Neckpinching for asymmetric surfaces moving by mean curvature

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(joint work with Zhou Gang, Israel Michael Sigal)

There is a folklore conjecture in geometric analysis which predicts that finite-time singularities of parabolic geometric PDE asymptotically become as symmetric as their topologies allow. How should this be interpreted?

A first observation is that the statement should be understood in the sense of quasi-isometries. For example, one may deform the standard round metric on $\mathbb{S}^3$ to a 'bumpy' metric g_0, making its isometry group trivial while preserving positive Ricci curvature. Now consider a solution $g(t)$ of Ricci flow with initial data g_0. By Kotschwar's result [14], the isometry group of each $g(t)$ remains trivial. But by Hamilton's seminal Ricci flow result [9], the actions of $\mathrm{O}(4)$ with respect to a fixed atlas become arbitrarily close to isometries as the singularity time is approached.

A second observation is that the statement must be understood locally. For example, asymptotics of Ricci flow neckpinch singularities [2] reveal that metrics possessing rotation and reflection symmetries asymptotically acquire an additional translational symmetry near the singular set by converging to the cylinder soliton.

A third observation is that the conjecture may be broken down into two parts. The first, which can be stated rigorously, is that dilations of finite-time singularities converge (at least modulo subsequences) to self-similar solutions, i.e. solitons. Indeed, Huisken's monotonicity formula [11] proves this for Type-I singularities of mean curvature flow (MCF). The second part of the conjecture is the heuristic expectation that solitons, as generalized fixed points of geometric heat flows, are in a sense 'maximally diffused' and hence possess symmetry groups that are as large as possible.

How might one investigate the full conjecture? The most powerful methods for studying finite-time singularities of parabolic geometric PDE in greatest generality are surgery programs. These exploit a 'canonical neighborhood' property — the fact that high-curvature regions of a solution have special properties which in some cases allow their classification. Two celebrated examples are Perelman's surgery program for Ricci flow [15, 16] (also see Hamilton's foundational work [10]) and the surgery program of Huisken and Sinestrari [12] for singularities of 2-convex hypersurfaces evolving by MCF. However, even these spectacularly successful surgery programs do not provide independence of subsequence, precisely because of their need to consider quite general solutions. This is an obstacle to showing that a solution asymptotically (locally, quasi-isometrically) approaches a unique singularity model. For example, given a family of hypersurfaces $\mathcal{M}_t^n \subset \mathbb{R}^{n+1}$ evolving by MCF and becoming singular as $t \nearrow T < \infty$, we call the set of points in the ambient space at which the solution becomes singular its *residue set*. The residue set of a cylinder is a line, and that of a rotationally and reflection symmetric neckpinch is a point [8]. For nonsymmetric neckpinches, however, it is not even known if the residue set is rectifiable (though it is conjecturally a point — see below).

The difficulties in proving independence of subsequence alluded to above are reflected in another conjecture, which we learned of from Ecker: *Do singularities of* MCF *have unique tangent flows?* This has recently been proved by Schulze [17] if one tangent flow consists of a closed, multiplicity-one, smoothly embedded self-similar shrinker, but the general case remains open.

Another approach to studying singularity formation involves matched asymptotic expansions. These generally require much stronger hypotheses than do surgery programs. But in turn, they provide statements that hold uniformly in suitable space-time neighborhoods of a developing singularity. Some examples (certainly not a comprehensive list!) of asymptotics for geometric PDE are work of King [13], Daskalopoulos–del Pino [5], and Daskalopoulos–Šešum [6] for logarithmic fast diffusion, $u_t = \Delta \log u$, which represents the evolution of the conformal factor for a noncompact 2-dimensional solution of Ricci flow encountering a Type-II singularity; work of Angenent–Velázquez [3] for Type-II MCF singularities; work of Angenent and an author [2] for Type-I Ricci flow singularities; work of two authors [8] for Type-I MCF singularities; and work [1] of Angenent–Isenberg and an author for Type-II Ricci flow singularities. Notably, all of these results — except for [6] — require a hypothesis of rotational symmetry, which invites the question: *Do singularities of geometric* PDE *become asymptotically rotationally symmetric?* In other words, is rotational symmetry stable in a suitable geometric sense?

This note is a report on the first step in a program intended to provide an affirmative answer to this question. In the first step [7], we remove the hypothesis of rotational symmetric for surfaces $\mathcal{M}_t^2 \subset \mathbb{R}^3$ evolving by MCF, replacing it by weaker discrete symmetries. We conclude that solutions starting sufficiently close to a standard rotationally-symmmetric neck become asymptotically rotationally symmetric in a precise sense (see below). This result provides another example in which the folklore conjecture outlined above can be made rigorous. In forthcoming work, we plan to remove the dimension restriction as well as the discrete symmetry assumptions. In light of the important result of Colding–Minicozzi [4] that spherical and cylindrical singularities are the only generic MCF singularities, a successful completion of this program will prove that rotationally symmetric neckpinch behavior is 'universal' in a precise sense. As a corollary, it will also prove a version of the conjecture that MCF neckpinch singularities have unique limiting cylinders. What follows is a brief outline of our methods and results in [7].

We study the evolution of graphs over a cylinder $\mathbb{S}^1 \times \mathbb{R}$ embedded in $\mathbb{R}^3$. In coordinates (x, y, z) for $\mathbb{R}^3$, we take as an initial datum a surface $\mathcal{M}_0^2$ around the x-axis, given by a map $\sqrt{y^2 + z^2} = u_0(x, \theta)$, where θ denotes the angle from the ray $y > 0$ in the (y, z)-plane. Then for as long as the flow remains a graph, all $\mathcal{M}_t^2$ are given by $\sqrt{y^2 + z^2} = u(x, \theta, t)$.

Analysis of rotationally symmetric neckpinch formation [8] leads one to expect that perturbations of rotationally symmetric necks should resemble spatially homogeneous cylinders $\sqrt{2(T - t)}$ in a space-time neighborhood of the developing singularity. So we apply adaptive rescaling, transforming the original variables x

and t into rescaled blowup variables $y(x,t) := \lambda^{-1}(t)x$ and $\tau(t) := \int_0^t \lambda^{-2}(s)\,\mathrm{d}s$, respectively. (Reflection symmetry fixes the center of the neck at $x = 0$.) What distinguishes this approach from standard parabolic rescaling (e.g. [3] or [2]) is that we do not fix $\lambda(t)$ but instead consider it as a free parameter to be determined from the evolution itself. We study a rescaled radius $v(y,\theta,\tau)$ defined by $v(y(x,t),\theta,\tau(t)) := \lambda^{-1}(t)\,u(x,\theta,t)$. Then in commuting (y,θ,τ) variables, the quantity v evolves by $v_\tau = A_v(v) - (\lambda\lambda_t)v - v^{-1}$, where A_v is a quasilinear elliptic operator. The formal adiabatic approximate solution of this equation is given by $V_{\alpha,\beta}(y) := \sqrt{\alpha(2+\beta y^2)}$ for positive parameters α and β.

We assume that the initial surface $v_0(y,\theta) = v(y,\theta,0)$ is sufficiently C^3-close to some $V_{\alpha_0,\beta_0}(y)$. We then prove that for such initial data, the solution $v(y,\theta,\tau)$ of MCF becomes singular at some $T < \infty$ and converges locally to a rotationally symmetric solution $V_{\alpha(\tau),\beta(\tau)}(y)$, where $\alpha \approx 1$ and $\beta \approx (-\log(T-t))^{-1}$ as $t \nearrow T$. The solution's residue set is a point. An interested reader should consult [7] for precise statements of our assumptions, estimates, and convergence results.

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