k_{0k} - \Upsilon^j_{0j}\big).$$

But $\hat{\Gamma}^k_{jj} = \hat{G}^{kk}\langle\hat{\nabla}_{F_j}F_j, F_k\rangle$, and $\hat{\nabla}_{F_j}F_j = -(\mathrm{ad}F_j)^*F_j = 0$, because (F_1, F_2, F_3) is a Milnor frame (see [4, Chapter 1.4]). Therefore, all $R_{0jjk} = 0$, and so $R_{0k} = 0$.

A.3. **Evolution of warped Berger metrics by Ricci flow.** The calculations in Section A.2 show that the Ricci endormorphism is diagonal in the coordinates induced by $(F_0, \ldots, F_3)$, with $R^\gamma_\alpha = 0$ if $\gamma \neq \alpha$, and $R^\alpha_\alpha = \sum_{\beta\neq\alpha}\kappa_{\alpha\beta}$. Hence the warped Berger metric *Ansatz* is preserved under Ricci flow.

We now abuse notation and allow f, g, h and the gauge ρ (hence s) to depend on time as well as the spatial variable ξ. Then Ricci flow of G is equivalent to the system

$$f_t = f_{ss} + \left(\frac{g_s}{g} + \frac{h_s}{h}\right)f_s - f\big(\hat{\kappa}_{12} + \hat{\kappa}_{31}\big), \tag{55a}$$

$$g_t = g_{ss} + \left(\frac{f_s}{f} + \frac{h_s}{h}\right)g_s - g\big(\hat{\kappa}_{12} + \hat{\kappa}_{23}\big), \tag{55b}$$

$$h_t = h_{ss} + \left(\frac{f_s}{f} + \frac{g_s}{g}\right)h_s - h\big(\hat{\kappa}_{23} + \hat{\kappa}_{31}\big), \tag{55c}$$

along with the evolution equation

$$(\log\rho)_t = -\big(\kappa_{01} + \kappa_{02} + \kappa_{03}\big) \tag{56}$$

satisfied by the gauge ρ. Our choice of gauge means that space and time derivatives do not commute; instead one has the commutator

$$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = \big(\kappa_{01} + \kappa_{02} + \kappa_{03}\big)\frac{\partial}{\partial s}.$$

Finally, one has to impose boundary conditions at $\xi_\pm$ in order to get a smooth metric on some topology. In this paper, we study metrics on $\mathcal{S}^1 \times \mathcal{S}^3$, so we take $[\xi_-, \xi_+] = [-\pi, \pi]$ and stipulate that everything in sight is 2π-periodic in space. This allows us in the body of the paper to regard ξ as a coordinate on $\mathcal{S}^1$, with s representing arclength from a fixed but arbitrary point ξ_0.

A.4. **A simplified *Ansatz*.** We conjecture that Ricci flow solutions satisfying the general system (55) become asymptotically rotationally symmetric if they develop neckpinch singularities. In this paper, we prove the conjecture in the special case that $g = h$ initially. The following result shows that this condition is preserved.

Lemma 25. *For these metrics, any ordering, e.g., $f \leq g$ or $g \leq h$, that holds initially is preserved by Ricci flow.*

Proof. Without loss of generality, it suffices to show that the condition $g \leq h$ is preserved. We set $z := g - h$. Then a straightforward computation shows that

$$z_t = z_s + \frac{f_s}{f} z_s + \left\{ \frac{g_s h_s}{gh} - \hat{\kappa}_{23} + \zeta \right\} z,$$

where

$$\zeta = 2\frac{f^2 - gh}{f^2 gh} - 3\frac{g^2 + gh + h^2}{f^2 gh} - \frac{f^4 - 2f^2(g^2 + gh + h^2) + g^4 + g^3 h + g^2 h^2 + gh^3 + h^4}{(fgh)^2}.$$

By the parabolic maximum principle, the condition $z \leq 0$ is preserved if it holds initially; the same is true of $z \geq 0$. □

Remark 4. Unsurprisingly, ordering is also preserved by the Ricci flow ODE system on SU(2); see [4, Chapter 1.5].

Appendix B. Initial data that result in local singularities

Here we show that there are (non-unique) open sets of warped Berger initial data giving rise to solutions that satisfy Assumption 1 or Assumption 2 and that develop local neckpinch singularities.

To begin, we consider metrics of the form (1) on $\mathbb{R} \times \mathcal{S}^3$, with $g(s) = \gamma(s)$ and $f = \eta\gamma(s)$, where

$$\gamma(s) := \sqrt{\alpha + \beta s^2}.$$

Here, α, β, and $\eta \leq 1$ are positive constants. It is easy to check that $\gamma_s = \beta s/\gamma$ satisfies the bound $|\gamma_s| \leq \sqrt{\beta}$, and that $\gamma_{ss} = \alpha\beta/\gamma^3$. Then, observing that the curvatures of these metrics are

$$\kappa_{12} = \kappa_{31} = \frac{\eta^2\gamma^2 - \beta^2 s^2}{\gamma^4} \qquad \text{and} \qquad \kappa_{23} = \frac{(4 - 3\eta^2)\gamma^2 - \beta^2 s^2}{\gamma^4},$$

and

$$\kappa_{01} = \kappa_{02} = \kappa_{03} = -\frac{\alpha\beta}{\gamma^4},$$

one computes easily that the scalar curvature is

$$R = \frac{(4 - \eta^2 - 3\beta)}{\gamma^2},$$

which is positive if β is sufficiently small.

Next, working on $\mathcal{S}^1 \times \mathcal{S}^3$, we set $\rho = \Lambda$, where Λ is a large constant. It follows that $s \in [-\Lambda\pi, \Lambda\pi]$. We now choose $g(s) = \bar{\gamma}(s)$ and $f = \eta\bar{\gamma}(s)$, where $\bar{\gamma}$ is the piecewise smooth function

$$\bar{\gamma}(s) := \begin{cases} \gamma(s) & \text{if } |s| \leq \Lambda, \\ \gamma(\Lambda) & \text{if } |s| > \Lambda. \end{cases}$$

For $|s| > \Lambda$, one has $R = (4 - \eta^2)/\gamma(\Lambda)^2$, which is positive. (As noted above, R is also positive for $|s| \leq \Lambda$.)

We smooth the "corner" that $\bar{\gamma}$ has at $s = \Lambda$ in two steps. First we construct $\tilde{\gamma}$, which agrees with $\bar{\gamma}$ outside intervals $I_\delta := \{|s| \in (\Lambda - \delta, \Lambda + \delta)\}$ and has $\tilde{\gamma}_{ss}$ constant in each I_δ. We choose the constant $-\beta(\Lambda - \delta)/\big(2\delta\sqrt{\alpha + \beta(\Lambda - \delta)^2}\big)$, so that $\tilde{\gamma}$ is C^1. Because $\tilde{\gamma}_{ss} < 0$ in each I_δ and $|\tilde{\gamma}_s| \leq \sqrt{\beta}$ everywhere, the metric induced by $\tilde{\gamma}$ continues to have positive scalar curvature everywhere it is smooth.

Now $\tilde{\gamma}$ is piecewise smooth, and $\tilde{\gamma}_{ss}$ has simple jump discontinuities at $|s| = \Lambda \pm \delta$. So in the final step, we smooth $\tilde{\gamma}$, obtaining a C^∞ function $\tilde{\tilde{\gamma}}$ that agrees with $\bar{\gamma}$ outside intervals $I_{2\delta}$. It is clear that this can be done so that $|\tilde{\tilde{\gamma}}_s| \leq 2\sqrt{\beta}$. For $\alpha \in (0, \alpha^*)$, $\beta \in (0, \beta^*)$, and $\eta \in (0,1)$, this produces a family $\mathcal{G}$ of initial data with $f < g$, positive scalar curvature, and uniform curvature bounds, depending only on α^*, β^*, and δ. Hence the first two conditions of Assumption 1 are satisfied.

Each initial metric $G_0 \in \mathcal{G}$ has a "pseudo-neck" at $s = 0$ of radius $\eta\alpha$ and a "pseudo-bump" at $|s| = \Lambda\pi$ of height $\gamma(\Lambda)$. A solution originating from G_0 must become singular at some $T < \infty$ and thus must satisfy the third condition of Assumption 1; indeed, Lemma 3 shows there exists a constant c such that the singular time T satisfies

$$T \leq \frac{\check{M}^2(0)}{c} = \frac{\eta^2 \alpha}{c}. \tag{57}$$

Finally, we note that it follows from Corollary 8 together with this upper estimate for T that for $s \geq \Lambda + 2\delta$, one has

$$f^2(s,T) \geq \eta^2 \left\{ \alpha - \frac{C}{c}\alpha + \beta\Lambda^2 \right\}.$$

We observe that the constant c — which comes from estimate (57) — and the constant C — which comes from the estimate for $f^2(s,t)$ in Corollary 8, and can be traced back to the curvature estimate in Lemma 6 — depend only on the ratio $f/g = \eta$ and on bounds for the curvatures, all of which are independent of $\Lambda \geq 1$. So by taking Λ sufficiently large, we can ensure that $f^2(s,T) > 0$, hence that the singularity is local.

Remark 5. It is clear from this construction that there is a neighborhood $\mathcal{G}_1$ of $\mathcal{G}$ in $\mathfrak{Met}(\mathcal{S}^1 \times \mathcal{S}^3)$ such that all warped Berger solutions originating in this open set satisfy Assumption 1 and develop local singularities in finite time.

Remark 6. It is also clear from the construction that by taking β sufficiently close to 0 and η sufficiently close to 1, we obtain a family $\mathcal{G}'$ of initial data that satisfy Assumption 2, as do all warped Berger solutions originating in a neighborhood $\mathcal{G}_2$ of $\mathcal{G}'$ in $\mathfrak{Met}(\mathcal{S}^1 \times \mathcal{S}^3)$. Because our construction is reflection-symmetric, it also produces initial data that satisfy Assumption 3.

Appendix C. Parabolically rescaled equations

C.1. **Evolution equations in blow-up variables.** Given a singularity time T, we introduce parabolically dilated time and space variables

$$\tau := -\log(T - t) \qquad \text{and} \qquad \sigma := e^{\tau/2} s,$$

respectively. We parabolically dilate the metric, considering [15]

$$u := \frac{1}{2} e^{\tau/2} f \qquad \text{and} \qquad v := \frac{1}{2} e^{\tau/2} g.$$

[15] The fraction $\frac{1}{2}$ corresponds to the factor $1/\sqrt{2(n-1)}$ in [2] and simplifies what follows.

One computes that $f_t = 2e^{\tau/2}\{u_\tau + \sigma_\tau u_\sigma - \frac{1}{2}u\}$, $f_s = 2u_\sigma$, and $f_{ss} = 2e^{\tau/2}u_{\sigma\sigma}$, where $\sigma_\tau = \frac{1}{2}\sigma + \mathcal{I}$, with $\mathcal{I}$ the nonlocal term

$$\mathcal{I}(\sigma,\tau) = \int_0^\sigma \left(\frac{u_{\bar\sigma\bar\sigma}}{u} + 2\frac{v_{\bar\sigma\bar\sigma}}{v}\right) \mathrm{d}\bar\sigma.$$

The nonlocal quantity $\mathcal{I}$ is necessary for σ and τ to be commuting variables, i.e. for us to interpret τ derivatives as time derivatives taken with σ rather than ξ fixed. Of course, analogous formulas hold for g_t, g_s, and g_{ss}.

With these rescalings imposed , system (14) becomes

$$\begin{aligned} u_\tau &= u_{\sigma\sigma} - \left(\frac{\sigma}{2} + \mathcal{I}\right) u_\sigma + 2\frac{v_\sigma}{v} u_\sigma + \frac{1}{2}\left(u - \frac{u^3}{v^4}\right), \\ v_\tau &= v_{\sigma\sigma} - \left(\frac{\sigma}{2} + \mathcal{I}\right) v_\sigma + \left(\frac{u_\sigma}{u} + \frac{v_\sigma}{v}\right) v_\sigma + \frac{1}{2}\left(v - \frac{2v^2 - u^2}{v^3}\right), \end{aligned}$$

which reduces to the equation studied in [2] if $u = v$.

C.2. **Linearization at the cylinder.** In a space-time neighborhood of the singular set, our results in Lemma 16 show that the solution is close to the self-similarly shrinking cylinder soliton near the developing neckpinch, so that $u \approx 1$ and $v \approx 1$. Accordingly, we introduce (locally small) quantities φ and ψ defined by

$$\varphi := u - 1 \qquad \text{and} \qquad \psi := v - 1.$$

Linearizing near $u = 1$ and $v = 1$, one finds that

$$\varphi_\tau = \left\{\varphi_{\sigma\sigma} - \frac{\sigma}{2}\varphi_\sigma - \varphi + 2\psi\right\} + N_1(\varphi, \psi),$$

where the operator in braces is linear, and $N_1(\varphi,\psi)$ is the nonlinear term

$$\begin{aligned} N_1 = &\left(\frac{2\psi_\sigma}{1+\psi} - \mathcal{I}\right)\varphi_\sigma + (\varphi - 2\psi)\frac{(1+\psi)^4 - 1}{(1+\psi)^4} \\ &+ \frac{-3\varphi^2 + 4\varphi\psi + 6\psi^2 - \varphi^3 + 6\varphi\psi^2 + 4\psi^3 + 4\varphi\psi^3 + \psi^4 + \varphi\psi^4}{2(1+\psi)^4}. \end{aligned}$$

In the same way, one finds that

$$\psi_\tau = \left\{\psi_{\sigma\sigma} - \frac{\sigma}{2}\psi_\sigma + \varphi\right\} + N_2(\varphi, \psi),$$

where the operator in braces is linear, and

$$N_2 = \left(\frac{\varphi_\sigma}{1+\varphi} + \frac{\psi_\sigma}{1+\psi} - \mathcal{I}\right)\psi_\sigma - \varphi\frac{(1+\psi)^3 - 1}{(1+\psi)^3} + \frac{\varphi^2 + 4\psi^2 + 4\psi^3 + \psi^4}{2(1+\psi)^3}.$$

For clarity of exposition, we have not simplified N_1, N_2 as much as possible here.

Examination of the linearized system for φ and ψ reveals that it is easily decoupled by introducing new quantities

$$x := \varphi - \psi \qquad \text{and} \qquad y = \varphi + 2\psi. \tag{58}$$

Then, neglecting nonlinear terms, one finds that x and y evolve by

$$\begin{aligned} x_\tau &= x_{\sigma\sigma} - \frac{\sigma}{2}x_\sigma - 2x + \cdots = (\mathcal{A} - 3)x + \cdots, \\ y_\tau &= y_{\sigma\sigma} - \frac{\sigma}{2}y_\sigma + y + \cdots = \quad \mathcal{A}y + \cdots, \end{aligned}$$

where $\mathcal{A}$ is the elliptic operator the generates the quantum harmonic oscillator,

$$\mathcal{A} = \frac{\partial^2}{\partial \sigma^2} - \frac{\sigma}{2}\frac{\partial}{\partial \sigma} + 1.$$

The spectrum of $-\mathcal{A}$ is $\{\mu_k = \frac{k}{2} - 1,\ k \geq 0\}$, with associated eigenfunctions the Hermite polynomials h_k, normalized here so that $h_k(\sigma) = \sigma^k + \mathcal{O}(\sigma^{k-2})$. Clearly, the spectrum of $-\mathcal{A} + 3$ is $\{\nu_k = \frac{k}{2} + 2,\ k \geq 0\}$.

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