

CONVERGENCE AND STABILITY OF LOCALLY \mathbb{R}^N -INVARIANT SOLUTIONS OF RICCI FLOW

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ABSTRACT. Valuable models for immortal solutions of Ricci flow that collapse with bounded curvature come from locally \mathcal{G} -invariant solutions on bundles $\mathcal{G}^N \hookrightarrow \mathcal{M} \xrightarrow{\pi} \mathcal{B}^n$, with \mathcal{G} a nilpotent Lie group. In this paper, we establish convergence and asymptotic stability, modulo smooth finite-dimensional center manifolds, of certain \mathbb{R}^N -invariant model solutions. In case $N + n = 3$, our results are relevant to work of Lott classifying the asymptotic behavior of all 3-dimensional Ricci flow solutions whose sectional curvatures and diameters are respectively $\mathcal{O}(t^{-1})$ and $\mathcal{O}(t^{1/2})$ as $t \rightarrow \infty$.

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1. INTRODUCTION

There are many interesting open questions regarding the geometric and analytic properties of immortal solutions $(\mathcal{M}, g(t))$ of Ricci flow that collapse. One does not expect such solutions to converge smoothly in a naive sense. Instead, one may expect $(\mathcal{M}, g(t))$ to exhibit Gromov–Hausdorff convergence to a lower-dimensional geometric object. (See [11], for example.) An impelling motivation for studying such solutions when the dimension of the total space is three is to obtain a better understanding of the role of smooth collapse in the Ricci flow approach to geometrization. See [19] and [14] for a broader discussion of some geometric and analytic problems related to collapse and their motivations.

In a recent paper [20], John Lott makes significant progress in understanding the long-time behavior of Ricci flow on a compact 3-manifold. He proves that any solution $(\mathcal{M}^3, g(t))$ of Ricci flow that exists for all $t \geq 0$ with sectional curvatures that are $\mathcal{O}(t^{-1})$ and diameter that is $\mathcal{O}(t^{1/2})$ as $t \rightarrow \infty$ converges (after pull-back to the universal cover and modification by diffeomorphisms) to an expanding homogeneous soliton. A key component in the proof is the analysis of locally \mathcal{G} -invariant solutions of Ricci flow, where \mathcal{G} is a connected N -dimensional Abelian Lie group. Such solutions are expected to constitute valuable models for the behavior of Type-III (immortal) solutions of Ricci flow in all dimensions, especially solutions undergoing collapse. We now review the basic setup from [20].

Let $\mathbb{R}^N \hookrightarrow \mathcal{M} \xrightarrow{\pi} \mathcal{B}^n$ be a fiber bundle over a connected oriented compact base \mathcal{B} , and let $\mathbb{R}^N \hookrightarrow \mathcal{E} \xrightarrow{p} \mathcal{B}^n$ be a flat vector bundle over \mathcal{B} . Let $\mathcal{E} \times_{\mathcal{B}} \mathcal{M}$ denote the fiber space $\bigcup_{b \in \mathcal{B}} (\mathcal{E}_b \times \mathcal{M}_b)$. Assume that there exists a smooth map $\mathcal{E} \times_{\mathcal{B}} \mathcal{M} \rightarrow \mathcal{M}$ such that over each basepoint $b \in \mathcal{B}$, the map $\mathcal{E}_b \times \mathcal{M}_b \rightarrow \mathcal{M}_b$ is a free transitive action. Assume also that this action is consistent with the flat connection on \mathcal{E} in the sense that for any open set $\mathcal{U} \subseteq \mathcal{B}$ small enough that $\mathcal{E}|_{\mathcal{U}} \approx \mathcal{U} \times \mathbb{R}^N$ is a local trivialization, the set $\pi^{-1}(\mathcal{U})$ has a free \mathbb{R}^N action, hence is the total space of a principal \mathbb{R}^N bundle over \mathcal{U} . In this way, \mathcal{M} may be regarded as an \mathbb{R}^N -principal bundle over \mathcal{B} twisted by the flat vector bundle \mathcal{E} . One can define a connection A on \mathcal{M} with the property that any restriction $A|_{\pi^{-1}(\mathcal{U})}$ is an \mathbb{R}^N -valued connection.

Let $\mathcal{U} \subseteq \mathcal{B}$ be an open set small enough such that $\mathcal{E}|_{\mathcal{U}} \approx \mathcal{U} \times \mathbb{R}^N$ is a local trivialization and such that there exist a local parameterization $\rho : \mathbb{R}^n \rightarrow \mathcal{U}$ and a local section $\sigma : \mathcal{U} \rightarrow \pi^{-1}(\mathcal{U})$. Then any choice of basis (e_1, \dots, e_N) for \mathbb{R}^N yields local coordinates on $\pi^{-1}(\mathcal{U})$ via

$$\mathbb{R}^n \times \mathbb{R}^N \ni (x^\alpha, x^i) \mapsto (x^i e_i) \cdot \sigma(\rho(x^\alpha)),$$

where $\{x^\alpha\}_{\alpha=1}^n$ are coordinates on \mathcal{U} , $\{x^i\}_{i=1}^N$ are coordinates on \mathbb{R}^N , and \cdot denotes the free \mathbb{R}^N -action described above.

Let \bar{g} denote a Riemannian metric on \mathcal{M} with the property that this action is a local isometry. Then with respect to these coordinates, one may write

$$(1.1) \quad \bar{g} = \sum_{\alpha, \beta=1}^n \bar{g}_{\alpha\beta} dx^\alpha dx^\beta + \sum_{i, j=1}^N \bar{G}_{ij} (dx^i + \sum_{\alpha=1}^n \bar{A}_\alpha^i dx^\alpha) (dx^j + \sum_{\beta=1}^n \bar{A}_\beta^j dx^\beta).$$

Observe here that for $b \in \mathcal{U} \subseteq \mathcal{B}$, $\sum_{\alpha, \beta=1}^n \bar{g}_{\alpha\beta}(b) dx^\alpha dx^\beta$ is the local expression of a Riemannian metric on \mathcal{B} , $\bar{A}_\alpha^i(b) dx^\alpha$ is locally the pullback by σ of a connection on $\pi^{-1}(\mathcal{U}) \rightarrow \mathcal{U}$, and $\sum_{i, j=1}^N \bar{G}_{ij}(b) dx^i dx^j$ gives a Euclidean inner product on the fiber over b .

A one-parameter family $(\mathcal{M}, \bar{\mathbf{g}}(t) : t \in \mathcal{I})$ of such metrics evolving by Ricci flow in a nonempty time interval \mathcal{I} constitutes a *locally \mathbb{R}^N -invariant Ricci flow solution*. In [20], Lott shows that such a solution is equivalent, modulo diffeomorphisms of \mathcal{M} and modifications of \bar{A} by exact forms, to the system¹

$$(1.2a) \quad \frac{\partial}{\partial t} \bar{g}_{\alpha\beta} = -2\bar{R}_{\alpha\beta} + \frac{1}{2} \bar{G}^{ik} \bar{G}^{j\ell} \bar{\nabla}_\alpha \bar{G}_{ij} \bar{\nabla}_\beta \bar{G}_{k\ell} + \bar{g}^{\gamma\delta} \bar{G}_{ij} (d\bar{A})_{\alpha\gamma}^i (d\bar{A})_{\beta\delta}^j,$$

$$(1.2b) \quad \frac{\partial}{\partial t} \bar{A}_\alpha^i = -(\bar{\delta} d\bar{A})_\alpha^i + \bar{G}^{ij} \bar{\nabla}^\beta \bar{G}_{jk} (d\bar{A})_{\beta\alpha}^k,$$

$$(1.2c) \quad \frac{\partial}{\partial t} \bar{G}_{ij} = \bar{\Delta} \bar{G}_{ij} - \bar{G}^{k\ell} \bar{\nabla}_\alpha \bar{G}_{ik} \bar{\nabla}^\alpha \bar{G}_{\ell j} - \frac{1}{2} \bar{g}^{\alpha\gamma} \bar{g}^{\beta\delta} \bar{G}_{ik} \bar{G}_{j\ell} (d\bar{A})_{\alpha\beta}^k (d\bar{A})_{\gamma\delta}^\ell.$$

Abusing notation, we denote a solution of (1.2) by $\bar{\mathbf{g}}(t) = (\bar{g}(t), \bar{A}(t), \bar{G}(t))$. In order to study the long-time behavior of such systems, we engineer a transformation that turns certain model solutions into fixed points whose asymptotic stability can be investigated. To facilitate this, we assume that \mathcal{M} admits a flat connection, allowing us to regard \bar{A} as an \mathbb{R}^N -valued 1-form. Let a function $s(t)$ and a constant c be given. Let $\sigma(t)$ be any positive antiderivative of s . Consider the transformation

$$(\bar{g}(t), \bar{A}(t), \bar{G}(t)) = \bar{\mathbf{g}}(t) \mapsto \mathbf{g}(\tau) = (g(\tau), A(\tau), G(\tau)),$$

where

$$g = \sigma^{-1} \bar{g}, \quad A = \sigma^{-\frac{1+c}{2}} \bar{A}, \quad G = \sigma^c \bar{G}, \quad \tau = \int_{t_0}^t \sigma^{-1}(\bar{t}) d\bar{t}, \quad (t_0 \in \mathcal{I}).$$

Observe that the exponent $-(1+c)/2$ above is necessary so that no factors of σ appear in the transformed system (1.3) below. Examples 1–3 below illustrate why this is desirable. Under the transformation $\bar{\mathbf{g}}(t) \mapsto \mathbf{g}(\tau)$, system (1.2) becomes²

$$(1.3a) \quad \frac{\partial}{\partial \tau} g_{\alpha\beta} = -2R_{\alpha\beta} + \frac{1}{2} G^{ik} G^{j\ell} \nabla_\alpha G_{ij} \nabla_\beta G_{k\ell} + g^{\gamma\delta} G_{ij} (dA)_{\alpha\gamma}^i (dA)_{\beta\delta}^j - s g_{\alpha\beta},$$

$$(1.3b) \quad \frac{\partial}{\partial \tau} A_\alpha^i = -(\delta dA)_\alpha^i + G^{ij} \nabla^\beta G_{jk} (dA)_{\beta\alpha}^k - \frac{1+c}{2} s A_\alpha^i,$$

$$(1.3c) \quad \frac{\partial}{\partial \tau} G_{ij} = \Delta G_{ij} - G^{k\ell} \nabla_\alpha G_{ik} \nabla^\alpha G_{\ell j} - \frac{1}{2} g^{\alpha\gamma} g^{\beta\delta} G_{ik} G_{j\ell} (dA)_{\alpha\beta}^k (dA)_{\gamma\delta}^\ell + c s G_{ij}.$$

We call this system a *rescaled locally \mathbb{R}^N -invariant Ricci flow*.

Example 1. On a solution $(\mathbb{R}^N \times \mathcal{B}, \bar{\mathbf{g}}(t))$ of (1.2) that is a Riemannian product over a nonflat Einstein base (\mathcal{B}, \bar{g}) , one may choose coordinates so that \bar{G} is constant, \bar{A} vanishes, and $\bar{g}(t) = -\varkappa t \bar{g}(-\varkappa)$, where $\varkappa = \pm 1$ is the Einstein constant such that $2\text{Rc}(\bar{g}(-\varkappa)) = \varkappa \bar{g}(-\varkappa)$. (Note that $\bar{g}(t)$ exists for $t < 0$ if $\varkappa = 1$ and for $t > 0$ if $\varkappa = -1$.) The choices $s = -\varkappa$, $c = 0$, and $t_0 = -\varkappa$ transform $\bar{\mathbf{g}}(t_0)$ into a stationary solution $\mathbf{g}(0)$ of the autonomous system (1.3).

¹Here $\bar{\nabla}_\alpha = \frac{\partial}{\partial x^\alpha}$, $\bar{\nabla}^\alpha = \bar{g}^{\alpha\beta} \frac{\partial}{\partial x^\beta}$, $(d\bar{A})_{\alpha\beta}^i = \bar{\nabla}_\alpha \bar{A}_\beta^i - \bar{\nabla}_\beta \bar{A}_\alpha^i$, $(\bar{\delta} d\bar{A})_\alpha^i = -\bar{\nabla}^\beta (d\bar{A})_{\beta\alpha}^i$, and $\bar{\Delta} \bar{G}_{ij} = \bar{g}^{\alpha\beta} \bar{\nabla}_\alpha \bar{\nabla}_\beta \bar{G}_{ij} = \bar{g}^{\alpha\beta} (\frac{\partial^2}{\partial x^\alpha \partial x^\beta} \bar{G}_{ij} - \bar{\Gamma}_{\alpha\beta}^\gamma \frac{\partial}{\partial x^\gamma} \bar{G}_{ij})$, where $\bar{\Gamma}$ represents the Levi-Civita connection of \bar{g} .

²Here $\nabla_\alpha = \frac{\partial}{\partial x^\alpha}$, $\nabla^\alpha = g^{\alpha\beta} \frac{\partial}{\partial x^\beta}$, $(dA)_{\alpha\beta}^i = \nabla_\alpha A_\beta^i - \nabla_\beta A_\alpha^i$, $(\delta dA)_\alpha^i = -\nabla^\beta (dA)_{\beta\alpha}^i$, and $\Delta G_{ij} = g^{\alpha\beta} \nabla_\alpha \nabla_\beta G_{ij} = g^{\alpha\beta} (\frac{\partial^2}{\partial x^\alpha \partial x^\beta} G_{ij} - \Gamma_{\alpha\beta}^\gamma \frac{\partial}{\partial x^\gamma} G_{ij})$, where Γ represents the Levi-Civita connection of g .

Example 2. A somewhat more general normalization is as follows. Suppose that $g(\tau)$ may be regarded as a well-defined metric on \mathcal{B} . Let $V(\tau)$ denote the volume of $(\mathcal{B}, g(\tau))$ and define

$$r = R - \frac{1}{4} |\nabla G|^2 - \frac{1}{2} |dA|^2,$$

where everything is computed with respect to g . Because $\frac{dV}{d\tau} = -\int_{\mathcal{B}} r \, d\mu - \frac{n}{2} sV(\tau)$, it follows that V is fixed if and only if³

$$s = -\frac{2}{n} \oint_{\mathcal{B}} r \, d\mu.$$

Consider a Riemannian product solution over an Einstein base. Such a solution may be written as $\bar{\mathbf{g}}(t) = (\bar{g}(t), \bar{A}, \bar{G})$, where \bar{A} vanishes identically and \bar{G} is fixed in space and time. For any t_0 in its time domain of existence, taking $c = 0$ makes $\mathbf{g}(0) = (\sigma^{-1}(t_0)\bar{g}(t_0), 0, \bar{G})$ into a stationary solution of (1.3) for any choice of $\sigma^{-1}(t_0) > 0$.

Example 3. The mapping torus \mathcal{M}_f of $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is $([0, 1] \times \mathbb{R}^N) / \sim$, where $(0, p) \sim (1, f(p))$. With respect to coordinates $x^0 \in \mathcal{S}^1$ and $(x^1, \dots, x^N) \in \mathbb{R}^N$ on the circle bundle $\mathbb{R}^N \hookrightarrow \mathcal{M}_f \xrightarrow{\pi} \mathcal{S}^1$, a locally \mathbb{R}^N -invariant metric has the form

$$\mathbf{g} = u \, dx^0 \otimes dx^0 + A_i (dx^0 \otimes dx^i + dx^i \otimes dx^0) + G_{ij} \, dx^i \otimes dx^j,$$

where u , A^i , and G depend only on x^0 , and $A_i = G_{ij} A^j$. System (1.3) reduces to

$$(1.4a) \quad \frac{\partial}{\partial \tau} u = \frac{1}{2} |\partial_0 G|^2 - su,$$

$$(1.4b) \quad \frac{\partial}{\partial \tau} A^i = -\frac{1+c}{2} s A^i,$$

$$(1.4c) \quad \frac{\partial}{\partial \tau} G_{ij} = u^{-1} (\nabla_0^2 G_{ij} - G^{k\ell} \partial_0 G_{ik} \partial_0 G_{\ell j}) + cs G_{ij},$$

where $|\partial_0 G|^2 = G^{ik} G^{j\ell} \partial_0 G_{ij} \partial_0 G_{k\ell}$ and $\nabla_0^2 = \partial_0^2 - \Gamma_{00}^0 \partial_0$. A motivating case occurs when $N = 2$. Compact 3-manifolds with sol geometry are mapping tori of automorphisms $\mathcal{T}^2 \rightarrow \mathcal{T}^2$ induced by $\Lambda \in \text{SL}(2, \mathbb{Z})$ with eigenvalues $\lambda^{-1} < 1 < \lambda$. Now let \mathbf{g} be a locally homogeneous sol-Gowdy metric of the kind considered as initial data in [16] and [17], so that $A = 0$. Without loss of generality, one may parameterize the initial data by arc length (thereby setting $u = 1$ at $t = 0$) and thereby write $\mathbf{g} = dx^0 \otimes dx^0 + G_{ij} \, dx^i \otimes dx^j$, where one identifies G with the matrix

$$\begin{pmatrix} e^{F+W} & 0 \\ 0 & e^{F-W} \end{pmatrix}.$$

Here F is an arbitrary constant and $\partial_0 W = 2 \log \lambda$ is a topological constant determined by the holonomy Λ around $\mathcal{S}^1 = [0, 1] / \sim$. It is easy to check that this \mathbf{g} becomes a stationary solution of (1.4) if and only if $s = (2 \log \lambda)^2 > 0$ and $c = 0$. For the corresponding unscaled solution $\bar{\mathbf{g}} = (\bar{u}(t), \bar{A}(t), \bar{G}(t))$, one has $\bar{u} = 1 + st$, with $\bar{A} = 0$ and \bar{G} fixed in time.

Our purpose in this note is to investigate the stability of certain natural fixed points of system (1.3) like those in Examples (1)–(3). More precisely, we establish convergence and asymptotic stability of rescaled Ricci flow, modulo smooth finite-dimensional center manifolds, for all initial data sufficiently near certain model

³Given any smooth function $f : \mathcal{B} \rightarrow \mathbb{R}$, we define $\oint_{\mathcal{B}} f \, d\mu = \frac{\int_{\mathcal{B}} f \, d\mu}{\int_{\mathcal{B}} d\mu}$.

solutions in the cases $N = 1$ or $n = 1$. These are the cases most amenable to study by linearization and also most relevant to applications when the total dimension is $N + n = 3$, e.g. to [20]. Because stability of flat metrics (modulo smooth finite-dimensional center manifolds) was established in [13], we assume in what follows that the model solutions in question are not flat.

A standard method for establishing stability of a nonlinear flow $\tilde{\mathbf{g}}_\tau = \mathbf{F}(\tilde{\mathbf{g}})$ near a stationary solution \mathbf{g} is to proceed in two steps. (1) Compute the linearized operator $\mathbf{L}_\mathbf{g}^\mathbb{C}$ and establish that it is sectorial with stable spectrum.⁴ (2) Deduce stability of the nonlinear flow in appropriate function spaces from properties of its linearization. For Ricci flow, this method is often not as easy as one might expect. There are several reasons for this. To apply existing theory such as [6], one requires the linearization to be elliptic. But because Ricci flow is invariant under the full infinite-dimensional diffeomorphism group, its linearization $\mathbf{L}_\mathbf{g}^\mathbb{C}$ is not elliptic unless one fixes a gauge, typically by the introduction of DeTurck diffeomorphisms. Even then, $\mathbf{L}_\mathbf{g}^\mathbb{C}$ may fail to be self adjoint. (For example, see [14] and Section 5 below.) Furthermore, in many cases of interest, such as those considered in [13] and in this paper, the spectrum of $\mathbf{L}_\mathbf{g}^\mathbb{C}$ has nonempty intersection with the imaginary axis. In general, this allows families of stationary or slowly changing solutions to compose local C^r center manifolds, which can shrink as r increases.⁵ Finally, the DeTurck diffeomorphisms can introduce instabilities that may not be present in the original equation. (For example, the DeTurck diffeomorphisms solve a harmonic map flow. But the identity map $\text{id} : \mathcal{S}^n \rightarrow \mathcal{S}^n$ of the round sphere is unstable under harmonic map flow for all $n \geq 3$. See Remark 2 below.) In spite of these obstacles, stability results are obtained in [13] and (by a somewhat different method) by Šešum in [23].

In this paper, we prove the following results that imply convergence in little-Hölder spaces to be defined in Section 2.1, below:

Theorem 1. *Let $\mathbf{g} = (g, A, u)$ be a locally \mathbb{R}^1 -invariant metric of the form (3.1) on a product $\mathbb{R}^1 \times \mathcal{B}$, where \mathcal{B} is compact and orientable. Suppose that g has constant sectional curvature $-1/2(n-1)$, A vanishes, and u is constant. Then for any $\rho \in (0, 1)$, there exists $\theta \in (\rho, 1)$ such that the following holds.*

There exists a $(1 + \theta)$ little-Hölder neighborhood \mathcal{U} of \mathbf{g} such that for all initial data $\tilde{\mathbf{g}}(0) \in \mathcal{U}$, the unique solution $\tilde{\mathbf{g}}(\tau)$ of \varkappa -rescaled locally \mathbb{R}^1 -invariant Ricci flow (3.3) exists for all $\tau \geq 0$ and converges exponentially fast in the $(2 + \rho)$ -Hölder norm to a limit metric $\mathbf{g}_\infty = (g_\infty, A_\infty, u_\infty)$ such that g_∞ is hyperbolic, A_∞ vanishes, and u_∞ is constant.

Theorem 2. *Let $\mathbf{g} = (g, A, u)$ be a locally \mathbb{R}^1 -invariant metric of the form (4.1) on a product $\mathbb{R}^1 \times \mathcal{S}^2$. Suppose that g has constant positive sectional curvature, A vanishes, and u is constant. Then for any $\rho \in (0, 1)$, there exists $\theta \in (\rho, 1)$ such that the following holds.*

There exists a $(1 + \theta)$ little-Hölder neighborhood \mathcal{U} of \mathbf{g} such that for all initial data $\tilde{\mathbf{g}}(0) \in \mathcal{U}$, the unique solution $\tilde{\mathbf{g}}(\tau)$ of volume-rescaled locally \mathbb{R}^1 -invariant Ricci flow (4.2) exists for all $\tau \geq 0$ and converges exponentially fast in the $(2 + \rho)$ -Hölder norm to a limit metric $\mathbf{g}_\infty = (g_\infty, A_\infty, u_\infty)$ such that g_∞ has constant positive sectional curvature, A_∞ vanishes, and u_∞ is constant.

⁴See Definition 1 on page 10.

⁵For example, the “inner layer” asymptotics of a Ricci flow neckpinch correspond to non-stationary solutions in the kernel of its linearization at the cylinder soliton. See [1].

Theorem 3. *Let $\mathbf{g} = (u, A, G)$ be a metric of the form (5.1) on the mapping torus $\mathbb{R}^N \hookrightarrow \mathcal{M}_\Lambda \xrightarrow{\pi} \mathcal{S}^1$ of a given nonzero $\Lambda \in \mathrm{Gl}(N, \mathbb{R})$ admitting a flat connection. Suppose that $u = 1$, A vanishes, and G is a harmonic-Einstein metric (5.5). Then for any $\rho \in (0, 1)$, there exists $\theta \in (\rho, 1)$ such that the following holds.*

There exists a $(1 + \theta)$ little-Hölder neighborhood \mathcal{U} of \mathbf{g} such that for all initial data $\tilde{\mathbf{g}}(0) \in \mathcal{U}$, the unique solution $\tilde{\mathbf{g}}(\tau)$ of holonomy-rescaled locally \mathbb{R}^N -invariant Ricci flow (5.2) exists for all $\tau \geq 0$ and converges exponentially fast in the $(2 + \rho)$ -Hölder norm to a limit metric $\mathbf{g}_\infty = (u_\infty, A_\infty, G_\infty)$ such that $u_\infty = 1$, A_∞ vanishes, and G_∞ is harmonic-Einstein.

This paper is organized as follows. In Section 2, we review a general theory of asymptotic stability for quasilinear PDE in the presence of center manifolds, then establish the context in which that theory applies here. In Sections 3, 4, and 5, we study and prove stability of \varkappa -rescaled flows, volume-rescaled flows, and holonomy-rescaled flows, respectively. Each of these is a suitably-chosen variant of (1.3). In Remarks 3 and 4, below, we provide (partial) explanations of these apparently *ad hoc* choices of normalization.

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2. CENTER MANIFOLD STABILITY THEORY FOR QUASILINEAR SYSTEMS

In this section, we recall relevant aspects of the theory that allows one to derive rigorous conclusions about the asymptotic behavior of the nonlinear system (1.3) from its linearization. Good sources are [6] and [25]. Our approach here mainly follows the latter. The reader familiar with [13] can safely skim this section.

2.1. Maximal regularity spaces. Because optimal asymptotic stability results are obtained in continuous interpolation spaces, we begin by recalling aspects of the *maximal regularity* theory of Da Prato and Grisvard [5]. (Section 2.2 below provides some motivation for our use of this theory.)

Let \mathcal{Y} be a compact Riemannian manifold, possibly with boundary $\partial\mathcal{Y}$. The main cases we have in mind in this note are: (1) \mathcal{Y} is a compact hyperbolic manifold; (2) \mathcal{Y} is the round sphere; or (3) $\mathcal{Y} = [0, 1]$.

Let $\Sigma_0(\mathcal{Y})$, $\Sigma_1(\mathcal{Y})$, and $\Sigma_2(\mathcal{Y})$ respectively denote the spaces of C^∞ functions, C^∞ 1-forms, and C^∞ symmetric $(2, 0)$ -tensor fields supported on \mathcal{Y} . If $\partial\mathcal{Y} \neq \emptyset$, as in case (3), we restrict to those functions, forms, and tensor fields satisfying prescribed linear boundary conditions like (5.8)–(5.10), respectively.

Given $r \in \mathbb{N}$ and $\rho \in (0, 1)$, let $\Sigma_0^{r+\rho}(\mathcal{Y})$, $\Sigma_1^{r+\rho}(\mathcal{Y})$, and $\Sigma_2^{r+\rho}(\mathcal{Y})$ denote the closures of $\Sigma_0(\mathcal{Y})$, $\Sigma_1(\mathcal{Y})$, and $\Sigma_2(\mathcal{Y})$, respectively, with respect to the relevant $r + \rho$ Hölder norms. These are the well-known *little-Hölder spaces*. (Recall that C^∞ representatives are not dense in the usual Hölder spaces.) Any little-Hölder

space $\mathfrak{h}^{r+\rho}$ is a Banach space. Moreover, if $s \leq r$ and $\sigma \leq \rho$, then there is a continuous and dense inclusion $\mathfrak{h}^{r+\rho} \hookrightarrow \mathfrak{h}^{s+\sigma}$.

Given a continuous dense inclusion $B_1 \hookrightarrow B_0$ of Banach spaces and $\theta \in (0, 1)$, the *continuous interpolation space* $(B_0, B_1)_\theta$ is the set of all $x \in B_0$ such that there exist sequences $\{y_n\} \subset B_0$ and $\{z_n\} \subset B_1$ satisfying

$$(2.1) \quad \begin{cases} x = y_n + z_n \\ \|y_n\|_{B_0} = o(2^{-n\theta}) \text{ as } n \rightarrow \infty \\ \|z_n\|_{B_1} = o(2^{n(1-\theta)}) \text{ as } n \rightarrow \infty. \end{cases}$$

Continuous interpolation spaces are introduced in [5]. By [9], they are equivalent in norm to the real interpolation spaces also found in the literature. In fact, the norm $\|x\|_\theta$ on $(B_0, B_1)_\theta$ is equivalent to

$$\inf \left\{ \sup_{n \in \mathbb{N}} \{2^{n\theta} \|y_n\|_{B_0}, 2^{-n(1-\theta)} \|z_n\|_{B_1}\} : \{y_n\} \text{ and } \{z_n\} \text{ satisfy (2.1)} \right\}.$$

The little-Hölder spaces are particularly well adapted to continuous interpolation. (See [27].) Indeed, let $\mathfrak{h}^{r+\rho}$ and $\mathfrak{h}^{s+\sigma}$ be little-Hölder spaces. Let $s \leq r \in \mathbb{N}$, $0 < \sigma < \rho < 1$, and $0 < \theta < 1$. If $\theta(r + \rho) + (1 - \theta)(s + \sigma) \notin \mathbb{N}$, then there is a Banach space isomorphism

$$(2.2) \quad (\mathfrak{h}^{s+\sigma}, \mathfrak{h}^{r+\rho})_\theta \cong \mathfrak{h}^{[\theta r + (1-\theta)s] + [\theta \rho + (1-\theta)\sigma]}$$

and there exists $C < \infty$ such that for all $\eta \in \mathfrak{h}^{r+\rho}$, one has

$$(2.3) \quad \|\eta\|_{(\mathfrak{h}^{s+\sigma}, \mathfrak{h}^{r+\rho})_\theta} \leq C \|\eta\|_{\mathfrak{h}^{s+\sigma}}^{1-\theta} \|\eta\|_{\mathfrak{h}^{r+\rho}}^\theta.$$

These properties make the little-Hölder spaces highly useful for our purposes.

2.2. A general center manifold theorem. To prove the results in this note, we invoke a special case of a theorem of Simonett [25], whose hypotheses we now recall. (Compare [13].) Our notation is as follows. If \mathbb{X} and \mathbb{Y} are Banach spaces, then $\mathcal{L}(\mathbb{X}, \mathbb{Y})$ is the set of bounded linear maps $\mathbb{X} \rightarrow \mathbb{Y}$. If (\mathbb{X}, d) is a metric space, then $B(\mathbb{X}, x, r)$ is the open ball of radius $r > 0$ centered at $x \in \mathbb{X}$. If \mathbf{L} is a linear operator on a real space, we denote its natural complexification by $\mathbf{L}^{\mathbb{C}}(u + iv) = \mathbf{L}u + i\mathbf{L}v$.

If $\mathbb{X} \hookrightarrow \mathbb{Y}$ is a continuous and dense inclusion of Banach spaces and $\mathbf{Q} : \mathbb{X} \rightarrow \mathbb{Y}$ is a nonlinear differential operator, $\mathbb{X} \ni \mathbf{g} \mapsto \mathbf{Q}(\mathbf{g}) \in \mathbb{Y}$, we denote its linearization at $\hat{\mathbf{g}} \in \mathbb{X}$ by $\mathbf{L}_{\hat{\mathbf{g}}} = \mathbf{Q}'(\hat{\mathbf{g}}) : \mathbb{D}(\mathbf{L}_{\hat{\mathbf{g}}}) \subseteq \mathbb{Y} \rightarrow \mathbb{Y}$. Our main assumptions below are that $\mathbf{g} \mapsto \mathbf{Q}(\mathbf{g})$ is a quasilinear differential operator and that $\mathbf{L}_{\hat{\mathbf{g}}} : \mathbf{g} \mapsto \mathbf{L}_{\hat{\mathbf{g}}}(\mathbf{g})$ generates an analytic strongly-continuous semigroup on $\mathcal{L}(\mathbb{Y}, \mathbb{Y})$.

To apply Simonett's stability theorem, one must verify certain technical hypotheses with respect to a given quasilinear parabolic equation and ordered Banach spaces satisfying

$$(2.4) \quad \mathbb{X}_1 \subset \mathbb{E}_1 \subset \mathbb{X}_0 \subset \mathbb{E}_0 \quad \text{and} \quad \mathbb{X}_1 \subset \mathbb{X}_\alpha \subset \mathbb{X}_\beta \subset \mathbb{X}_0,$$

whose precise relationships we describe below. The reasons for introducing this curious arrangement of Banach spaces come from the beautiful *maximal regularity* construction of Da Prato and Grisvard [5]. As motivation, consider a linear initial value problem

$$(*) \quad \begin{cases} \dot{u}(t) = Lu(t) + f(t), \\ u(0) = u_0, \end{cases}$$

posed on a Banach space \mathbb{Y}_0 , where the linear operator L generates a strongly-continuous analytic semigroup on $\mathcal{L}(\mathbb{Y}_0, \mathbb{Y}_0)$. Then L is a densely-defined closed operator whose domain, when equipped with the graph norm,⁶ naturally becomes a Banach space, which we may call \mathbb{Y}_1 . Thus $\mathbb{Y}_1 \hookrightarrow \mathbb{Y}_0$ is a continuous and dense inclusion. Now (for some fixed $T > 0$) suppose that $f : [0, T] \rightarrow \mathbb{Y}_0$ is a bounded continuous function. Then the *formal* solution of (*) is given by the integral formula

$$u(t) = e^{tL}u_0 + \int_0^t e^{(t-s)L}f(s) ds, \quad 0 < t \leq T.$$

However, the convolution term above may not have enough regularity to make this formal solution rigorous unless one imposes stronger hypotheses (like requiring, for example, that $f : [0, T] \rightarrow \mathbb{Y}_1$ be bounded and continuous). Da Prato and Grisvard's theory overcomes this difficulty by restriction to suitably chosen interpolation spaces. In these (necessarily non-reflexive) spaces, \dot{u} and Lu enjoy the same regularity as f , which justifies the language "maximal regularity".

Stability theory for nonlinear autonomous initial value problems

$$(**) \begin{cases} \dot{u}(t) = N(u(t)), \\ u(0) = u_0, \end{cases}$$

near a stationary solution $N(0) = 0$ often proceeds by linearization, in which one replaces (**) by (*) with $L = N'(0)$ and $f(t) = N(u(t)) - Lu(t)$. Here the regularity of f is of evident importance. This is where one exploits the (2.4) hierarchy. Suppose that (the complexification of) $L : \mathbb{E}_1 \rightarrow \mathbb{E}_0$ is a sectorial operator.⁷ The arguments Simonett uses to prove Theorem 4 below require the linear problem (*) to have a bounded continuous solution $u : [0, T] \rightarrow \mathbb{X}_1$ for every bounded continuous $f : [0, T] \rightarrow \mathbb{X}_0$. This will fail for general sectorial $\mathbb{E}_1 \rightarrow \mathbb{E}_0$ but can be achieved by restricting $L : \mathbb{X}_1 \rightarrow \mathbb{X}_0$ to well-chosen continuous interpolation spaces $\mathbb{X}_1 \hookrightarrow \mathbb{X}_0$.

A similar stability theorem is derived by Da Prato and Lunardi [6]. We use Simonett's theorem because it takes optimal advantage of the parabolic smoothing properties of the quasilinear equation (2.5) in continuous interpolation spaces [25]. With regard to (2.4), these properties allow one to show that invariant manifolds can be exponentially attractive in the norm of the smaller space \mathbb{X}_1 for solutions whose initial data are close to a fixed point in the larger interpolation space \mathbb{X}_α . In particular, such solutions immediately regularize and belong to \mathbb{X}_1 for all $t > 0$.

With this brief introduction in hand, we are ready to state the seven hypotheses one needs to apply the theorem. In our applications, we choose certain little-Hölder spaces whose properties, e.g. (2.2), greatly simplify verification of the hypotheses. While perusing those hypotheses, readers may wish to consult (2.6) and (2.7) below for specifics of how the various spaces are realized in the remainder of this paper.

Assumption 1. $\mathbb{X}_1 \hookrightarrow \mathbb{X}_0$ and $\mathbb{E}_1 \hookrightarrow \mathbb{E}_0$ are continuous dense inclusions of Banach spaces. For fixed $0 < \beta < \alpha < 1$, \mathbb{X}_α and \mathbb{X}_β are continuous interpolation spaces corresponding to the inclusion $\mathbb{X}_1 \hookrightarrow \mathbb{X}_0$.

Assumption 2. There is an autonomous quasilinear parabolic equation

$$(2.5) \quad \frac{\partial}{\partial \tau} \tilde{\mathbf{g}}(\tau) = \mathbf{Q}(\tilde{\mathbf{g}}(\tau)), \quad (\tau \geq 0),$$

⁶The graph norm with respect to \mathbb{Y}_0 of a suitable linear operator $L : D(L) \subseteq \mathbb{Y}_0 \rightarrow \mathbb{Y}_0$ is $\|x\|_{D(L)} = \|x\|_{\mathbb{Y}_0} + \|Lx\|_{\mathbb{Y}_0}$.

⁷See Definition 1 on page 10.

with the property that there exists a positive integer k such that for all $\widehat{\mathbf{g}}$ in some open set $\mathbb{G}_\beta \subseteq \mathbb{X}_\beta$, the domain $\mathbb{D}(\mathbf{L}_{\widehat{\mathbf{g}}})$ of $\mathbf{L}_{\widehat{\mathbf{g}}}$ contains \mathbb{X}_1 and the map $\widehat{\mathbf{g}} \mapsto \mathbf{L}_{\widehat{\mathbf{g}}}|_{\mathbb{X}_1}$ belongs to $C^k(\mathbb{G}_\beta, \mathcal{L}(\mathbb{X}_1, \mathbb{X}_0))$.

Assumption 3. For each $\widehat{\mathbf{g}} \in \mathbb{G}_\beta$, there exists an extension $\widehat{\mathbf{L}}_{\widehat{\mathbf{g}}}$ of $\mathbf{L}_{\widehat{\mathbf{g}}}$ to a domain $\widehat{\mathbb{D}}(\widehat{\mathbf{g}})$ that contains \mathbb{E}_1 (hence is dense in \mathbb{E}_0).

Assumption 4. For each $\widehat{\mathbf{g}} \in \mathbb{G}_\alpha = \mathbb{G}_\beta \cap \mathbb{X}_\alpha$, $\widehat{\mathbf{L}}_{\widehat{\mathbf{g}}}|_{\mathbb{E}_1} \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_0)$ generates a strongly-continuous analytic semigroup on $\mathcal{L}(\mathbb{E}_0, \mathbb{E}_0)$. (Observe that for $\widehat{\mathbf{g}} \in \mathbb{G}_\alpha$, this implies that $\widehat{\mathbb{D}}(\widehat{\mathbf{g}})$ becomes a Banach space when equipped with the graph norm with respect to \mathbb{E}_0 .)

Assumption 5. For each $\widehat{\mathbf{g}} \in \mathbb{G}_\alpha$, $\mathbf{L}_{\widehat{\mathbf{g}}}$ is the part of $\widehat{\mathbf{L}}_{\widehat{\mathbf{g}}}$ in \mathbb{X}_0 .⁸

Assumption 6. For each $\widehat{\mathbf{g}} \in \mathbb{G}_\alpha$, there exists $\theta \in (0, 1)$ such that $\mathbb{X}_0 \cong (\mathbb{E}_0, \widehat{\mathbb{D}}(\widehat{\mathbf{g}}))_\theta$ and $\mathbb{X}_1 \cong (\mathbb{E}_0, \widehat{\mathbb{D}}(\widehat{\mathbf{g}}))_{1+\theta}$, where $(\mathbb{E}_0, \widehat{\mathbb{D}}(\widehat{\mathbf{g}}))_{1+\theta} = \{\mathbf{g} \in \widehat{\mathbb{D}}(\widehat{\mathbf{g}}) : \widehat{\mathbf{L}}_{\widehat{\mathbf{g}}}(\mathbf{g}) \in (\mathbb{E}_0, \widehat{\mathbb{D}}(\widehat{\mathbf{g}}))_\theta\}$ as a set, endowed with the graph norm of $\widehat{\mathbf{L}}_{\widehat{\mathbf{g}}}$ with respect to $(\mathbb{E}_0, \widehat{\mathbb{D}}(\widehat{\mathbf{g}}))_\theta$.⁹

Assumption 7. $\mathbb{E}_1 \hookrightarrow \mathbb{X}_\beta \hookrightarrow \mathbb{E}_0$ is a continuous and dense inclusion such that there exist $C > 0$ and $\delta \in (0, 1)$ such that for all $\eta \in \mathbb{E}_1$, one has

$$\|\eta\|_{\mathbb{X}_\beta} \leq C \|\eta\|_{\mathbb{E}_0}^{1-\delta} \|\eta\|_{\mathbb{E}_1}^\delta.$$

Simonett obtains (a more general version of) the following result:

Theorem 4 (Simonett). Let $\mathbf{L}_{\mathbf{g}}^{\mathbb{C}}$ denote the complexification of the linearization $\mathbf{L}_{\mathbf{g}}$ of (2.5) at a stationary solution \mathbf{g} of (2.5).¹⁰ Suppose there exists $\lambda_s > 0$ such that the spectrum σ of $\mathbf{L}_{\mathbf{g}}^{\mathbb{C}}$ admit the decomposition $\sigma = \sigma_s \cup \{0\}$, where 0 is an eigenvalue of finite multiplicity and $\sigma_s \subseteq \{z : \operatorname{Re} z \leq -\lambda_s\}$. If Assumptions 1–7 hold, then:

- (1) For each $\alpha \in [0, 1]$, there is a direct-sum decomposition $\mathbb{X}_\alpha = \mathbb{X}_\alpha^s \oplus \mathbb{X}_\alpha^c$, where \mathbb{X}_α^c is the finite-dimensional algebraic eigenspace corresponding to the null eigenvalue of $\mathbf{L}_{\mathbf{g}}^{\mathbb{C}}$.
- (2) For each $r \in \mathbb{N}$, there exists $d_r > 0$ such that for all $d \in (0, d_r]$, there exists a bounded C^r map $\gamma_d^r : B(\mathbb{X}_1^c, \mathbf{g}, d) \rightarrow \mathbb{X}_1^s$ such that $\gamma_d^r(\mathbf{g}) = 0$ and $D\gamma_d^r(\mathbf{g}) = 0$. The image of γ_d^r lies in the closed ball $\bar{B}(\mathbb{X}_1^s, \mathbf{g}, d)$. Its graph is a local C^r center manifold $\Gamma_{\text{loc}}^r = \{(h, \gamma_d^r(h)) : h \in B(\mathbb{X}_1^c, \mathbf{g}, d)\} \subset \mathbb{X}_1$ satisfying $T_{\mathbf{g}}\Gamma_{\text{loc}}^r \cong \mathbb{X}_1^c$. Moreover, Γ_{loc}^r is invariant for solutions of (2.5) as long as they remain in $B(\mathbb{X}_1^c, \mathbf{g}, d) \times B(\mathbb{X}_1^s, 0, d)$.
- (3) Fix $\lambda \in (0, \lambda_s)$. Then for each $\alpha \in (0, 1)$, there exist $C > 0$ and $d \in (0, d_r]$ such that for each initial datum $\widetilde{\mathbf{g}}(0) \in B(\mathbb{X}_\alpha, \mathbf{g}, d)$ and all times $\tau \geq 0$ such that $\widetilde{\mathbf{g}}(\tau) \in B(\mathbb{X}_\alpha, \mathbf{g}, d)$, the center manifold Γ_{loc}^r is exponentially attractive in the stronger space \mathbb{X}_1 in the sense that

$$\|\pi^s \widetilde{\mathbf{g}}(\tau) - \gamma_d^r(\pi^c \widetilde{\mathbf{g}}(\tau))\|_{\mathbb{X}_1} \leq \frac{C_\alpha}{\tau^{1-\alpha}} e^{-\lambda\tau} \|\pi^s \widetilde{\mathbf{g}}(0) - \gamma_d^r(\pi^c \widetilde{\mathbf{g}}(0))\|_{\mathbb{X}_\alpha}.$$

⁸If \mathbb{X} is a Banach space with subspace \mathbb{Y} and $L : D(L) \subseteq \mathbb{X} \rightarrow \mathbb{X}$ is linear, then $L^{\mathbb{Y}}$, the part of L in \mathbb{Y} , is defined by the action $L^{\mathbb{Y}} : x \mapsto Lx$ on the domain $D(L^{\mathbb{Y}}) = \{x \in D(L) : Lx \in \mathbb{Y}\}$.

⁹To see that this assumption is equivalent to condition (II) in [25, Section 4], recall the fact (see e.g. [21, Propositions 2.2.2 and 2.2.4]) that if A is the generator of a C_0 analytic semigroup in a Banach space \mathbb{X} and $0 < \theta < 1$, then $(X, \mathbb{D}(A))_\theta \cong \mathbb{D}_A(\theta) = \{x \in \mathbb{X} : \lim_{t \searrow 0} t^{-\theta} (e^{tA}x - x) = 0\}$.

¹⁰Note that $\mathbf{L}_{\mathbf{g}}$ is the operator that appears in Assumption 2.

Here, $\tilde{\mathbf{g}}(\tau)$ is the unique solution of (2.5), while π^s and π^c denote the projections onto $\mathbb{X}_\alpha^s \cong (\mathbb{X}_1^s, \mathbb{X}_0^s)_\alpha$ and \mathbb{X}_α^c , respectively.

2.3. Prerequisites for application of the theorem. We now establish the context in which we apply Theorem 4.

Let $\tilde{\mathbf{g}}_\tau = \mathbf{Q}(\tilde{\mathbf{g}})$ denote the rescaled \mathbb{R}^N -invariant Ricci flow system (1.3), modified by DeTurck diffeomorphisms with respect to a background metric $\underline{\mathbf{g}}$ chosen as in Sections 3–5 below. We assume that \mathbf{g} is a smooth stationary solution, namely that $\mathbf{g}_\tau = \mathbf{Q}(\mathbf{g}) = 0$.

For fixed $0 < \sigma < \rho < 1$, consider the following nested spaces:

$$(2.6) \quad \begin{aligned} \mathbb{E}_0 &= \Sigma_0^{0+\sigma}(\mathcal{Y}) \times \Sigma_1^{0+\sigma}(\mathcal{Y}) \times \Sigma_2^{0+\sigma}(\mathcal{Y}) \\ &\cup \\ \mathbb{X}_0 &= \Sigma_0^{0+\rho}(\mathcal{Y}) \times \Sigma_1^{0+\rho}(\mathcal{Y}) \times \Sigma_2^{0+\rho}(\mathcal{Y}) \\ &\cup \\ \mathbb{E}_1 &= \Sigma_0^{2+\sigma}(\mathcal{Y}) \times \Sigma_1^{2+\sigma}(\mathcal{Y}) \times \Sigma_2^{2+\sigma}(\mathcal{Y}) \\ &\cup \\ \mathbb{X}_1 &= \Sigma_0^{2+\rho}(\mathcal{Y}) \times \Sigma_1^{2+\rho}(\mathcal{Y}) \times \Sigma_2^{2+\rho}(\mathcal{Y}) \end{aligned}$$

For fixed $1/2 \leq \beta < \alpha < 1$ and $\varepsilon > 0$ to be chosen, define

$$(2.7) \quad \mathbb{X}_\beta = (\mathbb{X}_0, \mathbb{X}_1)_\beta \quad \text{and} \quad \mathbb{X}_\alpha = (\mathbb{X}_0, \mathbb{X}_1)_\alpha$$

and

$$(2.8) \quad \mathbb{G}_\beta = B(\mathbb{X}_\beta, \mathbf{g}, \varepsilon) \quad \text{and} \quad \mathbb{G}_\alpha = \mathbb{G}_\beta \cap \mathbb{X}_\alpha.$$

For each $\hat{\mathbf{g}} \in \mathbb{G}_\beta$, let $\widehat{\mathbf{L}}_{\hat{\mathbf{g}}}$ denote the linearization of \mathbf{Q} at $\hat{\mathbf{g}}$, regarded as an unbounded linear operator on \mathbb{E}_0 with dense domain $\widehat{\mathbb{D}}(\hat{\mathbf{g}}) = \mathbb{E}_1$. Let $\mathbf{L}_{\hat{\mathbf{g}}}$ denote the corresponding operator on \mathbb{X}_0 with dense domain $\mathbb{D}(\mathbf{L}_{\hat{\mathbf{g}}}) = \mathbb{X}_1$.

Recall the following:

Definition 1. A densely-defined linear operator \mathbf{L} on a complex Banach space $\mathbb{X} \neq \{0\}$ is said to be *sectorial* if there exist $\alpha \in (\pi, 2\pi)$, $\omega \in \mathbb{R}$, and $C > 0$ such that the “sector”

$$(2.9) \quad S_{\alpha, \omega} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \alpha\}$$

is contained in the resolvent set $\rho(\mathbf{L})$, and such that

$$(2.10) \quad \|(\lambda \mathbf{I} - \mathbf{L})^{-1}\|_{\mathcal{L}(\mathbb{X}, \mathbb{X})} \leq \frac{C}{|\lambda - \omega|}$$

holds for all $\lambda \in S_{\alpha, \omega} \subseteq \rho(\mathbf{L})$.

The following lemmas verify the technical hypotheses needed for Theorem 4.

Lemma 1.

- (1) $\hat{\mathbf{g}} \mapsto \mathbf{L}_{\hat{\mathbf{g}}}$ is an analytic map $\mathbb{G}_\beta \rightarrow \mathcal{L}(\mathbb{X}_1, \mathbb{X}_0)$.
- (2) $\hat{\mathbf{g}} \mapsto \widehat{\mathbf{L}}_{\hat{\mathbf{g}}}$ is an analytic map $\mathbb{G}_\alpha \rightarrow \mathcal{L}(\mathbb{E}_1, \mathbb{E}_0)$.
- (3) If $\widehat{\mathbf{L}}_{\hat{\mathbf{g}}}^{\mathbb{C}}$ is sectorial, then there exists $\varepsilon > 0$ such that for all $\hat{\mathbf{g}}$ in the set \mathbb{G}_α defined by (2.8), $\widehat{\mathbf{L}}_{\hat{\mathbf{g}}}^{\mathbb{C}}$ is the infinitesimal generator of an analytic C_0 -semigroup on $\mathcal{L}(\mathbb{E}_0, \mathbb{E}_0)$.

Proof. Statements (1)–(2) are proved in Lemma 3.3 of [13].

To prove statement (3), first observe that $\widehat{\mathbf{L}}_{\mathbf{g}}^{\mathbb{C}}$ generates an analytic C_0 -semigroup on $\mathcal{L}(\mathbb{E}_0, \mathbb{E}_0)$: it is a standard fact that a sectorial operator generates an analytic semigroup; one knows that semigroup is strongly continuous because $\widehat{\mathbf{L}}_{\mathbf{g}}^{\mathbb{C}}$ is densely defined by construction. Now if $\widehat{\mathbf{g}} \in \mathbb{G}_\alpha$, then by statement (2), we can choose $\varepsilon > 0$ small enough so that

$$\left\| \widehat{\mathbf{L}}_{\widehat{\mathbf{g}}}^{\mathbb{C}} - \widehat{\mathbf{L}}_{\mathbf{g}}^{\mathbb{C}} \right\|_{\mathcal{L}(\mathbb{E}_1, \mathbb{E}_0)} < \frac{1}{C+1},$$

where $C > 0$ is the constant in (2.10) corresponding to $\widehat{\mathbf{L}}_{\mathbf{g}}^{\mathbb{C}}$. As is well known, this implies that $\widehat{\mathbf{L}}_{\widehat{\mathbf{g}}}^{\mathbb{C}}$ is sectorial as well. (See [21, Proposition 2.4.2], for example.) Statement (3) follows easily. \square

Lemma 2. *If $\widehat{\mathbf{L}}_{\mathbf{g}}^{\mathbb{C}}$ is sectorial, then the choices made in (2.6)–(2.8) ensure that Assumptions 1–7 are satisfied for the system $\widetilde{\mathbf{g}}_\tau = \mathbf{Q}(\widetilde{\mathbf{g}})$ given in (1.3).*

Proof. Assumption 1 holds by construction. Assumption 2 follows directly from statement (2) of Lemma 1, because the “off-diagonal terms” in (1.3) are all contractions involving at most one derivative of $\widetilde{\mathbf{g}}$. Assumptions 3 and 5 hold by construction of $\widehat{\mathbf{L}}_{\widetilde{\mathbf{g}}}$ and $\mathbf{L}_{\widetilde{\mathbf{g}}}$. Assumption 4 is a consequence of the hypothesis on $\widehat{\mathbf{L}}_{\mathbf{g}}^{\mathbb{C}}$ and statement (3) of Lemma 1. To verify Assumption 6, first observe that putting $\theta = (\rho - \sigma)/2 \in (0, 1)$ into the isomorphism identity (2.2) yields $\mathbb{X}_0 \cong (\mathbb{E}_0, \mathbb{E}_1)_\theta$; then recall that standard Schauder theory implies that the graph norm of $\widehat{\mathbf{L}}_{\widetilde{\mathbf{g}}}$ with respect to \mathbb{E}_0 is equivalent to the \mathbb{E}_1 norm. (So the graph norm of $\mathbf{L}_{\widetilde{\mathbf{g}}}$ with respect to \mathbb{X}_0 is equivalent to the \mathbb{X}_1 norm.) Finally, Assumption 7 is implied by (2.3). \square

3. THE CASE $N = 1$: \varkappa -RESCALED LOCALLY \mathbb{R}^1 -INVARIANT RICCI FLOW

To consider evolving Riemannian metrics on a bundle $\mathbb{R}^1 \hookrightarrow \mathcal{M} \xrightarrow{\pi} \mathcal{B}^n$, it is convenient to change notation. Let (x^1, \dots, x^n) be local coordinates on \mathcal{B}^n and let x^0 denote the coordinate on an \mathbb{R}^1 fiber. If $(\mathcal{M}, \widetilde{\mathbf{g}}(t) : t \in \mathcal{I})$ is a locally \mathbb{R}^1 -invariant solution (1.1), then there exist a Riemannian metric \bar{g} , a 1-form \bar{A} , and a function \bar{u} , all defined on $\mathcal{B}^n \times \mathcal{I}$, such that we can write $\widetilde{\mathbf{g}}$ in local coordinates as

$$(3.1) \quad \widetilde{\mathbf{g}} = e^{2\bar{u}} dx^0 \otimes dx^0 + e^{2\bar{u}} \bar{A}_k (dx^0 \otimes dx^k + dx^k \otimes dx^0) + (e^{2\bar{u}} \bar{A}_i \bar{A}_j + \bar{g}_{ij}) dx^i \otimes dx^j.$$

We again abuse notation by writing $\widetilde{\mathbf{g}}(t) = (\bar{g}(t), \bar{A}(t), \bar{u}(t))$. In the coordinates of this section, system (1.2) takes the form

$$(3.2a) \quad \frac{\partial}{\partial t} \bar{g}_{ij} = -2\bar{R}_{ij} + e^{2\bar{u}} \bar{g}^{k\ell} (d\bar{A})_{ik} (d\bar{A})_{j\ell} + 2\bar{\nabla}_i u \bar{\nabla}_j u,$$

$$(3.2b) \quad \frac{\partial}{\partial t} \bar{A}_k = \bar{g}^{ij} \{ \bar{\nabla}_i (d\bar{A})_{jk} + 2(\bar{\nabla}_i u) (d\bar{A})_{jk} \},$$

$$(3.2c) \quad \frac{\partial}{\partial t} \bar{u} = \bar{\Delta} u - \frac{1}{4} e^{2\bar{u}} |d\bar{A}|^2.$$

We are interested in the stability of metrics of this type on trivial (i.e. product) bundles. It turns out that the most practical rescaling depends on the sign of the curvatures of (\mathcal{B}, g) . We explain this in Remarks 3–4 below.

3.1. \varkappa -rescaled flow. Suppose that (\mathcal{M}, \bar{g}) is a Riemannian product over an Einstein manifold with $2\text{Rc}(\bar{g}(-\varkappa)) = \varkappa\bar{g}(-\varkappa)$ for some $\varkappa = \pm 1$. Then the renormalization in Example 1 yields

$$(3.3a) \quad \frac{\partial}{\partial \tau} g_{ij} = -2R_{ij} + e^{2u} g^{k\ell} (dA)_{ik} (dA)_{j\ell} + 2\nabla_i u \nabla_j u + \varkappa g_{ij},$$

$$(3.3b) \quad \frac{\partial}{\partial \tau} A_k = g^{ij} \{ \nabla_i (dA)_{jk} + 2(\nabla_i u)(dA)_{jk} \} + \frac{\varkappa}{2} A_k,$$

$$(3.3c) \quad \frac{\partial}{\partial \tau} u = \Delta u - \frac{1}{4} e^{2u} |dA|^2.$$

We call this system (in which all quantities are computed with respect to g) the \varkappa -rescaled locally \mathbb{R}^1 -invariant Ricci flow. It is most satisfactory for studying the case $\varkappa = -1$. (In Section 4, we apply a different rescaling that is more suitable for the case $\varkappa = +1$.)

3.2. Linearization at a stationary solution of \varkappa -rescaled flow. Any Riemannian product $(\mathbb{R} \times \mathcal{B}, e^{2u} dx^0 \otimes dx^0 + g_{ij} dx^i \otimes dx^j)$ with $2\text{Rc}(g) = \varkappa g$, A identically zero, and u constant in space is clearly a stationary solution of (3.3). Let $(g + h, B, u + v)$ denote a perturbation of such a fixed point $\mathbf{g} = (g, 0, u)$. Define $H = \text{tr}_g h$, and let Δ_ℓ denote the Lichnerowicz Laplacian acting on symmetric $(2, 0)$ -tensor fields. In coordinates,

$$(3.4) \quad \Delta_\ell h_{ij} = \Delta h_{ij} + 2R_{ipqj} h^{pq} - R_i^k h_{kj} - R_j^k h_{ik}.$$

Lemma 3. *The linearization of (3.3) at a fixed point $\mathbf{g} = (g, 0, u)$ with $2\text{Rc} = \varkappa g$ and u constant acts on (h, B, v) by*

$$(3.5a) \quad \frac{\partial}{\partial \tau} h_{ij} = \Delta_\ell h_{ij} + \{ \nabla_i (\delta h)_j + \nabla_j (\delta h)_i + \nabla_i \nabla_j H \} + \varkappa h_{ij},$$

$$(3.5b) \quad \frac{\partial}{\partial \tau} B_k = -(\delta dB)_k + \frac{\varkappa}{2} B_k,$$

$$(3.5c) \quad \frac{\partial}{\partial \tau} v = \Delta v.$$

Proof. For the convenience of the reader, we begin by recalling a few classical variation formulas (all of which may be found in [2, Chapter 1, Section K], for instance). Let $\tilde{g}(\varepsilon)$ be a smooth one-parameter family of Riemannian metrics such that

$$\tilde{g}(0) = g \quad \text{and} \quad \left. \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} \tilde{g} = h.$$

Using tildes to denote geometric quantities associated to \tilde{g} and undecorated characters to denote quantities associated to g , one computes that

$$\begin{aligned} \left. \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} \tilde{g}^{ij} &= -h^{ij} \quad (= -g^{ik} g^{j\ell} h_{k\ell}), \\ \left. \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} \tilde{\Gamma}_{ij}^k &= \frac{1}{2} (\nabla_i h_j^k + \nabla_j h_i^k - \nabla^k h_{ij}), \\ \left. \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} \tilde{R}_{ij} &= -\frac{1}{2} \{ \Delta_\ell h_{ij} + \nabla_i (\delta h)_j + \nabla_j (\delta h)_i + \nabla_i \nabla_j H \}. \end{aligned}$$

The lemma follows by careful but straightforward application of these formulas. \square

The linear system (3.5) is autonomous but not quite parabolic. We impose parabolicity by the DeTurck trick [7, 8]. Fix a background connection $\underline{\Gamma}$ and define a 1-parameter family of vector fields $W(\tau)$ along a solution $\mathbf{g}(\tau)$ of (3.3) by

$$(3.6a) \quad W^0 = \delta A,$$

$$(3.6b) \quad W^k = g^{ij}(\Gamma_{ij}^k - \underline{\Gamma}_{ij}^k) \quad (1 \leq k \leq n).$$

The solution of the \varkappa -rescaled \mathbb{R}^1 -invariant Ricci–DeTurck flow is the 1-parameter family of metrics $\psi_\tau^* \mathbf{g}(\tau)$, where the diffeomorphisms ψ_τ are generated by $W(\tau)$, subject to the initial condition $\psi_0 = \text{id}$. In what follows, we take $\underline{\Gamma}$ to be the Levi-Civita connection of the stationary solution around which we linearize. Observe that a stationary solution $\mathbf{g} = (g, 0, u)$ of (3.3) with $2\text{Rc} = \varkappa g$ and u constant is then also a stationary solution of the \varkappa -rescaled Ricci–DeTurck flow.

Lemma 4. *The linearization of the \varkappa -rescaled Ricci–DeTurck flow at a fixed point $\mathbf{g} = (g, 0, u)$ with $2\text{Rc} = \varkappa g$ is the autonomous, self-adjoint, strictly parabolic system*

$$\frac{\partial}{\partial \tau} \begin{pmatrix} h & B & v \end{pmatrix} = \mathbf{L} \begin{pmatrix} h & B & v \end{pmatrix} = \begin{pmatrix} \mathbf{L}_2 h & \mathbf{L}_1 B & \mathbf{L}_0 v \end{pmatrix},$$

where

$$(3.7a) \quad \mathbf{L}_2 h = \Delta_\ell h + \varkappa h,$$

$$(3.7b) \quad \mathbf{L}_1 B = \Delta_1 B + \frac{\varkappa}{2} B,$$

$$(3.7c) \quad \mathbf{L}_0 v = \Delta_0 v.$$

Here $-\Delta_p = d\delta + \delta d$ denotes the Hodge–de Rham Laplacian acting on p -forms.

Proof. The Christoffel symbols Γ for a fixed point $\mathbf{g} = (g, 0, u)$ vanish if any index is zero and satisfy $\Gamma_{ij}^k = \Gamma_{ij}^k$, otherwise, where Γ_{ij}^k are the Christoffel symbols for g . Hence the DeTurck correction terms to the linearization (3.5) of (3.3) are

$$\begin{aligned} (\mathcal{L}_W \mathbf{g})_{00} &= 0, \\ (\mathcal{L}_W \mathbf{g})_{0k} &= e^{2u} (d\delta B)_k, \\ (\mathcal{L}_W \mathbf{g})_{ij} &= \nabla_i (\delta h)_j + \nabla_j (\delta h)_i + \nabla_i \nabla_j H, \end{aligned}$$

whence the result is immediate from (3.1) and (3.5). \square

3.3. Linear stability of \varkappa -rescaled flow. Now let us make the stronger assumption that $\mathbf{g} = (g, 0, u)$ is a fixed point of the \varkappa -rescaled \mathbb{R}^1 -invariant Ricci–DeTurck flow with u constant and g a metric of constant sectional curvature $-1/2(n-1)$.

We say a linear operator L is *weakly (strictly) stable* if its spectrum is confined to the half plane $\text{Re } z \leq 0$ (and is uniformly bounded away from the imaginary axis).

Because \mathbf{L} is diagonal, we can determine its stability by examining its component operators. The conclusions we obtain here will hold below when we extend \mathbf{L} to a complex-valued operator on a larger domain in which smooth representatives are dense.

Lemma 5. *Let $\mathbf{g} = (g, 0, u)$ be a metric of the form (3.1) such that u is constant and g has constant sectional curvature $-1/2(n-1)$. Then the linear system (3.7) has the following stability properties:*

The operator \mathbf{L}_0 is weakly stable; constant functions form its null eigenspace.

The operator \mathbf{L}_1 is strictly stable.

If $n \geq 3$, then \mathbf{L}_2 is strictly stable.

If $n = 2$, then \mathbf{L}_2 is weakly stable. On an orientable surface \mathcal{B} of genus $\gamma \geq 2$, the null eigenspace of $\Delta_\ell - 1$ is the $(6\gamma - 6)$ -dimensional space of holomorphic quadratic differentials.

Proof. The statements about \mathbf{L}_0 and \mathbf{L}_1 are clear.

Let $Q = Q(h)$ denote the contraction $Q = R_{ijkl}h^{il}h^{jk}$ and define a tensor field $T = T(h)$ by

$$(3.8) \quad T_{ijk} = \nabla_k h_{ij} - \nabla_i h_{jk}.$$

Then integrating by parts and applying Koiso's Bochner formula [18], one obtains

$$(3.9) \quad \begin{aligned} (\mathbf{L}_2 h, h) &= -\|\nabla h\|^2 + 2 \int_{\mathcal{B}} Q(h) \, d\mu \\ &= -\frac{1}{2} \|T\|^2 - \|\delta h\|^2 - \frac{1}{2} \|h\|^2 + \int_{\mathcal{B}} Q(h) \, d\mu \\ &= -\frac{1}{2} \|T\|^2 - \|\delta h\|^2 - \frac{1}{2(n-1)} \|H\|^2 - \frac{n-2}{2(n-1)} \|h\|^2. \end{aligned}$$

This proves that (3.7a) is strictly stable for all $n > 2$.

When $n = 2$, it follows from (3.9) that a tensor field h belongs to the nullspace of $\mathbf{L}_2 = \Delta_\ell + R$ if and only if h is trace free and divergence free, with $T(h) = 0$ vanishing identically. We first verify that the latter condition is superfluous. In normal coordinates at an arbitrary point $p \in \mathcal{B}$, all components of $T(h)$ vanish except possibly $T_{112} = -T_{211} = \partial_2 h_{11} - \partial_1 h_{12}$ and $T_{221} = -T_{122} = \partial_1 h_{22} - \partial_2 h_{21}$. But one has $\partial_1 h_{11} + \partial_2 h_{21} = 0$ and $\partial_1 h_{12} + \partial_2 h_{22} = 0$ because h is divergence free and $\partial_1 h_{11} + \partial_1 h_{22} = 0$ and $\partial_2 h_{11} + \partial_2 h_{22} = 0$ because h is trace free. It follows easily that T vanishes at p , hence everywhere.

Now let h be a trace-free and divergence-free tensor field on an orientable surface \mathcal{B} of genus $\gamma \geq 2$. To identify the nullspace of \mathbf{L}_2 , it is most convenient to work in local complex coordinates. The trace-free condition implies that

$$h = f(z) \, dz \, d\bar{z} + \bar{f}(z) \, d\bar{z} \, dz$$

for some function $f : \mathcal{B} \rightarrow \mathbb{C}$, while the divergence-free condition implies that f is holomorphic. Hence h is a holomorphic quadratic differential. If \mathcal{R} is a Riemann surface obtained from a closed surface by deleting p disjoint closed discs and q isolated points not on any disc, the Riemann–Roch theorem implies that the space of holomorphic quadratic differentials on \mathcal{R} has real dimension $6(\gamma - 1) + 3p + 2q$. The result follows. \square

When $n = 2$, the nullspace for the system is classically identified with the cotangent space to Teichmüller space.

3.4. Convergence and stability of \varkappa -rescaled flow. We now prove Theorem 1. To obtain asymptotic stability of \varkappa -rescaled \mathbb{R}^1 -invariant Ricci flow (3.2) with $\varkappa = -1$ from its Ricci–DeTurck linearization (3.7), we shall freely use the theory reviewed in Section 2 and in particular the maximal regularity and interpolation spaces chosen in Section 2.3.

Proof of Theorem 1. Our notation is as follows. $\mathbf{g} = (g, A, u)$ is a locally \mathbb{R}^1 -invariant metric (3.1) on a product $\mathbb{R}^1 \times \mathcal{B}$, with \mathcal{B} compact and orientable, such

that g has constant sectional curvature $-1/2(n-1)$, A vanishes, and u is constant. $\widehat{\mathbf{g}}(\tau)$ is the unique solution of \varkappa -rescaled locally \mathbb{R}^1 -invariant Ricci–DeTurck flow corresponding to an initial datum $\widehat{\mathbf{g}}(0)$. Then there are diffeomorphisms φ_τ with $\varphi_0 = \text{id}$ such that the unique solution $\widetilde{\mathbf{g}}(\tau)$ of \varkappa -rescaled locally \mathbb{R}^1 -invariant Ricci flow (3.3) with initial datum $\widetilde{\mathbf{g}}(0)$ is given by

$$\widetilde{\mathbf{g}}(\tau) = \varphi_\tau^*(\widehat{\mathbf{g}}(\tau)).$$

The proof consists of four steps.

STEP 1. We prove that (the complexification of) the linearization $\mathbf{L} \equiv \mathbf{L}_{\mathbf{g}}$ of \varkappa -rescaled locally \mathbb{R}^1 -invariant Ricci–DeTurck flow is sectorial. Observe that \mathbf{L} is strictly elliptic and self adjoint. By Lemma 5, one may fix $\omega > 0$ such that its spectrum $\sigma(\mathbf{L}^{\mathbb{C}})$ satisfies $\sigma(\mathbf{L}^{\mathbb{C}}) \setminus \{0\} \subset (-\infty, -\omega)$. Standard Schauder theory then implies that $\mathbf{L}^{\mathbb{C}}$ is sectorial. (See Lemma 3.4 in [13] for a more detailed argument.)

STEP 2. By Step 1 and Lemma 2, we may apply Theorem 4 to $\widehat{\mathbf{g}}(\tau)$. Recall that $\mathbb{X}_\alpha = (\mathbb{X}_0, \mathbb{X}_1)_\alpha$, where \mathbb{X}_0 and \mathbb{X}_1 are defined by (2.6). Theorem 4 implies the following three statements:

- (1) For each $\alpha \in [0, 1]$, there is a direct sum decomposition $\mathbb{X}_\alpha = \mathbb{X}_\alpha^s \oplus \mathbb{X}_\alpha^c$, where \mathbb{X}_α^c is the nullspace of \mathbf{L} of dimension

$$\dim \mathbb{X}_\alpha^c = \begin{cases} 1 + 6(\gamma - 1) & \text{if } n = 2, \\ 1 & \text{if } n \geq 3. \end{cases}$$

(Recall that if $n = 2$, then \mathcal{B} must be an orientable surface of genus $\gamma \geq 2$.)

- (2) For each $r \in \mathbb{N}$, there exist d and a local C^r center manifold $\Gamma_{\text{loc}}^r = \text{graph}(\gamma_d^r : B(\mathbb{X}_1^c, \mathbf{g}, d) \rightarrow \mathbb{X}_1^s)$ which is invariant for solutions of Ricci–DeTurck flow as long as they remain in $B(\mathbb{X}_1^c, \mathbf{g}, d) \times B(\mathbb{X}_1^s, 0, d)$.
- (3) For each $\alpha \in (0, 1)$, there exist positive constants $M = M(\alpha)$ and $d = d(\alpha, r)$ such that for each initial datum $\widehat{\mathbf{g}}(0) \in B(\mathbb{X}_\alpha, \mathbf{g}, d)$ and all times $\tau > 0$ such that $\widehat{\mathbf{g}}(\tau) \in B(\mathbb{X}_\alpha, \mathbf{g}, d)$, the center manifold Γ_{loc}^r is exponentially attractive in the stronger space \mathbb{X}_1 in the sense that

$$(3.10) \quad \|\pi^s \widehat{\mathbf{g}}(\tau) - \gamma_d^r(\pi^c \widehat{\mathbf{g}}(\tau))\|_{\mathbb{X}_1} \leq \frac{M}{\tau^{1-\alpha}} e^{-\omega\tau} \|\pi^s \widehat{\mathbf{g}}(0) - \gamma_d^r(\pi^c \widehat{\mathbf{g}}(0))\|_{\mathbb{X}_\alpha},$$

where π^s and π^c are projections onto $\mathbb{X}_\alpha^s \cong (\mathbb{X}_1^s, \mathbb{X}_0^s)_\alpha$ and \mathbb{X}_α^c , respectively.

(All constants introduced here and below implicitly depend on \mathbf{g} .)

STEP 3. We prove that the local center manifolds Γ_{loc}^r coincide for all r , namely, that there is a unique smooth center manifold $\Gamma = \text{graph}(\gamma : B(\mathbb{X}_1^c, \mathbf{g}, d_0) \rightarrow \mathbb{X}_1^s)$ consisting of fixed points of \varkappa -rescaled locally \mathbb{R}^1 -invariant Ricci flow (3.3). First observe that for all $n \geq 2$ and all $c \in \mathbb{R}$, $(g, 0, c)$ is a stationary solution both of (3.3) and Ricci–DeTurck flow. By (3.10), any such metric sufficiently near $\mathbf{g} = (g, 0, u)$ must belong to all Γ_{loc}^r . When $n = 2$, Teichmüller theory shows that there is a $6(\gamma - 1)$ -dimensional space Γ' of metrics $\mathbf{g}' = (g', 0, u)$ near \mathbf{g} such that g' is hyperbolic. Each such metric is a fixed point of (3.3), hence evolves under Ricci–DeTurck flow only by diffeomorphisms. By diffeomorphism invariance, $g'(\tau)$ remains hyperbolic, so $\mathbf{g}'(\tau)$ remains in Γ' . By (3.10), $\mathbf{g}'(\tau)$ must belong to all Γ_{loc}^r .

STEP 4. Fix $\alpha \in (1/2, 1 - \rho/2)$. Then by the interpolation isomorphism (2.2), $\|\cdot\|_{\mathbb{X}_\alpha}$ is equivalent to a $(1 + \theta)$ -Hölder norm, with $\theta \in (\rho, 1)$. For all $\widehat{\mathbf{g}}(0)$ sufficiently near \mathbf{g} , we now prove that $\widehat{\mathbf{g}}(\tau)$ converges in the \mathbb{X}_α norm to an element of Γ . (We give the proof in general, even though certain steps are much easier when $n > 2$.)

Fix $\lambda \in (0, \omega)$. Then by (3.10) and Step 3 there exists $C = C(M, \lambda)$ such that

$$(3.11) \quad \|\pi^s \widehat{\mathbf{g}}(\tau) - \gamma(\pi^c \widehat{\mathbf{g}}(\tau))\|_{\mathbb{X}_1} \leq C e^{-\lambda\tau} \|\widetilde{\mathbf{g}}(0) - \mathbf{g}\|_{\mathbb{X}_\alpha}$$

for all $\tau > 0$ such that $\widehat{\mathbf{g}}(\tau) \in B(\mathbb{X}_\alpha, \mathbf{g}, d)$. Let δ, ε be positive constants to be determined so that $0 < \varepsilon < \delta < d$, and suppose that $\widetilde{\mathbf{g}}(0) \in B(\mathbb{X}_\alpha, \mathbf{g}, \varepsilon)$. Then it follows from (3.11) and (3.3) that

$$(3.12) \quad \left\| \frac{\partial}{\partial \tau} \widetilde{\mathbf{g}}(\tau) \right\|_{0+\rho} \leq C_n C e^{-\lambda\tau} \varepsilon.$$

If δ/ε is sufficiently large, this implies that

$$\|\widetilde{\mathbf{g}}(\tau) - \mathbf{g}\|_{\mathbb{X}_1} \leq \varepsilon \left(1 + \frac{C_n C}{\lambda} \right) < \delta,$$

uniformly in time. Because $\widetilde{\mathbf{g}}(\tau) = \varphi_\tau^*(\widehat{\mathbf{g}}(\tau))$, the only way that the solution $\widehat{\mathbf{g}}(\tau)$ of Ricci–DeTurck flow could leave $B(\mathbb{X}_\alpha, \mathbf{g}, \delta)$ is by diffeomorphisms. But as was observed by Hamilton [15], the diffeomorphisms φ_τ satisfy a harmonic map flow

$$\frac{\partial}{\partial \tau} \varphi_\tau = \Delta_{\widetilde{\mathbf{g}}(\tau), \mathbf{g}} \varphi_\tau.$$

with domain metric $\widetilde{\mathbf{g}}(\tau)$ and codomain metric \mathbf{g} . Because $\widetilde{\mathbf{g}}(\tau) \in B(\mathbb{X}_1, \mathbf{g}, \delta)$ for all $\tau \geq 0$, mild generalizations of standard estimates for harmonic map heat flow into negatively curved targets imply that the diffeomorphisms φ_τ exist for all $\tau \geq 0$ and satisfy

$$\|\varphi_\tau - \text{id}\|_{\mathbb{X}_1} \leq c$$

for $c = c(\delta)$. (See [10] and [24], for instance.) Note that c does not increase if one makes δ smaller. It follows that for $\delta \in (0, d)$ and $\varepsilon = \varepsilon(\delta) \in (0, \delta)$ sufficiently small, one has $\widehat{\mathbf{g}}(\tau) \in B(\mathbb{X}_\alpha, \mathbf{g}, d)$ for all $\tau \geq 0$. By (3.12), the result follows. \square

4. THE CASE $N = 1$: VOLUME-RESCALED LOCALLY \mathbb{R}^1 -INVARIANT RICCI FLOW

In case $(\mathcal{M}, \bar{\mathbf{g}})$ is a Riemannian product over an Einstein manifold of positive Ricci curvature such that

$$(4.1) \quad \bar{\mathbf{g}} = e^{2\bar{u}} dx^0 \otimes dx^0 + e^{2\bar{u}} \bar{A}_k (dx^0 \otimes dx^k + dx^k \otimes dx^0) + (e^{2\bar{u}} \bar{A}_i \bar{A}_j + \bar{g}_{ij}) dx^i \otimes dx^j,$$

the rescaled \mathbb{R}^1 -invariant Ricci flow system (1.3) corresponding to the normalization in Example 2 is a more suitable choice.

4.1. Volume-rescaled flow. *Volume-rescaled locally \mathbb{R}^1 -invariant Ricci flow* is the system

$$(4.2a) \quad \frac{\partial}{\partial \tau} g_{ij} = -2R_{ij} + e^{2u} g^{k\ell} (dA)_{ik} (dA)_{j\ell} + 2\nabla_i u \nabla_j u + \frac{2}{n} \left(\int_{\mathcal{B}} r \, d\mu \right) g_{ij},$$

$$(4.2b) \quad \frac{\partial}{\partial \tau} A_k = g^{ij} \{ \nabla_i (dA)_{jk} + 2(\nabla_i u) (dA)_{jk} \} + \frac{1}{n} \left(\int_{\mathcal{B}} r \, d\mu \right) A_k,$$

$$(4.2c) \quad \frac{\partial}{\partial \tau} u = \Delta u - \frac{1}{4} e^{2u} |dA|^2,$$

where all geometric quantities are computed with respect to g , and

$$(4.3) \quad r = R - \frac{1}{2} e^{2u} |dA|^2 - |\nabla u|^2.$$

4.2. Linearization at a stationary solution of volume-rescaled flow. Fix a background connection $\underline{\Gamma}$ and define a 1-parameter family of vector fields $W(\tau)$ along a solution $\mathbf{g}(\tau)$ of (4.2) by

$$(4.4) \quad \begin{aligned} W^0 &= \delta A, \\ W^k &= g^{ij}(\Gamma_{ij}^k - \underline{\Gamma}) \quad (1 \leq k \leq n). \end{aligned}$$

The solution of the *volume-rescaled \mathbb{R}^1 -invariant Ricci-DeTurck flow* is the 1-parameter family of metrics $\psi_\tau^* \mathbf{g}(\tau)$, where the diffeomorphisms ψ_τ are generated by $W(\tau)$, subject to the initial condition $\psi_0 = \text{id}$. Below, we take $\underline{\Gamma}$ to be the Levi-Civita connection of the metric about which we linearize.

Any Riemannian product $(\mathbb{R} \times \mathcal{B}, e^{2u} dx^0 \otimes dx^0 + g_{ij} dx^i \otimes dx^j)$ with g an Einstein metric, A identically zero, and u constant in space is clearly a stationary solution of the volume-rescaled \mathbb{R}^1 -invariant Ricci-DeTurck flow. Let $\mathbf{g} = (g, 0, u)$ be such a fixed point, with $\text{Rc} = Kg$ and hence $r = nK$. Again, let $(g+h, B, u+v)$ denote a perturbation of \mathbf{g} . We continue to write $H = \text{tr}_g h$ and to denote the Lichnerowicz Laplacian (3.4) by Δ_ℓ . Proceeding as we did for the \varkappa -rescaled flow, one obtains the following.

Lemma 6. *The linearization of the volume-rescaled \mathbb{R}^1 -invariant Ricci-DeTurck flow at a fixed point $\mathbf{g} = (g, 0, u)$ with $\text{Rc} = Kg$ and u constant is the autonomous, self-adjoint, strictly parabolic system*

$$\frac{\partial}{\partial \tau} \begin{pmatrix} h & B & v \end{pmatrix} = \mathbf{L} \begin{pmatrix} h & B & v \end{pmatrix} = \begin{pmatrix} \mathbf{L}_2 h & \mathbf{L}_1 B & \mathbf{L}_0 v \end{pmatrix},$$

where

$$(4.5a) \quad \mathbf{L}_2 h = \Delta_\ell h + 2K \left\{ h - \frac{1}{n} \bar{H} g \right\},$$

$$(4.5b) \quad \mathbf{L}_1 B = \Delta_1 B + KB,$$

$$(4.5c) \quad \mathbf{L}_0 v = \Delta_0 v.$$

Here $\bar{H} = \int_{\mathcal{B}} H d\mu$ and $-\Delta_p = d\delta + \delta d$ denotes the Hodge-de Rham Laplacian acting on p -forms.

Proof. The argument here needs a few more variation formulas in addition to those we recalled in the proof of Lemma 3. Once again, let $\tilde{g}(\varepsilon)$ be a smooth one-parameter family of Riemannian metrics such that

$$\tilde{g}(0) = g \quad \text{and} \quad \left. \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} \tilde{g} = h.$$

Using tildes to denote geometric quantities associated to \tilde{g} and undecorated characters to denote quantities associated to g , one recalls from [13, Lemma 2.2] that

$$\begin{aligned} \left. \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} \tilde{R} &= -(\Delta H - \delta^2 h + \langle \text{Rc}, h \rangle), \\ \left. \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} d\tilde{\mu} &= \frac{1}{2} H d\mu, \\ \left. \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} \int_{\mathcal{B}} \tilde{R} d\tilde{\mu} &= \int_{\mathcal{B}} \left\{ \frac{1}{2} \left(R - \int R d\mu \right) H - \langle \text{Rc}, h \rangle \right\} d\mu. \end{aligned}$$

These formulas simplify nicely when $\text{Rc} = Kg$. Using this fact, the result follows by careful but straightforward calculation. \square

4.3. Linear stability of volume-rescaled flow. Now let us make the stronger assumption that $\mathbf{g} = (g, 0, u)$ is a fixed point of the volume-rescaled \mathbb{R}^1 -invariant Ricci–DeTurck flow with u constant and g a metric of constant sectional curvature $k > 0$ and hence $K = (n - 1)k$.

By passing to a covering space if necessary, we may assume that \mathcal{B}^n is the round n -sphere of radius $\sqrt{1/k}$. Recall that the spectrum of the Laplacian Δ_0 acting on scalar functions on (\mathcal{B}^n, g) is $\{\lambda_{j,k}\}_{j \geq 0}$, where

$$\lambda_{j,k} = -jk(n + j - 1).$$

The eigenspace $\Lambda_{j,k}$ consists of the restriction to $\mathcal{B}^n \subset \mathbb{R}^{n+1}$ of all j -homogeneous polynomials in the coordinate functions (x^1, \dots, x^{n+1}) of \mathbb{R}^{n+1} .

Given a symmetric $(2, 0)$ -tensor field h , let $f = f(h)$ denote its trace-free part defined by

$$(4.6) \quad h = \frac{1}{n}Hg + f.$$

It is then convenient to rewrite (4.5a) and (4.5b) as

$$(4.7a) \quad \mathbf{L}_2 h = \Delta h - 2kf + 2\frac{n-1}{n}k(H - \bar{H})g,$$

$$(4.7b) \quad \mathbf{L}_1 B = \Delta_1 B + (n-1)kB,$$

respectively.¹¹

As above, the fact that \mathbf{L} is diagonal lets us determine its stability by examining its component operators. The conclusions we obtain will continue to hold when we extend \mathbf{L} to a larger domain in which smooth representatives are dense.

Lemma 7. *Let $\mathbf{g} = (g, 0, u)$ be a metric of the form (3.1) such that u is constant and g has constant sectional curvature $k > 0$. Then the linear system (4.5) has the following stability properties:*

\mathbf{L}_0 is weakly stable; its null eigenspace consists of constant functions.

\mathbf{L}_1 is strictly stable.

If $n = 2$, then \mathbf{L}_2 is weakly stable; its null eigenspace is

$$\{\varphi g : \varphi \in \Lambda_{0,k} \cup \Lambda_{1,k}\}.$$

If $n \geq 3$, then \mathbf{L}_2 is unstable. The sole unstable eigenvalue is $(n-2)k$ with eigenspace $\{\varphi g : \varphi \in \Lambda_{1,k}\}$. The null eigenspace is $\{\varphi g : \varphi \in \Lambda_{0,k}\} = \{cg : c \in \mathbb{R}\}$.

Proof. Weak stability of \mathbf{L}_0 is clear.

It is well known (see [12], for example) that the spectrum of Δ_1 is

$$\{\lambda_{j,k}\}_{j \geq 1} \cup \{\lambda_{j,k} - (n-2)k\}_{j \geq 1}.$$

Because $n \geq 2$, it follows that the largest eigenvalue of \mathbf{L}_1 is $-k$.

Using (4.6), one obtains the decomposition $\mathbf{L}_2 h = \frac{1}{n}L'_0 H + L'_2 f$, where

$$L'_0 H = \Delta_0 H + 2(n-1)k(H - \bar{H}),$$

$$L'_2 f = \Delta f - 2kf.$$

The operator L'_2 is evidently strictly stable. Because

$$(L'_0 H, H) = -\|\nabla H\|^2 + 2(n-1)k(\|H\|^2 - V\bar{H}^2),$$

¹¹Recall that Δ is the rough Laplacian.

where $V = \text{Vol}(\mathcal{B}, g)$, one sees that the spectrum of L'_0 is bounded from above by that of $\Delta_0 + 2(n-1)k$. Because $\lambda_{2,k} = -2(n+1)k$, the only possible unstable eigenfunctions for L'_0 are elements of $\Lambda_{0,k}$ and $\Lambda_{1,k}$. It is easy to see that elements of $\Lambda_{0,k}$ (i.e. constant functions) belong to the nullspace of L'_0 . And if $H \in \Lambda_{1,k}$, then one has $\bar{H} = 0$ and hence $(L'_0 H, H) = \{\lambda_{1,k} + 2(n-1)k\} \|H\|^2 = (n-2)k \|H\|^2$. Because $\|h\|^2 = \frac{1}{n} \|H\|^2 + \|f\|^2$, the result follows. \square

Remark 1. The elements of $\{\varphi g : \varphi \in \Lambda_{1,k}\}$ are infinitesimal conformal diffeomorphisms (Möbius transformations): it is a standard fact that any $h \in \{\varphi g : \varphi \in \Lambda_{1,k}\}$ satisfies

$$h = \varphi g = -\frac{1}{2k} \mathcal{L}_{\text{grad } \varphi} g.$$

Remark 2. The apparent instability of \mathbf{L}_2 for $n \geq 3$ is an accident of the DeTurck trick rather than an essential feature of Ricci flow. Indeed, Hamilton observed [15] that the DeTurck diffeomorphisms solve a harmonic map heat flow, with an evolving domain metric and a fixed target metric. It is well known that the identity map of round spheres \mathcal{S}^n is an unstable harmonic map for all $n \geq 3$. On the other hand, one can show by different methods that spherical space forms are attractive fixed points for normalized Ricci flow in any dimension. See [3] and [4].

In any case, the instability of \mathbf{L}_2 is in a sense a red herring, i.e. geometrically insignificant. Indeed, Moser [22] has shown that if g and \tilde{g} are any metrics on a compact manifold \mathcal{N} such that $\text{Vol}(\mathcal{N}, \tilde{g}) = \text{Vol}(\mathcal{N}, g)$, then $\tilde{g} = \psi^* \hat{g}$ for some diffeomorphism $\psi : \mathcal{N} \rightarrow \mathcal{N}$ and some metric \hat{g} on \mathcal{N} with $d\tilde{\mu} = d\mu$. So, up to diffeomorphism, it suffices to consider perturbations that preserve the volume element $d\mu$ of a stationary solution g . These are exactly those h with $H = 0$ pointwise. Restricted to such perturbations, \mathbf{L}_2 is strictly stable.

We can now at least partially explain our *ad hoc* choices of normalization.

Remark 3. If $\mathbf{g} = (g, 0, u)$ is a metric of the form (3.1) such that u is constant and g has constant sectional curvature $1/2(n-1) > 0$, then the operator \mathbf{L}_2 in the linearization (3.7) of \varkappa -rescaled Ricci–DeTurck flow (3.3) satisfies

$$(\mathbf{L}_2 h, h) = \frac{1}{n} (\|\nabla H\|^2 + \|H\|^2) - \|\nabla f\|^2 - \frac{1}{n-1} \|f\|^2.$$

So 1 is an eigenvalue with eigenspace $\{cg : c \in \mathbb{R}\}$.

Remark 4. If $\mathbf{g} = (g, 0, u)$ is a metric of the form (3.1) such that u is constant and g has constant sectional curvature $k < 0$, then Koiso's Bochner formula [18] in the form

$$\|\nabla h\|^2 = \frac{1}{2} \|T\|^2 + \|\delta h\|^2 - nk \|f\|^2$$

implies that the operator \mathbf{L}_2 in the linearization (4.5) of the volume-rescaled Ricci–DeTurck flow (4.2) satisfies

$$(\mathbf{L}_2 h, h) = -\frac{1}{2} \|T\|^2 - \|\delta h\|^2 + (n-2)k \|f\|^2 + 2\frac{n-1}{n} k \{\|H\|^2 - V\bar{H}^2\}.$$

Hölder's inequality implies that the quantity in braces is nonnegative and vanishes exactly when $|H|$ is constant a.e. So \mathbf{L}_2 has a null eigenvalue with eigenspace $\{cg : c \in \mathbb{R}\}$ in all dimensions. (The nullspace is of course larger when $n = 2$).

4.4. Convergence and stability of volume-rescaled flow. Here we exploit the fact that the identity map of the round 2-sphere is a weakly stable harmonic map.

Proof of Theorem 2. The proof is entirely analogous to the proof of Theorem 1, except for Step 4. Here we must uniformly bound diffeomorphisms $\{\varphi_\tau\}$ solving

$$\begin{aligned} \frac{\partial}{\partial \tau} \varphi_\tau &= \Delta_{\tilde{\mathbf{g}}(\tau), \mathbf{g}} \varphi_\tau, \\ \varphi_0 &= \text{id} : (\mathcal{S}^2, \tilde{\mathbf{g}}(0)) \rightarrow (\mathcal{S}^2, \mathbf{g}), \end{aligned}$$

where \mathbf{g} is a metric of constant positive curvature and $\tilde{\mathbf{g}}(\tau) \in B(\mathbb{X}_1, \mathbf{g}, \delta)$ for all $\tau \geq 0$. In this case, the required bound $\|\varphi_\tau - \text{id}\|_{\mathbb{X}_1} \leq c(\delta)$ follows from a result of Topping [26]. The remainder of the proof goes through without modification. \square

5. THE CASE $n = 1$: HOLONOMY-RESCALED LOCALLY \mathbb{R}^N -INVARIANT RICCI FLOW

We now consider evolving metrics on the mapping torus $\mathbb{R}^N \hookrightarrow \mathcal{M}_\Lambda \xrightarrow{\pi} \mathcal{S}^1$ of a given $\Lambda \in \text{Gl}(N, \mathbb{R})$. We let x^0 denote the coordinate on $\mathcal{S}^1 \approx \mathbb{R}/\mathbb{Z}$ and take (x^1, \dots, x^N) as local coordinates on the fibers. Let c and s be constants and let V be a 1-form to be determined. Then the evolution of

$$(5.1) \quad \mathbf{g}(\tau) = u dx^0 \otimes dx^0 + A_i (dx^0 \otimes dx^i + dx^i \otimes dx^0) + G_{ij} dx^i \otimes dx^j$$

by rescaled locally \mathbb{R}^N -invariant Ricci flow (1.3) modified by diffeomorphisms generated by the vector field V^\sharp is equivalent to the system

$$(5.2a) \quad \frac{\partial}{\partial \tau} u = \frac{1}{2} |\partial_0 G|^2 - su + (\mathcal{L}_{V^\sharp} \mathbf{g})_{00},$$

$$(5.2b) \quad \frac{\partial}{\partial \tau} A^i = -\frac{1+c}{2} s A^i + G^{ij} (\mathcal{L}_{V^\sharp} \mathbf{g})_{j0},$$

$$(5.2c) \quad \frac{\partial}{\partial \tau} G_{ij} = u^{-1} \{ \nabla_0^2 G_{ij} - (\partial_0 G_{ik}) G^{k\ell} (\partial_0 G_{j\ell}) \} + cs G_{ij} + (\mathcal{L}_{V^\sharp} \mathbf{g})_{ij}.$$

We define

$$(5.3) \quad V = \frac{C}{2} (\partial_0 u) dx^0 + (G_{ij} \partial_0 A^j) dx^i,$$

where $C > 0$ is a constant to be chosen below. Notice that V implements a DeTurck trick, as was done in Sections 3–4.

Hereafter, we drop the boldface font and write g for \mathbf{g} . For later use, we note that the Christoffel symbols of g are

$$(5.4a) \quad \Gamma_{00}^0 = \frac{1}{2} g^{00} \partial_0 u + g^{0i} \partial_0 (G_{ij} A^j),$$

$$(5.4b) \quad \Gamma_{00}^k = \frac{1}{2} g^{k0} \partial_0 u + g^{ki} \partial_0 (G_{ij} A^j),$$

$$(5.4c) \quad \Gamma_{i0}^0 = \frac{1}{2} g^{0j} \partial_0 G_{ij},$$

$$(5.4d) \quad \Gamma_{ij}^0 = -\frac{1}{2} g^{00} \partial_0 G_{ij},$$

$$(5.4e) \quad \Gamma_{0j}^k = \frac{1}{2} g^{k\ell} \partial_0 G_{j\ell},$$

$$(5.4f) \quad \Gamma_{ij}^k = -\frac{1}{2} g^{k0} \partial_0 G_{ij}.$$

5.1. Holonomy-rescaled flow. Following Lott [19], we say G is a *harmonic-Einstein metric* if

$$(5.5) \quad \nabla_0^2 G_{ij} = (\partial_0 G_{ik}) G^{k\ell} (\partial_0 G_{\ell j}).$$

We henceforth assume that \mathcal{M}_Λ admits a metric g of the form (5.1) with G harmonic-Einstein. By an initial reparameterization, we may assume without loss of generality that $u = 1$ at $t = 0$. As we shall observe in Lemma 8 below, the harmonic-Einstein condition implies that $|\partial_0 G|^2$ is constant in space. Motivated by Example 3, we choose constants $c = 0$ and $s = \frac{1}{2} |\partial_0 G|^2$. Observe that the constant s is determined by the holonomy of \mathcal{M}_Λ , because for any fiber metric G , one has

$$(5.6) \quad G_{ij}|_{x^0=1} = \Lambda_i^k \Lambda_j^\ell G_{k\ell}|_{x^0=1}.$$

So when c , s , and V are chosen in this way, we call system (5.2) *holonomy-rescaled locally \mathbb{R}^N -invariant Ricci flow* and denote a solution by $g(\tau) = (u(\tau), A(\tau), G(\tau))$.

Lemma 8. *Let g be a metric of the form (5.1) such that $u = 1$, A vanishes, and G is a harmonic-Einstein metric. Then g is a stationary solution of (5.2).*

Proof. The hypotheses on u and A imply that $V = 0$ at $\tau = 0$. The fact that G is harmonic-Einstein implies that $|\partial_0 G|^2$ is constant in space. Hence $\frac{1}{2} |\partial_0 G|^2 = su$. The result follows. \square

Notice that the sol-geometry manifolds in Example 3 satisfy the hypotheses of the lemma. There s is the topological constant $s = 4(\log \lambda)^2$. Clearly, $s > 0$ whenever \mathcal{M}_Λ has nontrivial holonomy.

5.2. Linearization at a stationary solution of holonomy-rescaled flow. Let $(u + v, B, G + h)$ be a perturbation of a stationary solution $g = (1, 0, G)$ of the type considered in Lemma 8.

Lemma 9. *The linearization of holonomy-rescaled locally \mathbb{R}^N -invariant Ricci flow (5.2) about a stationary solution $(1, 0, G)$ with G harmonic-Einstein is the autonomous, strictly parabolic system*

$$\frac{\partial}{\partial \tau} \begin{pmatrix} v & B & h \end{pmatrix} = \mathbf{L} \begin{pmatrix} v & B & h \end{pmatrix} = \begin{pmatrix} \mathbf{L}_0 v + \mathbf{F}h & \mathbf{L}_1 B & \mathbf{L}_2 h \end{pmatrix},$$

where

$$(5.7a) \quad \mathbf{L}_0 v = C \partial_0^2 v - sv,$$

$$(5.7b) \quad \mathbf{F}h = \langle \partial_0 G, \partial_0 h \rangle - \langle \partial_0^2 G, h \rangle,$$

$$(5.7c) \quad (\mathbf{L}_1 B)^i = \partial_0^2 B^i - \frac{s}{2} B^i,$$

$$(5.7d) \quad (\mathbf{L}_2 h)_{ij} = \partial_0^2 h_{ij} - G^{k\ell} (\partial_0 h_{ik} \partial_0 G_{\ell j} + \partial_0 G_{ik} \partial_0 h_{\ell j}) + \partial_0 G_{ik} h^{k\ell} \partial_0 G_{\ell j}.$$

Proof. One may suppose that $\frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} u = v$, $\frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} A = B$, and $\frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} G = h$. We first compute that

$$\frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} V = \frac{C}{2} \partial_0 v dx^0 + G_{ij} \partial_0 B^j dx^i$$

and

$$\begin{aligned}\frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} (\mathcal{L}_V g)_{00} &= C \partial_0^2 v, \\ \frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} (\mathcal{L}_V g)_{j0} &= G_{ij} \partial_0^2 B^i, \\ \frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} (\mathcal{L}_V g)_{ij} &= \frac{1}{2} (\partial_0 v) (\partial_0 G_{ij}).\end{aligned}$$

Note that we used (5.4) to get the last equality. Using the fact that G is harmonic-Einstein, we obtain

$$\frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} |\partial_0 G|^2 = 2 \langle \partial_0 G, \partial_0 h \rangle - 2 \langle \partial_0^2 G, h \rangle,$$

where the inner products are taken with respect to G . Observing that

$$\frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} \nabla_0^2 G_{ij} = \partial_0^2 h_{ij} - (\partial_0 G_{ij}) \left(\frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} \Gamma_{00}^0 \right) = \partial_0^2 h_{ij} - \frac{1}{2} (\partial_0 v) (\partial_0 G_{ij}),$$

we conclude that

$$\begin{aligned}\frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} \{ u^{-1} [\nabla_0^2 G_{ij} - (\partial_0 G_{ik}) G^{k\ell} (\partial_0 G_{j\ell})] \} \\ = \partial_0^2 h_{ij} - G^{k\ell} (\partial_0 h_{ik} \partial_0 G_{\ell j} + \partial_0 G_{ik} \partial_0 h_{\ell j}) + \partial_0 G_{ik} h^{k\ell} \partial_0 G_{\ell j} - \frac{1}{2} (\partial_0 v) (\partial_0 G_{ij}).\end{aligned}$$

The result follows. \square

5.3. Linear stability of holonomy-rescaled flow. To investigate linear stability of (5.7), it is convenient to ‘cut’ the bundle \mathcal{M}_Λ and consider tensor fields depending on $x^0 \in [0, 1]$. By our assumption that \mathcal{M}_Λ admits a flat connection, we may henceforth assume that the \mathbb{R}^N -valued 1-form A has trivial holonomy, i.e. that $A^i|_{x^0=1} = A^i|_{x^0=0}$, while G satisfies (5.6) for a nontrivial action of $\Lambda \in \text{Gl}(N, \mathbb{R})$. Notice that these assumptions are satisfied by sol^3 -twisted torus bundles, of which Example 3 is a special case.

Corresponding to these assumptions, we require v and B to satisfy the boundary conditions

$$(5.8) \quad v|_{x^0=1} = v|_{x^0=0},$$

$$(5.9) \quad B^i|_{x^0=1} = B^i|_{x^0=0},$$

respectively. We require h to satisfy the linear compatibility condition

$$(5.10) \quad h_{ij}|_{x^0=1} = \Lambda_i^k \Lambda_j^\ell h_{k\ell}|_{x^0=0}$$

derived from (5.6).

For now, we apply the differential operators in (5.7) only to smooth data satisfying these boundary conditions. Below, we will specify larger domains in which smooth representatives are dense. With this intention, we now investigate the linear stability of (5.7). We will eventually have to consider the entire action of the lower-triangular operator \mathbf{L} , but we begin with observations about its component operators. The first two are self evident.

Lemma 10. \mathbf{L}_0 is self adjoint with respect to the inner product

$$(\varphi, \psi) = \int_0^1 \varphi \psi \, d\xi$$

and is strictly linearly stable if $s > 0$.

Lemma 11. \mathbf{L}_1 is self adjoint with respect to the inner product

$$(\varphi, \psi) = \int_0^1 \varphi^i \psi^j \delta_{ij} \, d\xi$$

and is strictly linearly stable if $s > 0$.

Lemma 12. \mathbf{L}_2 is self adjoint with respect to the inner product

$$(\varphi, \psi) = \int_0^1 \langle \varphi, \psi \rangle \, d\xi = \int_0^1 \varphi_{ij} \psi_{k\ell} G^{ik} G^{j\ell} \, d\xi$$

and is weakly linearly stable. Its null eigenspace is

$$\{GM : M \in \mathfrak{gl}(N, \mathbb{R}) \text{ commutes with all } G(\cdot)\}.$$

Proof. If φ and ψ satisfy (5.10) and are smooth, then $\partial_0 \varphi_{ij}|_{x^0=1} = \Lambda_i^k \Lambda_j^\ell \partial_0 \varphi_{k\ell}|_{x^0=0}$ and $\partial_0 \psi_{ij}|_{x^0=1} = \Lambda_i^k \Lambda_j^\ell \partial_0 \psi_{k\ell}|_{x^0=0}$. So if G satisfies (5.6), then all boundary terms arising from integrations by parts vanish. Using this fact, it is easy to verify that

$$(\mathbf{L}_2 \varphi, \psi) = (\varphi, \mathbf{L}_2 \psi).$$

Now we write $(\mathbf{L}_2 h, h) = I_1 - 2I_2 + I_3$, where (using primes to denote differentiation with respect to $x^0 = \xi$ inside an integral) the distinct terms are

$$\begin{aligned} I_1 &= \int_0^1 h''_{ij} h_{k\ell} G^{ik} G^{j\ell} \, d\xi, \\ I_2 &= \int_0^1 h'_{ip} G^{pq} G'_{qj} h_{k\ell} G^{ik} G^{j\ell} \, d\xi, \\ I_3 &= \int_0^1 G'_{ip} G^{pq} h_{qj} G'_{kr} G^{rs} h_{s\ell} G^{i\ell} G^{jk} \, d\xi. \end{aligned}$$

An integration by parts shows that

$$I_1 = -\|\partial_0 h\|^2 + 2I_2.$$

The integral I_3 , which may be rewritten as

$$I_3 = (\partial_0 G \circ h, h \circ \partial_0 G),$$

requires more work. Integrating it by parts produces seven terms:

$$I_3 = 2I_3 + 2\|\partial_0 G \circ h\|^2 - 2(\partial_0 G \circ h, \partial_0 h) - (\partial_0^2 G \circ h, h),$$

Because G satisfies the harmonic-Einstein equation (5.5), the final term above may be rewritten as

$$(\partial_0^2 G \circ h, h) = \|\partial_0 G \circ h\|^2.$$

Collecting terms and applying Cauchy-Schwarz, we obtain

$$\begin{aligned} (\mathbf{L}_2 h, h) &= -\|\partial_0 h\|^2 - \|\partial_0 G \circ h\|^2 + 2(\partial_0 G \circ h, \partial_0 h) \\ &\leq -\|\partial_0 h\|^2 - \|\partial_0 G \circ h\|^2 + 2\|\partial_0 G \circ h\| \|\partial_0 h\| \leq 0. \end{aligned}$$

One has equality if and only if $h_j^k = G^{k\ell} h_{\ell j}$ is independent of x^0 , hence if and only if $h_{ij} = G_{ik} M_j^k$ for some $M \in \mathfrak{gl}(N, \mathbb{R})$ that commutes with all $G(x^0)$. \square

Remark 5. An element GM of the nullspace of \mathbf{L}_2 will itself satisfy the harmonic-Einstein equation (5.5) if it is a metric, i.e. if M is invertible.

Remark 6. For the sol-Gowdy metrics considered in Example 3, it is easy to verify that the nullspace of \mathbf{L}_2 is two-dimensional.

Now we extend the L^2 inner products above to vectors $(v \ B \ h)$ in the natural way, i.e.

$$((v \ B \ h), (\alpha \ \beta \ \gamma)) = (v, \alpha) + (B, \beta) + (h, \gamma).$$

The lemma below is the main observation of this subsection. Observe that it does not directly imply convergence of the nonlinear system (5.2), because \mathbf{L} is not self adjoint. In the next subsection, after introducing suitable function spaces, we shall overcome this difficulty by exhibiting \mathbf{L} as a sectorial perturbation of the self-adjoint operator

$$(5.11) \quad \mathbf{L}_{\text{sa}} : (v \ B \ h) \mapsto (\mathbf{L}_0 v \ \mathbf{L}_1 B \ \mathbf{L}_2 h).$$

Lemma 13. *Assume that G has nontrivial holonomy $\Lambda \in \text{Gl}(N, \mathbb{R})$ but admits a flat connection. Then there exists $C > 0$ depending only on G such that the linearization \mathbf{L} of holonomy-rescaled locally \mathbb{R}^N -invariant Ricci flow is weakly linearly stable with respect to the DeTurck diffeomorphisms induced by $V = V(C)$ defined in (5.3). Its nullspace is*

$$\{GM : M \in \text{gl}(N, \mathbb{R}) \text{ commutes with all } G(\cdot)\}.$$

Proof. First suppose that h belongs to the null eigenspace of \mathbf{L}_2 . Then because G is harmonic-Einstein and $h = GM$, one has $\langle \partial_0^2 G, h \rangle = \langle \partial_0 G, \partial_0 h \rangle$. Hence $\mathbf{F}h = 0$ and so

$$(\mathbf{L}(v \ B \ h), (v \ B \ h)) = -C \|\partial_0 v\|^2 - s \|v\|^2 - \|\partial_0 B\|^2 - \frac{s}{2} \|B\|^2.$$

This is strictly negative unless v and B vanish identically.

For the general case, write $h = h^\circ + h^\perp$, where h° belongs to the nullspace of \mathbf{L}_2 and h^\perp belongs to its orthogonal complement. The self-adjoint elliptic operator \mathbf{L}_2 has pure point spectrum with eigenvalues of finite multiplicity. So there exists $\lambda > 0$ depending only on G such that $(\mathbf{L}_2 h^\perp, h^\perp) \leq -\lambda \|h^\perp\|^2$. Hence

$$\begin{aligned} (\mathbf{L}(v \ B \ h), (v \ B \ h)) &= -C \|\partial_0 v\|^2 - s \|v\|^2 + (\mathbf{F}h^\perp, v) \\ &\quad - \|\partial_0 B\|^2 - \frac{s}{2} \|B\|^2 - \lambda \|h^\perp\|^2. \end{aligned}$$

Now for *any* h , one integrates by parts and uses the harmonic-Einstein identity (5.5) to see that

$$(\mathbf{F}h, v) = - \int_0^1 \langle G', h \rangle v' d\xi.$$

There exists κ depending only on G such that $|\langle G', h \rangle| \leq \kappa |h|$ pointwise. Thus one estimates that

$$\begin{aligned} (\mathbf{L}(v \ B \ h), (v \ B \ h)) &\leq -s \|v\|^2 - \frac{s}{2} \|B\|^2 - \frac{\lambda}{2} \|h^\perp\|^2 \\ &\quad - \left\{ C \|\partial_0 v\|^2 - \kappa \|\partial_0 v\| \|h^\perp\| + \frac{\lambda}{2} \|h^\perp\|^2 \right\}. \end{aligned}$$

The quantity in braces is nonnegative as long as $C \geq \kappa^2/(2\lambda)$. The result follows. \square

5.4. Convergence and stability of holonomy-rescaled flow.

Proof of Theorem 3. We again follow the proof of Theorem 1, *mutatis mutandis*, except for two critical steps.

STEP 1. The complexification $\mathbf{L}_{\text{sa}}^{\mathbb{C}}$ of the operator defined in (5.11) is strictly elliptic and self adjoint with bounded spectrum, hence is sectorial by standard Schauder estimates. Observe that $\mathbf{L}^{\mathbb{C}} - \mathbf{L}_{\text{sa}}^{\mathbb{C}} = \mathbf{F}^{\mathbb{C}}$, where $\mathbf{F}^{\mathbb{C}} \in \mathcal{L}(\Sigma_2^{1+\alpha}, \Sigma_2^{0+\alpha})$ for all $\alpha \in (0, 1)$. Namely, $\mathbf{F}^{\mathbb{C}}$ is a bounded operator from intermediate spaces $\mathbb{X}_{\vartheta} = (\mathbb{X}_0, \mathbb{X}_1)_{\vartheta}$ and $\mathbb{E}_{\vartheta} = (\mathbb{E}_0, \mathbb{E}_1)_{\vartheta}$ to \mathbb{X}_0 and \mathbb{E}_0 , respectively, with its bound depending only on the stationary solution about which we are linearizing. The fact that $\mathbf{L}^{\mathbb{C}}$ is sectorial then follows from classical perturbation results. For instance, see [21, Proposition 2.4.1].

STEP 4. The only change in this step, once again, is how one controls the diffeomorphism φ_{τ} . Here the argument is much easier, because the base \mathcal{S}^1 is flat. By (5.3), the vector fields V that generate the diffeomorphisms ψ_{τ} satisfy

$$\|V\|_{1+\rho} \leq C_N \|\pi^{\text{s}}\widehat{\mathbf{g}}(\tau) - \gamma(\pi^{\text{c}}\widehat{\mathbf{g}}(\tau))\|_{\mathbb{X}_1} \leq C_N C e^{-\lambda\tau} \|\tilde{\mathbf{g}}(0) - \mathbf{g}\|_{\mathbb{X}_{\alpha}}.$$

It is then not hard to see that the diffeomorphisms ψ_{τ} and φ_{τ} must converge. Indeed, this is proved in [13, Lemma 3.5]. The remainder of the argument goes through without modification. \square

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