Quasi-convergence of the Ricci flow

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Abstract

We study a collection of Riemannian metrics which collapse under the Ricci flow, and show that the quasi-convergence equivalence class of an arbitrary metric in this collection contains a 1-parameter family of locally homogeneous metrics.

1 Introduction and statement of main theorem

In [1], Hamilton and Isenberg studied the Ricci flow of a family of solv-geometry metrics on twisted torus bundles. This family contains no Einstein metrics, so the (normalized) Ricci flow cannot converge. Hamilton–Isenberg introduced the concept of *quasi-convergence* to describe its behavior, writing

> *"...the Ricci flow of all metrics in this family asymptotically approaches the flow of a sub-family of locally homogeneous metrics..."*

The intent of this paper is to make that statement more precise. In so doing, we answer a question of Hamilton, who asked whether an arbitrary metric in this class would converge to a unique locally homogeneous limit or would exhibit a more nuanced behavior.

1.1 Definition If g, h are evolving Riemannian metrics on a manifold $\mathcal{M}^n$, we say g *quasi-converges* to h if for any $\varepsilon > 0$ there is a time t_ε such that

$$\sup_{\mathcal{M}^n \times [t_\varepsilon, \infty)} |g - h|_h < \varepsilon.$$

Quasi-convergence is an equivalence relation. Indeed, the standard fact that $|U(V,V)| \leq |U|_h |V|_h^2$ for any symmetric 2-tensor U and vector field V implies that g quasi-converges to h if and only if for all $t \geq t_\varepsilon$,

$$(1-\varepsilon)\ h(V,V) \leq g(V,V) \leq (1+\varepsilon)\ h(V,V).$$

We now state our result, using notation defined in [1] and to be reviewed in §2 below.

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1.2 Theorem *If g is any* solv*-Gowdy metric on a twisted torus bundle M^3_Λ, there is a locally homogeneous metric h in its quasi-convergence equivalence class $[g]$. Moreover, if h corresponds to the data $(\alpha(\theta), \Omega, F)$, the locally homogeneous metrics in $[g]$ are exactly those with data $(\ell + \alpha(\theta), \Omega, F)$, $\ell \in \mathbb{R}$.*

1.3 Remark Similar quasi-convergence of the Ricci flow to a 1-parameter family was conjectured for a class of $\mathcal{T}^3$ metrics studied in [2].

The paper is organized as follows. §2 describes the bundles $\mathcal{T}^2 \to \mathcal{M}^3_\Lambda \to \mathcal{S}^1$ and the solv-Gowdy metrics under study. It turns out that at large times, an arbitrary solv-Gowdy metric g behaves much like locally homogeneous metrics. §3 quantifies this observation and explicitly constructs a family h_ε of locally homogeneous metrics existing for all $t \geq 0$ which approximate g for times $t \geq t_\varepsilon$. In §4, we show that this family enjoys a certain compactness property which allows us to prove the existence part of the main theorem. The heuristic here is that g resembles a single locally homogeneous metric closely enough that the metrics h_ε are not too far apart at $t = 0$. §5 completes the main theorem by explaining the very special sort of non-uniqueness which can occur: distinct locally homogeneous metrics define distinct equivalence classes unless they differ only by a dilation of the base circle.

1.4 Acknowledgement I wish to thank Richard Hamilton for his helpful and encouraging comments.

2 Review of solv-Gowdy geometries

We begin by briefly recalling some notation and results of [1]. Readers familiar with that paper may skip this section.

To construct an arbitrary solv-Gowdy metric g, take $\Lambda \in \mathrm{SL}(2, \mathbb{Z})$ with eigenvalues $\lambda_+ > 1 > \lambda_-$. In coordinates θ, x, y on $\mathbb{R}^3$, chosen so that the x, y axes coincide with the eigenvectors of Λ, define

$$g \doteqdot e^{2A}\, d\theta \otimes d\theta + e^{F+W}\, dx \otimes dx + e^{F-W}\, dy \otimes dy, \tag{1}$$

where F is constant and A, W depend only on θ. Clearly, g descends to a metric on the product of the line and the torus $\mathcal{T}^2$. Let Λ act on $\mathbb{R} \times \mathcal{T}^2$ by $(\theta, x, y) \mapsto (\theta + 2\pi, \lambda_- x, \lambda_+ y)$. If

$$A(\theta + 2\pi) = A(\theta) \tag{2}$$

and

$$W(\theta + 2\pi) = W(\theta) + 2 \log \lambda_+, \tag{3}$$

then Λ is an isometry, and g becomes a well defined metric on the mapping torus $\mathcal{M}^3_\Lambda$, regarded as a twisted $\mathcal{T}^2$ bundle over $\mathcal{S}^1$. Notice that A governs the length of the base circle, while F and W respectively describe the scale and skew of the fibers. We denote arc length by

$$s(\theta) \doteqdot \int_0^\theta e^{A(u)} du \tag{4}$$

and set

$$Z \doteqdot \frac{\partial}{\partial s} W. \tag{5}$$

Then we can write the Ricci tensor as

$$\mathrm{Rc} = -\frac{1}{2} e^{2A} Z^2 \, d\theta \otimes d\theta - \frac{1}{2} e^{F+W} \frac{\partial Z}{\partial s} \, dx \otimes dx + \frac{1}{2} e^{F-W} \frac{\partial Z}{\partial s} \, dy \otimes dy. \tag{6}$$

The locally homogeneous solv-Gowdy metrics are easily characterized.

2.1 Lemma *A* solv-*Gowdy metric g is locally homogeneous if and only if W depends linearly on arc length.*

Proof. If g is locally homogeneous, then $R = -\frac{1}{2} Z^2$ is constant in space. Since Z is continuous, it follows that $\partial^2 W / \partial s^2 = 0$.

If Z is constant in space, let $P_0 = (\theta_0, x_0, y_0)$, $P_1 = (\theta_1, x_1, y_1)$ be points in $\mathcal{M}_\Lambda^3$. It will suffice to construct a diffeomorphism $\Phi : \mathcal{U}_0 \to \mathcal{U}_1$, where $\mathcal{U}_0$, $\mathcal{U}_1$ are neighborhoods of P_0, P_1 respectively, such that $\Phi(P_0) = P_1$ and $\Phi^* g = g$. If Φ is given in coordinates (θ, x, y) by

$$\Phi(\theta, x, y) = (\tau(\theta, x, y), \xi(\theta, x, y), \eta(\theta, x, y)),$$

the pullback condition $\Phi^* g = g$ is equivalent to the system

$$e^{2A(\theta)} = \left(\frac{\partial \tau}{\partial \theta}\right)^2 e^{2A(\tau)} + \left(\frac{\partial \xi}{\partial \theta}\right)^2 e^{F+W(\tau)} + \left(\frac{\partial \eta}{\partial \theta}\right)^2 e^{F-W(\tau)} \tag{7a}$$

$$e^{F+W(\theta)} = \left(\frac{\partial \tau}{\partial x}\right)^2 e^{2A(\tau)} + \left(\frac{\partial \xi}{\partial x}\right)^2 e^{F+W(\tau)} + \left(\frac{\partial \eta}{\partial x}\right)^2 e^{F-W(\tau)} \tag{7b}$$

$$e^{F-W(\theta)} = \left(\frac{\partial \tau}{\partial y}\right)^2 e^{2A(\tau)} + \left(\frac{\partial \xi}{\partial y}\right)^2 e^{F+W(\tau)} + \left(\frac{\partial \eta}{\partial y}\right)^2 e^{F-W(\tau)}. \tag{7c}$$

Note that $s(\theta)$ is invertible, because $\partial s / \partial \theta = e^{A(\theta)} > 0$, and define

$$\begin{aligned} \tau(\theta, x, y) &= s^{-1}(s(\theta) + s(\theta_1) - s(\theta_0)) \\ \xi(\theta, x, y) &= x_1 + e^{-\frac{Z}{2}(s(\theta_1) - s(\theta_0))}(x - x_0) \\ \eta(\theta, x, y) &= y_1 + e^{\frac{Z}{2}(s(\theta_1) - s(\theta_0))}(y - y_0). \end{aligned}$$

Clearly, $\Phi : P_0 \mapsto P_1$. Equation (7a) is satisfied, because

$$\frac{\partial \tau}{\partial \theta} = \frac{\partial \theta}{\partial s}(\tau) \cdot \frac{\partial s}{\partial \theta}(\theta) = e^{-A(\tau) + A(\theta)}.$$

To see that (7b) is satisfied, let ω denote W regarded as a linear function of arc length, so that $W(\theta) = \omega(s(\theta))$. Then we can write

$$\begin{aligned}\log\left(\left(\frac{\partial\xi}{\partial x}\right)^2 e^{W(\tau)}\right) &= -Z\cdot(s(\theta_1)-s(\theta_0))+\omega(s(\theta)+s(\theta_1)-s(\theta_0)) \\ &= \omega(s(\theta)) = W(\theta).\end{aligned}$$

Equation (7c) is verified in a similar fashion. ■

2.2 Remark When studying a single locally homogeneous solv-Gowdy metric, one can always make A constant in space by a reparameterization of $\mathcal{S}^1$; but it will not be convenient for us to do so.

If an arbitrary solv-Gowdy metric g evolves by the Ricci flow

$$\frac{\partial}{\partial t}g = -2\,\mathrm{Rc}, \tag{8}$$

we shall abuse notation and allow the quantities introduced above to depend also on time. We find that g remains a solv-Gowdy metric and that (8) is equivalent to the system

$$\frac{\partial}{\partial t}A = \frac{1}{2}Z^2 \tag{9a}$$

$$\frac{\partial}{\partial t}W = \frac{\partial}{\partial s}Z \tag{9b}$$

$$\frac{\partial}{\partial t}F = 0, \tag{9c}$$

whose solution exists for all $t \geq 0$. It is most convenient to study Z and recover A and W by integration. Z evolves by

$$\frac{\partial}{\partial t}Z = \frac{\partial^2}{\partial s^2}Z - \frac{1}{2}Z^3, \tag{10}$$

where the operator $\partial^2/\partial s^2$ plays the role of the Laplacian and evolves according to the commutator

$$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = -\frac{1}{2}Z^2\frac{\partial}{\partial s}. \tag{11}$$

For all $t \geq 0$, we identify $\mathcal{S}^1$ with the circle $x = 0$, $y = 0$ and denote its length by

$$L(t) \doteqdot \int_{\mathcal{S}^1} ds = \int_0^{2\pi} e^{A(\theta,t)}d\theta. \tag{12}$$

Notice that (3) implies the important *integral condition*

$$\int_{\mathcal{S}^1} Z\,ds = 2\log\lambda_+, \tag{13}$$

which is preserved by the flow.

If an evolving solv-Gowdy metric is locally homogeneous at $t = 0$, it remains so under the Ricci flow. For such metrics, Z is the function of time alone

$$Z(t) = \frac{1}{\sqrt{t + 1/\zeta^2}}, \tag{14}$$

where $\zeta \doteqdot Z(0)$ is positive by (13). The sub-family of locally homogeneous solv-Gowdy metrics can thus be indexed by $(\alpha(\theta), \Omega, F)$, where

$$\alpha(\theta) \doteqdot A(\theta, 0) \tag{15a}$$

$$\Omega \doteqdot W(0, 0). \tag{15b}$$

We now summarize the estimates we shall use from [1]. Let g be a solution to the Ricci flow whose initial data $g(\cdot, 0)$ is a C^2 solv-Gowdy metric. Hamilton–Isenberg organize the proof of their main theorem into four steps. In *Step 1*, they show there is $C > 0$ depending on $Z(\cdot, 0)$ such that for all $t > 0$,

$$|Z(\cdot, t)| \leq \frac{1}{\sqrt{t + C}} < \frac{1}{\sqrt{t}}. \tag{16}$$

By *Step 2,* there is a time $T > 0$ and constants $m \doteqdot Z_{\min}(T)$, $M \doteqdot Z_{\max}(T)$ depending on $L(0)$, $Z(\cdot, 0)$ and satisfying $0 < m \leq M < 1/\sqrt{T}$ such that for all $t \geq T$,

$$\frac{1}{\sqrt{t - T + 1/m^2}} \leq Z(\cdot, t) \leq \frac{1}{\sqrt{t - T + 1/M^2}}. \tag{17}$$

By *Step 1* again, there are $C, C' > 0$ depending on $L(0)$, $Z(\cdot, 0)$ such that for all $t \geq T + 1$,

$$C\sqrt{t - T} \leq L(t) \leq C'\sqrt{t - T}. \tag{18}$$

By *Step 4,* there is $C > 0$ depending on $L(0)$, $Z(\cdot, 0)$ such that for all $t \geq T$,

$$\left| \frac{\partial}{\partial s} Z(\cdot, t) \right| \leq \frac{C}{(1 + m^2(t - T))^2}. \tag{19}$$

3 Construction of approximating metrics

As a first step in proving the existence part (Theorem 4.1) of our main theorem, we find times t_ε and construct locally homogeneous metrics h_ε with the following properties: h_ε is in a sense the average of g at t_ε; h_ε remains ε-close to g for all times $t \geq t_\varepsilon$; and most importantly, h_ε exists for all $t \geq 0$.

3.1 Proposition *For any $\varepsilon > 0$, there is a time $t_\varepsilon > 0$ and a locally homogeneous* solv-*Gowdy metric h_ε evolving by the Ricci flow for $0 \leq t < \infty$ such that*

$$\sup_{\mathcal{M}^3_\Lambda \times [t_\varepsilon, \infty)} |g - h_\varepsilon|_{h_\varepsilon} < \varepsilon.$$

Before proving this, we collect some technical observations.

3.2 Lemma *For any $\varepsilon > 0$, there is $t_\varepsilon > 0$ such that Z satisfies the pinching estimate*

$$Z_{\max}(t) - Z_{\min}(t) \le \frac{\varepsilon}{L(t)}, \tag{20}$$

and the decay estimate

$$\frac{1}{\sqrt{t - t_\varepsilon + 1/m_\varepsilon^2}} \le Z(\cdot, t) \le \frac{1}{\sqrt{t - t_\varepsilon + 1/M_\varepsilon^2}}, \tag{21}$$

for all $t \ge t_\varepsilon$, where m_ε, M_ε are defined by

$$0 < m_\varepsilon \doteqdot Z_{\min}(t_\varepsilon) \le Z_{\max}(t_\varepsilon) \doteqdot M_\varepsilon < \infty \tag{22}$$

and satisfy

$$m_\varepsilon \le M_\varepsilon \le m_\varepsilon + \varepsilon \qquad \text{and} \qquad M_\varepsilon^2 \le (1+\varepsilon)\, m_\varepsilon^2. \tag{23}$$

Moreover, we can choose t_ε so that

$$\int_{t_\varepsilon}^{\infty} \left| \frac{\partial Z}{\partial s} \right| dt \le \varepsilon.$$

Proof. Let T, m, M be as in (17) and let C be the constant in (19). Let $t_* \doteqdot \max\left\{T + C/\left(m^4 \varepsilon\right), T+1\right\}$ and suppose $t \ge t_*$. Then (19) implies

$$\int_{t_*}^{\infty} \left| \frac{\partial Z}{\partial s} \right| dt \le \int_0^\infty \frac{C}{m^4 (t + t_* - T)^2}\, dt = \frac{C}{m^4 (t_* - T)} \le \varepsilon,$$

and (18) implies there is $C' > 0$ such that

$$L(t) \le C' \sqrt{t - T}.$$

Hence for such times

$$Z_{\max}(t) - Z_{\min}(t) \le \int_{S^1} \left| \frac{\partial Z}{\partial s} \right| ds \le CC' \frac{\sqrt{t-T}}{\left(1 + m^2 (t-T)\right)^2}.$$

Choose $t_\varepsilon \ge t_*$ large enough that (20) holds for $t \ge t_\varepsilon$, and that (23) holds for m_ε, M_ε defined by (22). This is possible, because

$$\left(\frac{Z_{\max}(t)}{Z_{\min}(t)} \right)^2 \le \frac{t - T + 1/m^2}{t - T + 1/M^2} \le 1 + \frac{1}{m^2 (t-T)}.$$

Then since $\frac{\partial}{\partial t} Z = \frac{\partial^2}{\partial s^2} Z - \frac{1}{2} Z^3$, we observe that

$$\frac{d}{dt} Z_{\min} \ge -\frac{1}{2} Z_{\min}^3 \qquad \text{and} \qquad \frac{d}{dt} Z_{\max} \le -\frac{1}{2} Z_{\max}^3.$$

A routine use of the maximum principle (proved in [3]) now establishes (21) for all $t \ge t_\varepsilon$. ■

3.3 Remark The proof shows that for $t \geq T+1$,

$$Z_{\max} - Z_{\min} = O\left(t-T\right)^{-3/2},$$

a result which also follows directly from (17).

3.4 Lemma *Let $\varepsilon > 0$ be given and let t_ε, m_ε, M_ε be as in Lemma 3.2. Then there is a locally homogeneous* solv-*Gowdy metric*

$$h_\varepsilon = e^{2A_\varepsilon}\, d\theta \otimes d\theta + e^{F_\varepsilon + W_\varepsilon}\, dx \otimes dx + e^{F_\varepsilon - W_\varepsilon}\, dy \otimes dy$$

evolving by the Ricci flow for $0 \leq t < \infty$ so that for $t \geq t_\varepsilon$,

$$\frac{1}{\sqrt{t - t_\varepsilon + 1/m_\varepsilon^2}} \leq Z_\varepsilon\left(t\right) \leq \frac{1}{\sqrt{t - t_\varepsilon + 1/M_\varepsilon^2}},$$

where $Z_\varepsilon = \frac{\partial W_\varepsilon}{\partial s_\varepsilon} = e^{-A_\varepsilon} \frac{\partial W_\varepsilon}{\partial \theta}$. Moreover, h_ε is constructed so that for all $\theta \in \mathcal{S}^1$, $A_\varepsilon\left(\theta, t_\varepsilon\right) = A\left(\theta, t_\varepsilon\right)$ and $\left|W\left(\theta, t_\varepsilon\right) - W_\varepsilon\left(\theta, t_\varepsilon\right)\right| \leq \varepsilon$.

Proof. Define

$$Z_\varepsilon\left(t\right) \doteqdot \frac{1}{\sqrt{t + \left(1/\zeta_\varepsilon^2 - t_\varepsilon\right)}}, \tag{24}$$

where

$$\zeta_\varepsilon \doteqdot \int_{\mathcal{S}^1} Z\, ds \Big/ \int_{\mathcal{S}^1} ds, \tag{25}$$

with the RHS evaluated at t_ε. Observe that Z_ε is well defined for all $t \geq 0$, because $\left|Z\left(t\right)\right| < 1/\sqrt{t}$ by (16), whence

$$1/\zeta_\varepsilon^2 - t_\varepsilon \geq 1/Z_{\max}^2\left(t_\varepsilon\right) - t_\varepsilon > 0.$$

Now recall that locally homogeneous solv-Gowdy metrics form a 3-parameter family and define

$$\alpha_\varepsilon\left(\theta\right) \doteqdot A\left(\theta, t_\varepsilon\right) - \frac{1}{2}\int_0^{t_\varepsilon} Z_\varepsilon^2\, dt \tag{26a}$$

$$\Omega_\varepsilon \doteqdot W\left(0, t_\varepsilon\right) \tag{26b}$$

$$F_\varepsilon \doteqdot F. \tag{26c}$$

Notice that h_ε is well defined; indeed, the identities

$$2\log\lambda_+ = \int_{\mathcal{S}^1} Z\, ds = \zeta_\varepsilon \int_{\mathcal{S}^1} ds = \int_{\mathcal{S}^1} \zeta_\varepsilon e^{A_\varepsilon}\, d\theta = \int_{\mathcal{S}^1} Z_\varepsilon\, ds_\varepsilon$$

show that the integral condition (13) is satisfied at t_ε, hence for all time.

The first assertion of the lemma is verified by the elementary observation

$$m_\varepsilon = Z_{\min}\left(t_\varepsilon\right) \leq \zeta_\varepsilon \leq Z_{\max}\left(t_\varepsilon\right) = M_\varepsilon,$$

which follows from (25). The second assertion is trivial; to prove the third, simply notice that

$$|W(\theta,t_\varepsilon)-W_\varepsilon(\theta,t_\varepsilon)|\le\int_{\mathcal{S}^1}|Z-\zeta_\varepsilon|\,ds\le(Z_{\max}-Z_{\min})(t_\varepsilon)\cdot L(t_\varepsilon)\le\varepsilon.$$

■

Proof of Proposition 3.1. Without loss of generality, assume $0<\varepsilon\le 1/6$.

Let $t\ge t_\varepsilon$ and observe that

$$\begin{aligned}|(A-A_\varepsilon)(\theta,t)|&=\frac{1}{2}\left|\int_{t_\varepsilon}^{t}\left(Z^2-Z_\varepsilon^2\right)(\theta,\tau)\,d\tau\right|\\&\le\frac{1}{2}\int_{t_\varepsilon}^{t}\left(\frac{1}{\tau-t_\varepsilon+1/M_\varepsilon^2}-\frac{1}{\tau-t_\varepsilon+1/m_\varepsilon^2}\right)d\tau\\&=\log\sqrt{\frac{1+M_\varepsilon^2(t-t_\varepsilon)}{1+m_\varepsilon^2(t-t_\varepsilon)}}.\end{aligned}$$

Then since $|e^u-1|\le e^U-1$ when $|u|\le U$, we have

$$\left|\left(e^{2A}-e^{2A_\varepsilon}\right)(\theta,t)\right|=e^{2A_\varepsilon}\left|e^{2(A-A_\varepsilon)}-1\right|\le e^{2A_\varepsilon}\frac{M_\varepsilon^2-m_\varepsilon^2}{m_\varepsilon^2}$$

and hence

$$\left((h_\varepsilon)^{\theta\theta}\right)^2\left(g_{\theta\theta}-(h_\varepsilon)_{\theta\theta}\right)^2\le\varepsilon^2.$$

Because W_ε is constant in time, we have

$$\begin{aligned}|(W-W_\varepsilon)(\theta,t)|&\le|W(\theta,t)-W(\theta,t_\varepsilon)|+|W(\theta,t_\varepsilon)-W_\varepsilon(\theta,t_\varepsilon)|\\&\le\left|\int_{t_\varepsilon}^{t}\frac{\partial Z}{\partial s}\,d\tau\right|+\varepsilon\\&\le 2\varepsilon,\end{aligned}$$

whence substituting $\delta=2\varepsilon\le 1/3$ in the crude estimate $e^\delta\le 1+\delta+\frac{e}{2}\delta^2$ (which holds for $0\le\delta\le 1$) gives

$$\left|\left(e^{F+W}-e^{F_\varepsilon+W_\varepsilon}\right)(\theta,t)\right|=e^{F_\varepsilon+W_\varepsilon}\left|e^{(W-W_\varepsilon)}-1\right|\le 3\varepsilon e^{F_\varepsilon+W_\varepsilon}$$

and thus

$$\left((h_\varepsilon)^{xx}\right)^2\left(g_{xx}-(h_\varepsilon)_{xx}\right)^2\le 9\varepsilon^2.$$

The estimate for $\left((h_\varepsilon)^{yy}\right)^2\left(g_{yy}-(h_\varepsilon)_{yy}\right)^2$ is entirely analogous. We have shown that

$$|g-h_\varepsilon|_{h_\varepsilon}^2=(h_\varepsilon)^{ac}(h_\varepsilon)^{bd}\left(g_{ab}-(h_\varepsilon)_{ab}\right)\left(g_{cd}-(h_\varepsilon)_{cd}\right)\le 19\varepsilon^2$$

for $t\ge t_\varepsilon$, which is clearly equivalent to the desired result. ■

4 Existence

We have seen that for any $\varepsilon > 0$, there is a natural choice h_ε of locally homogeneous metric approximating g for times $t \geq t_\varepsilon$. In view of our non-uniqueness result (Theorem 5.1), it is remarkable that these choices are close enough to one another that we can prove the existence of a locally homogeneous metric in $[g]$.

4.1 Theorem *There is a locally homogeneous* solv*-Gowdy metric h_∞ evolving by the Ricci flow for $0 \leq t < \infty$ such that for any $\varepsilon > 0$ there is a time $t_\varepsilon > 0$ with*

$$\sup_{\mathcal{M}^3_\Lambda \times [t_\varepsilon, \infty)} |g - h_\infty|_{h_\infty} < \varepsilon.$$

Again, we first obtain some preliminary results.

4.2 Lemma *Let $\{\varepsilon_j\}$ be a sequence with $\varepsilon_j \searrow 0$. For each j, let h_j denote the metric h_{ε_j} given by Proposition 3.1. Then there is a subsequence j_k and a locally homogeneous metric h_∞ with data $(\alpha_\infty(\theta), \Omega_\infty, F_\infty)$ such that*

$$(\alpha_{j_k}(\theta), \Omega_{j_k}, F_{j_k}) \to (\alpha_\infty(\theta), \Omega_\infty, F_\infty)$$

uniformly in θ. (Here, and throughout the proof, a subscript such as j denotes quantities corresponding to the metric $h_j \equiv h_{\varepsilon_j}$.)

Proof. The argument is constructed from four claims, as follows: Claim 4.3 bounds $\frac{\partial}{\partial\theta} A(\cdot, t_j)$, hence $\frac{\partial}{\partial\theta} A_j(\cdot, t_j)$ by construction, hence $\frac{\partial}{\partial\theta} A_j(\cdot, 0)$ by (27) and the local homogeneity of h_j. Combining this with Claim 4.4 proves $\{A_j(\cdot, 0)\}$ is bounded and equicontinuous. Since Claim 4.5 bounds $\frac{\partial}{\partial s_j} W_j(\cdot, 0)$, this lets us bound $\frac{\partial}{\partial\theta} W_j(\cdot, 0)$. Combining this with Claim 4.6 then proves $\{W_j(\cdot, 0)\}$ is bounded and equicontinuous. Because $F_j \equiv F$ by construction, this lets us extract a subsequence of the h_j whose initial data converge uniformly to the data of a locally homogeneous metric h_∞ existing for all $t \geq 0$.

Notice that if $j < k$, we may (and shall) assume $t_j \leq t_k$.

4.3 Claim *There is $C < \infty$ such that*

$$\sup_{\mathcal{M}^3_\Lambda \times [T, \infty)} \left| \frac{\partial A}{\partial \theta} \right| < C.$$

Compute

$$\frac{\partial}{\partial t}\left(\frac{\partial A}{\partial \theta}\right) = \frac{\partial}{\partial \theta}\left(\frac{1}{2} Z^2\right) = e^A Z \frac{\partial Z}{\partial s}. \tag{27}$$

Since by (17),

$$\frac{\partial}{\partial t} A \leq \frac{1}{2} \cdot \frac{1}{t - T + 1/M^2}$$

for $t \geq T$, there is $C' > 0$ such that

$$A(\cdot, t) \leq \log C' + \log \sqrt{t - T + 1/M^2}$$

for $t \geq T$. Then by (19), we have

$$\left| \frac{\partial}{\partial t} \left(\frac{\partial A}{\partial \theta} \right) \right| \leq C' \sqrt{t - T + 1/M^2} \frac{1}{\sqrt{t - T + 1/M^2}} \cdot \frac{C''}{\left(1 + m^2 (t - T)\right)^2}$$
$$\leq \frac{C'C''}{1 + m^4 (t - T)^2}$$

for all $t \geq T$. Since there is $B > 0$ depending only on the initial data such that $-B \leq \partial A / \partial \theta \leq B$ at $t = T$, the claim follows.

4.4 Claim *The sequence* $\{\alpha_j(\theta)\}$ *is bounded for each* $\theta \in S^1$.

Let $\theta \in \mathcal{S}^1$ be arbitrary. For $j < k$, consider

$$\alpha_j(\theta) - \alpha_k(\theta) = A(\theta, t_j) - \frac{1}{2} \int_0^{t_j} Z_j^2 \, dt - A(\theta, t_k) + \frac{1}{2} \int_0^{t_k} Z_k^2 \, dt$$
$$= \frac{1}{2} \int_{t_j}^{t_k} \left(Z_k^2 - Z^2 \right) dt + \frac{1}{2} \int_0^{t_j} \left(Z_k^2 - Z_j^2 \right) dt.$$

Since $1/\zeta_k^2 - t_k \geq 1/M_j^2 - t_j$, we obtain a familiar estimate for the first integral:

$$\frac{1}{2} \left| \int_{t_j}^{t_k} \left(Z_k^2 - Z^2 \right) dt \right| \leq \log \sqrt{\frac{1 + M_j^2 (t_k - t_j)}{1 + m_j^2 (t_k - t_j)}} \leq \log \sqrt{1 + \varepsilon_j}.$$

Write the second integral as

$$\frac{1}{2} \int_0^{t_j} \left(Z_k^2 - Z_j^2 \right) dt = \log \sqrt{\frac{1/\zeta_j^2 - t_j}{1/\zeta_k^2 - t_k}} + \log \sqrt{\frac{t_j + (1/\zeta_k^2 - t_k)}{1/\zeta_j^2}}$$
$$= \log \sqrt{P_{jk}},$$

where

$$P_{jk} \doteqdot \left(1 - \zeta_j^2 t_j\right) \left(1 + \frac{t_j}{1/\zeta_k^2 - t_k}\right) > 0. \tag{28}$$

Since

$$\frac{1/M^2 - T}{t_j + 1/M^2 - T} \leq 1 - \zeta_j^2 t_j \leq \frac{1/m^2 - T}{t_j + 1/m^2 - T}$$

and

$$\frac{t_j + 1/m^2 - T}{1/m^2 - T} \leq 1 + \frac{t_j}{1/\zeta_k^2 - t_k} \leq \frac{t_j + 1/M^2 - T}{1/M^2 - T},$$

we conclude that

$$\frac{1/M^2 - T}{1/m^2 - T} \leq P_{jk} \leq \frac{1/m^2 - T}{1/M^2 - T}.$$

4.5 Claim *There are* $0 < Z_* \leq Z^* < \infty$ *such that* $Z_j(0) \in [Z_*, Z^*]$ *for all* j.

Note how

$$1/Z_j^2(0) = 1/\zeta_j^2 - t_j \geq 1/Z_{\max}^2(t_j) - t_j \geq 1/M^2 - T > 0$$

by (16) and (17), and similarly

$$1/Z_j^2(0) = 1/\zeta_j^2 - t_j \leq 1/Z_{\min}^2(t_j) - t_j \leq 1/m^2 - T < \infty.$$

4.6 Claim *There are $\Omega_* \leq \Omega^*$ such that $\Omega_j \in [\Omega_*, \Omega^*]$ for all j.*

Suppose $j < k$. Then since $\Omega_j \doteqdot W(0, t_j)$, we have

$$|\Omega_k - \Omega_j| = |W(0, t_k) - W(0, t_j)| \leq \int_{t_j}^{t_k} \left| \frac{\partial W}{\partial t} \right| dt = \int_{t_j}^{t_k} \left| \frac{\partial Z}{\partial s} \right| dt \leq \varepsilon_j.$$

■

4.7 Lemma *If h_∞ is a locally homogeneous metric with data $(\alpha_\infty(\theta), \Omega_\infty, F)$ and $\{h_j\}$ is a sequence of locally homogeneous metrics with data $(\alpha_j(\theta), \Omega_j, F)$ converging to $(\alpha_\infty(\theta), \Omega_\infty, F)$ uniformly in θ, then for any $\varepsilon > 0$ there is J_ε such that for each $j \geq J_\varepsilon$*

$$\sup_{\mathcal{M}_\Lambda^3 \times [0,\infty)} |h_j - h_\infty|_{h_\infty} < \varepsilon.$$

Proof. The integral condition

$$\int_{\mathcal{S}^1} Z_\infty(0)\, e^{\alpha_\infty(\theta)}\, d\theta = 2 \log \lambda_+ = \int_{\mathcal{S}^1} Z_j(0)\, e^{\alpha_j(\theta)}\, d\theta$$

shows that $Z_j(0) \to Z_\infty(0)$. For $\delta > 0$ to be determined, choose J_ε large enough that

$$\sup_{\theta \in \mathcal{S}^1} |\alpha_\infty(\theta) - \alpha_j(\theta)| \leq \delta \qquad \text{and} \qquad \left| \frac{Z_\infty^2(0)}{Z_j^2(0)} - 1 \right| \leq \delta$$

for all $j \geq J_\varepsilon$, and consider

$$(A_\infty - A_j)(\theta, t) = (\alpha_\infty - \alpha_j)(\theta) + \frac{1}{2} \int_0^t \left(Z_\infty^2 - Z_j^2\right) dt.$$

For any $\lambda, \mu > 0$ we have the now-familiar inequality

$$\log\left(1 - \frac{|\mu - \lambda|}{\lambda}\right) \leq \int_0^t \left(\frac{1}{t+\lambda} - \frac{1}{t+\mu}\right) dt \leq \log\left(1 + \frac{|\mu - \lambda|}{\lambda}\right).$$

Since

$$\frac{1}{2} \int_0^t \left(Z_\infty^2 - Z_j^2\right) dt = \frac{1}{2} \int_0^t \left(\frac{1}{t + 1/Z_\infty^2(0)} - \frac{1}{t + 1/Z_j^2(0)}\right) dt$$

and

$$\frac{\left|1/Z_j^2(0) - 1/Z_\infty^2(0)\right|}{1/Z_\infty^2(0)} \leq \delta,$$

we get our first estimate:

$$|(A_\infty - A_j)(\theta, t)| \leq \delta + \log\sqrt{1+\delta}.$$

Next observe that when $0 < \delta \leq \log 2$ we have $e^\delta \leq 1 + 2\delta$ and thus obtain our second estimate:

$$\begin{aligned}
|(W_\infty - W_j)(\theta, t)| &= |W_\infty(\theta, 0) - W_j(\theta, 0)| \\
&= \left| \int_0^\theta Z_\infty(0) \cdot e^{\alpha_\infty(u)} du - \int_0^\theta Z_j(0) \cdot e^{\alpha_j(u)} du \right| \\
&\leq \int_0^\theta Z_\infty(0) \cdot e^{\alpha_\infty(u)} \left|1 - e^{\alpha_j(u) - \alpha_\infty(u)}\right| du \\
&+ \int_0^\theta Z_j(0) \cdot e^{\alpha_j(u)} \left| \frac{Z_\infty(0)}{Z_j(0)} - 1 \right| du \\
&\leq 3\delta(2\log\lambda_+).
\end{aligned}$$

As in the proof of Theorem 3.1, it follows that we can make $|h_\infty - h_j|_{h_\infty}$ as small as desired by choosing $\delta = \delta(\varepsilon)$ appropriately. ■

Proof of Theorem 4.1. Note that $|g - h_\infty|_{h_\infty}$ will be small if both $|g - h_j|_{h_j}$ and $|h_j - h_\infty|_{h_\infty}$ are. So take the subsequence of metrics h_{j_k} and times t_{j_k} given by Lemma 4.2 and pass to a further subsequence according to Lemma 4.7. ■

5 Uniqueness

Distinct locally homogeneous solv-Gowdy metrics belong to the same equivalence class if and only if they differ merely by a dilation of arc length. In that case, we shall see that they approach one another at the rate C/t, where the constant depends on the initial difference in length of the base circle.

5.1 Theorem *Let h and h_* be locally homogeneous metrics corresponding to the data $(\alpha(\theta), \Omega, F)$ and $(\alpha_*(\theta), \Omega_*, F_*)$ respectively. If for some constant ℓ we have $\alpha_* \equiv \alpha + \ell$ and $\Omega_* = \Omega$ and $F_* = F$, then h and h_* quasi-converge with*

$$|h_* - h|_h = O\left(\frac{1}{t}\right).$$

In all other cases, there are $\delta > 0$ and $\theta \in \mathcal{S}^1$ such that

$$|h_* - h|_h(\theta, t) \geq \delta$$

for all $t > 0$, so h and h_* do not quasi-converge.

Proof. We consider three cases.

5.2 Case $\alpha_* \equiv \alpha + \ell,\ \Omega_* = \Omega,\ F_* = F.$

Writing

$$Z(t) = \frac{1}{\sqrt{t + 1/\zeta^2}} \qquad \text{and} \qquad Z_*(t) = \frac{1}{\sqrt{t + 1/\zeta_*^2}},$$

we observe that $\ell = \log(\zeta/\zeta_*)$, since by the integral condition (13) we have

$$\frac{\zeta}{\zeta_*} = \frac{\int_{\mathcal{S}^1} e^{\alpha_*(\theta)}\, d\theta}{\int_{\mathcal{S}^1} e^{\alpha(\theta)}\, d\theta} = e^{\ell}. \tag{29}$$

It follows that the function

$$\omega(\theta) \doteqdot \int_0^\theta \left(\zeta_* e^{\alpha_*(u)} - \zeta e^{\alpha(u)}\right) du \tag{30}$$

is identically zero. So for all $\theta \in \mathcal{S}^1$ and $t \geq 0$ we have

$$(W_* - W)(\theta, t) = (W_* - W)(\theta, 0) = \Omega_* - \Omega + \omega(\theta) = 0.$$

Now notice that

$$(A_* - A)(\theta, t) = (\alpha_* - \alpha)(\theta) + \frac{1}{2}\int_0^t \left(Z_*^2(\tau) - Z^2(\tau)\right) d\tau = \ell + \phi(t),$$

where

$$\phi(t) \doteqdot \frac{1}{2}\log\frac{1 + \zeta_*^2 t}{1 + \zeta^2 t}. \tag{31}$$

It is clear by (29) that $A_* - A \to 0$ uniformly in θ as $t \to \infty$. In fact, this identifies the critical rate at which distinct locally homogeneous metrics h, h_* approach each other, because

$$\left(e^{2A_*} - e^{2A}\right)(\theta, t) = e^{2A(\theta,t)}\left(e^{2(\ell + \phi(t))} - 1\right)$$

and hence

$$|h_* - h|_h = \left|h^{\theta\theta}(h_* - h)_{\theta\theta}\right| = \left|e^{2(\ell+\phi(t))} - 1\right| = \frac{\left|1/\zeta_*^2 - 1/\zeta^2\right|}{t + 1/\zeta^2}.$$

5.3 Case $\alpha_* \equiv \alpha + \ell,\ \Omega_* = \Omega,\ F_* \neq F.$

Notice that $W_* - W \equiv 0$ and $A_* - A \to 0$ as above. Without loss of generality, suppose $F_* - F = \delta > 0$. Then for all $\theta \in \mathcal{S}^1$ and $t \geq 0$ we have

$$e^{F_* + W_*} - e^{F+W} = e^{F+W}\left(e^{F_* - F} - 1\right) > \delta e^{F+W}$$

and hence

$$|h_* - h|_h \geq |h^{xx}(h_* - h)_{xx}| > \delta > 0.$$

5.4 Case *Either* $\alpha_* \not\equiv \alpha + \ell$ *or* $\Omega_* \neq \Omega$.

Observe that we can always find θ with

$$(W_* - W)(\theta, 0) = \Omega_* - \Omega + \omega(\theta) \neq 0,$$

since ω cannot be identically zero if $\alpha_* \not\equiv \alpha + \ell$. Without loss of generality, assume $(W_* - W)(\theta, 0) = \delta > 0$. Then if $F_* \geq F$, we have

$$e^{F_* + W_*(\theta,t)} - e^{F + W(\theta,t)} = e^{F + W(\theta,t)} \left(e^{F_* - F} e^{\delta} - 1\right) \geq e^{F + W(\theta,t)} \left(e^{\delta} - 1\right)$$

for all $t \geq 0$ and hence

$$|h_* - h|_h (\theta, t) \geq |h^{xx} (h_* - h)_{xx}| (\theta, t) > \delta > 0.$$

On the other hand, if $F \geq F_*$ we obtain

$$e^{F_* - W_*(\theta,t)} - e^{F - W(\theta,t)} = e^{F - W(\theta,t)} \left(e^{F_* - F} e^{-\delta} - 1\right) \leq e^{F - W(\theta,t)} \left(e^{-\delta} - 1\right)$$

for all $t \geq 0$ and thus

$$|h_* - h|_h (\theta, t) \geq \left| h^{yy} (h_* - h)_{yy} \right| (\theta, t) > \frac{\delta}{1 + \delta} > 0.$$

■

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