

SPHERE BUNDLES WITH 1/4-PINCHED FIBERWISE METRICS

THOMAS FARRELL, ZHOU GANG, DAN KNOPF, AND PEDRO ONTANEDA

ABSTRACT. We prove that all smooth sphere bundles that admit fiberwise 1/4-pinched metrics are induced bundles of vector bundles, so their structure groups reduce from $\text{DIFF}(\mathbb{S}^n)$ to $O(n+1)$. This result implies the existence of many smooth \mathbb{S}^n -bundles over \mathbb{S}^k that do not support strictly 1/4-pinched positively curved Riemannian metrics on their fibers.

CONTENTS

Introduction and statement of results	1
Part 1. Topology	2
1. Notations and definitions	2
2. Ricci flow and families of round metrics	3
3. Proof of the Main Theorem	3
Part 2. Analysis	4
4. Outline of the argument	4
5. Ricci–DeTurck flow for a sequence of background metrics	8
6. Existence and uniqueness of admissible diffeomorphisms	11
7. Estimates for the diffeomorphisms	12
8. Existence of a convergent subsequence	15
9. Long-time existence	17
References	17

INTRODUCTION AND STATEMENT OF RESULTS

Let M be a smooth manifold. By a *smooth M -bundle over a space X* , we mean a locally trivial bundle over X whose structural group is $\text{DIFF}(M)$. Here $\text{DIFF}(M)$ is the group of self-diffeomorphisms of M , with the smooth topology.

In this paper, we consider smooth sphere bundles that can be equipped with a fiberwise 1/4-pinched metric, that is, a smoothly varying Riemannian metric on each fiber with sectional curvatures in the interval $(1/4, 1]$. The purpose of this work is to call attention to the fact that such bundles are precisely $O(n)$ -bundles; hence they are the induced bundles of vector bundles. Specifically, we obtain a suitable extension to general dimensions of results due to Smale [10] for $n = 2$ and Hatcher [8] for $n = 3$:

TF and PO thank NSF for support in DMS-1206622. ZG thanks NSF for support in DMS-1308985 and DMS-1443225. DK thanks NSF for support in DMS-1205270.

Main Theorem. *For $n \geq 3$, let $E \rightarrow X$ be a smooth n -sphere bundle over the locally compact space X . If E can be equipped with a $1/4$ -pinched fiberwise metric, then E is equivalent to an $O(n+1)$ n -sphere bundle over X .*

Hence if the fibers of E admit $1/4$ -pinched metrics, then the structural group of E can be reduced from $\text{DIFF}(\mathbb{S}^n)$ to $O(n+1)$.

Corollary. Suppose n is an odd integer, k is divisible by 4, and $n \geq 3k+4$. Then it is known [4] that $\pi_{k-1}(\text{DIFF}(\mathbb{S}^n)) = \mathbb{Z} \oplus \mathbb{Z} \oplus \{\text{finite}\}$, and $\pi_{k-1}O(n+1) = \mathbb{Z}$. Consequently, the Main Theorem implies the existence of many smooth \mathbb{S}^n -bundles over \mathbb{S}^k that do not support strictly $1/4$ -pinched positively curved Riemannian metrics on their fibers. Further examples result from Theorem 3''' (p. 59) of [5]. Namely, for every pair of positive integers k and n , with $n \equiv (3-k) \pmod{4}$ and $n \geq n_1(k)$, where $n_1(k)$ is an integer depending on k , there exists such a smooth bundle.

Remark 1. As noted above, the assumption $n \geq 3$ can be dropped from the Main Theorem, because $O(n+1)$ is a deformation retract of $\text{DIFF}(\mathbb{S}^n)$ when $n = 0, 1, 2, 3$, as was proved by Smale [10] for $n = 2$, and by Hatcher [8] for $n = 3$.

The proof of the main theorem consists of two steps. First we evolve the metrics on the fibers by normalized Ricci flow (NRF) to obtain fiberwise round metrics. Note that we cannot use standard results here, because we need uniform control on how far each metric in a family moves in the orbit of its diffeomorphism group. (See Remark 2 below.) Instead, we adapt methods developed in [6, 7] for controlling coordinate parameterizations of mean curvature flow solutions. Then we use a fiberwise version of the fact that each round metric is isometric to the canonical round metric on the sphere. We explain these steps below in more detail, starting with the second.

Part 1. Topology

1. NOTATIONS AND DEFINITIONS

Let $E_1 \rightarrow X$ and $E_2 \rightarrow X$ be two smooth M -bundles over the space X . Let $f : E_1 \rightarrow E_2$ be a *fiberwise map*, meaning that f sends the fiber in E_1 over each $x \in X$ diffeomorphically onto the fiber in E_2 over x . The expression of f in trivializing charts over an open set $U \subset X$ has the form $(x, p) \mapsto (x, f(x, p))$ for all $(x, p) \in U \times M$. This determines a map $U \rightarrow \text{DIFF}(M)$.

We say that the bundles E_1, E_2 are *smoothly equivalent* (or simply *equivalent*) if there is a fiberwise map $f : E_1 \rightarrow E_2$ such that all induced maps $U \rightarrow \text{DIFF}(M)$ are continuous. The map f is a (*smooth*) *bundle equivalence*.

The space of Riemannian metrics on the smooth manifold M , with the smooth topology, will be denoted by $\text{MET}(M)$. Let $E \rightarrow X$ be a smooth M -bundle over the space X . We write $\pi : E \rightarrow X$ for the bundle projection. A (*smooth*) *fiberwise metric on E* is a collection $\{g_x\}_{x \in X}$ of Riemannian metrics on the fibers of E that varies smoothly on X . By this we mean that g_x is a Riemannian metric on the fiber $E_x = \pi^{-1}(\{x\})$, and if $\alpha : \pi^{-1}(U) \rightarrow U \times M$ is a trivializing chart, then the maps $U \rightarrow \text{MET}(M)$, $x \mapsto \alpha_* g_x$ are continuous.

We denote by $\text{MET}^{1/4 < \text{sec} \leq 1}(\mathbb{S}^n)$ the space of Riemannian metrics on the n -sphere \mathbb{S}^n with sectional curvatures in the interval $(1/4, 1]$, considered with the smooth topology. Similarly, we denote by $\text{MET}^{\text{round}}(\mathbb{S}^n)$ the space of metrics on \mathbb{S}^n with all sectional curvatures equal to 1, i.e., the space of round metrics, again with the smooth topology. The canonical round metric on \mathbb{S}^n , that is, the one induced by the inclusion $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, is denoted by σ_n , or simply σ .

2. RICCI FLOW AND FAMILIES OF ROUND METRICS

The following is a “family version” of the fact that for every $g \in \text{MET}^{\text{round}}(\mathbb{S}^n)$, there is an isometry $\phi_g : (\mathbb{S}^n, g) \rightarrow (\mathbb{S}^n, \sigma)$.

Lemma 1. *There is a continuous map*

$$\Phi : \text{MET}^{\text{round}}(\mathbb{S}^n) \longrightarrow \text{DIFF}(\mathbb{S}^n)$$

such that $\Phi(g) : (\mathbb{S}^n, g) \rightarrow (\mathbb{S}^n, \sigma)$ is an isometry.

Proof. Fix $x \in \mathbb{S}^n$ and let \mathcal{B}_σ be a σ -orthonormal basis for $T_x\mathbb{S}^n$. For any $g \in \text{MET}^{\text{round}}(\mathbb{S}^n)$, the Gram–Schmidt process produces a g -orthonormal basis \mathcal{B}_g of $T_x\mathbb{S}^n$. Moreover the map $g \mapsto \mathcal{B}_g$ is continuous. Now using the frame \mathcal{B}_g and the exponential maps \exp^σ and \exp^g in the usual way, we get an isometry $\phi : (\mathbb{S}^n, g) \rightarrow (\mathbb{S}^n, \sigma)$. We put $\phi = \Phi(g)$. It is straightforward to check that the map Φ is continuous. \square

Henceforth we write $\phi_g = \Phi(g)$.

Remark 2. Let $(\mathbb{S}^n, g_\lambda^0)$ be a continuous one-parameter family of initial data indexed by a compact set $\lambda \in \Lambda$, with $g_{\lambda_0}^0$ the canonical round metric in some fixed atlas. If all g_λ^0 have strictly 1/4-pinched positive sectional curvatures and are evolved by normalized Ricci flow (NRF), it follows from Brendle–Schoen [2] that there exist for each $\lambda \in \Lambda$ and $k \in \mathbb{N}_+$, constants $\alpha_{\lambda,k} > 0$ and $C_{\lambda,k} < \infty$ such that $\sup_{\mathbb{S}^n} |\nabla^k \text{Rm}[g_\lambda]| \leq C_{\lambda,k} e^{-\alpha_{\lambda,k} t}$, where everything is sight in measured with respect to g_λ . So there exist constants $\alpha_\lambda > 0$ and $C_\lambda < \infty$ such that

$$\left| \frac{\partial}{\partial t} g_\lambda \right| \leq C_\lambda e^{-\alpha_\lambda t}.$$

Each solution $g_\lambda(t)$ converges to a round metric g_λ^\bullet in the sense that there exist scaling factors $s_\lambda > 0$ and diffeomorphisms φ_λ such that $\lim_{t \rightarrow \infty} g_\lambda(t) = s_\lambda \varphi_\lambda^*(g_{\lambda_0}^0)$. Even though $C_{\lambda_0,k} = C_{\lambda_0} = 0$ and $\alpha_{\lambda_0,k} = \alpha_{\lambda_0} = \infty$, the natural expectation that that C_λ and α_λ are uniformly bounded for $\lambda \in \Lambda$ requires proof. Standard regularity theory says that a solution of a quasilinear parabolic equation $u_t = Q_u(u)$ depends continuously on its initial data on any time interval $[0, T_*]$ on which it remains smooth. But if the solution becomes singular at $T \leq \infty$, the estimates that establish this continuity generally deteriorate as $T_* \nearrow T$; this is not resolved by normalizing Ricci flow, which simply rescales space-time to move T to ∞ . In essence, one must show that solutions do not move arbitrarily far along the orbit of the infinite-dimensional diffeomorphism group. This is not at all addressed by standard Ricci flow geometric analysis, which only considers diffeomorphism-invariant quantities.

So the key fact that we need to prove the Main Theorem is as follows.

Theorem 1. *Let X be compact, and let $F : X \rightarrow \text{MET}^{1/4 < \text{sec} \leq 1}(\mathbb{S}^n)$ be continuous. Then the map $F^\bullet : X \rightarrow \text{MET}^{\text{sec}=1}(\mathbb{S}^n)$ induced by NRF is also continuous, where $F^\bullet(g) = (F(g))^\bullet$.*

We prove Theorem 1 in Part 2 below.

3. PROOF OF THE MAIN THEOREM

Using Theorem 1, we prove our Main Theorem as follows.

Proof of Main Theorem. Let $E \rightarrow X$ be a smooth \mathbb{S}^n -bundle. We write $\pi : E \rightarrow X$ for the bundle projection, and set $E_x = \pi^{-1}(x)$. We assume that there is a fiberwise 1/4-pinched metric $\{g_x\}_{x \in X}$ on E .

Let $\{(U_i, \alpha_i)\}_{i \in \mathcal{J}}$ be an atlas of trivializing bundle charts of E . Since X is locally compact, we can assume that all \bar{U}_i are compact. For the change of charts, we write $\alpha_{ij} : U_{ij} \times \mathbb{S}^n \rightarrow U_{ij} \times \mathbb{S}^n$, $\alpha_{ij} = \alpha_j \circ \alpha_i^{-1}$. Here $U_{ij} = U_i \cap U_j$. We write $\alpha_{ij}(x, p) = (x, \alpha_{ij}(x)(p))$. Note that

$$(1) \quad \alpha_{ij}(x) : (\mathbb{S}^n, (\alpha_i)_* g_x) \longrightarrow (\mathbb{S}^n, (\alpha_j)_* g_x)$$

is an isometry.

Now we evolve each metric g_x on E_x by Ricci flow to obtain a fiberwise metric $\{g_x^\bullet\}_{x \in X}$ on E such that each g_x^\bullet is a round metric. Note that the family $\{g_x^\bullet\}_{x \in X}$ is smooth: on each U_i , Theorem 1 ensures that the map $U_i \rightarrow \text{MET}^{\text{sec}=1}(\mathbb{S}^n)$ given by $x \mapsto (\alpha_i)_* g_x^\bullet$ is continuous. Furthermore, since Ricci flow preserves isometries, we find that all the maps $U_i \rightarrow \text{MET}^{\text{sec}=1}(\mathbb{S}^n)$ patch to give the round fiberwise metric $\{g_x^\bullet\}_{x \in X}$ on E . Define $G_i : U_i \rightarrow \text{MET}^{\text{sec}=1}(\mathbb{S}^n)$ by $G_i(x) = (\alpha_i)_* g_x^\bullet$. As in (1), we similarly obtain that

$$(2) \quad \alpha_{ij}(x) : (\mathbb{S}^n, (\alpha_i)_* g_x^\bullet) \longrightarrow (\mathbb{S}^n, (\alpha_j)_* g_x^\bullet)$$

is an isometry.

Next we apply the map Φ from Lemma 1 to each G_i to obtain $\Phi \circ G_i : U_i \rightarrow \text{DIFF}(\mathbb{S}^n)$, with

$$x \mapsto \phi_{(\alpha_i)_* g_x^\bullet}.$$

Let $f_i : U_i \times \mathbb{S}^n \rightarrow U_i \times \mathbb{S}^n$ be given by

$$(3) \quad f_i(x, v) = (x, \phi_{(\alpha_i)_* g_x^\bullet}(v)).$$

Finally for each i, j , define $\beta_{i,j} = f_j \circ \alpha_{ij} \circ f_i^{-1}$. Then, by definition, the following diagram commutes:

$$(4) \quad \begin{array}{ccc} U_{ij} \times \mathbb{S}^n & \xrightarrow{f_i} & U_{ij} \times \mathbb{S}^n \\ \alpha_{ij} \downarrow & & \downarrow \beta_{ij} \\ U_{ij} \times \mathbb{S}^n & \xrightarrow{f_j} & U_{ij} \times \mathbb{S}^n \end{array}$$

Therefore the f_i patch to give a smooth bundle equivalence $f : E \rightarrow E'$, where E' is the smooth \mathbb{S}^n -bundle constructed using the $\{\beta_{ij}\}_{i,j}$. Write $\beta_{ij}(x, p) = (x, \beta_{ij}(x)(p))$. Then, by definition of the β_{ij} (see diagram (4)) and the definition of the f_i given in (3), we have

$$(5) \quad \beta_{ij}(x) = \phi_{(\alpha_j)_* g_x^\bullet} \circ \alpha_{ij} \circ \phi_{(\alpha_i)_* g_x^\bullet}^{-1}.$$

But by Lemma 1, both maps ϕ in (5) are isometries, and the map α_{ij} appearing in the middle of (5) is the same map from equation (2), so is also an isometry. Hence

$$\beta_{ij}(x) : (\mathbb{S}^n, \sigma) \longrightarrow (\mathbb{S}^n, \sigma)$$

is an isometry. So $\beta_{ij}(x) \in O(n+1)$. This proves that the bundle E' is an $O(n+1)$ bundle. \square

Part 2. Analysis

4. OUTLINE OF THE ARGUMENT

Let \mathcal{S}_2 denote the space of smooth sections of the bundle of symmetric $(2,0)$ -tensor fields over \mathbb{S}^n , and let \mathcal{S}_2^+ denote the open convex cone of positive-definite tensor fields (i.e. Riemannian metrics). Then \mathcal{S}_2 with the C^∞ topology is a Fréchet space. There is a natural right action of the group $\mathcal{D} := \text{DIFF}(\mathbb{S}^n)$ of smooth diffeomorphisms of \mathbb{S}^n on \mathcal{S}_2^+ given by $(g, \psi) \mapsto \psi^* g$. So it is often convenient to regard \mathcal{S}_2^+ as a union of infinite-dimensional orbits \mathcal{O}_g .

Ebin's slice theorem [3] reveals that \mathcal{S}_2^+ almost has the structure of an infinite-dimensional manifold with an exponential map. More precisely, the theorem states that for any metric g ,

(i) there exists a map $\chi : \mathcal{U} \rightarrow \mathcal{D}$ of a neighborhood \mathcal{U} of g in \mathcal{O}_g such that $(\chi(\psi^*g))^*g = \psi^*g$ for all $\psi^*g \in \mathcal{U}$; and (ii) there exists a submanifold \mathcal{H} of \mathcal{S}_2^+ containing g such that the map $\mathcal{U} \times \mathcal{H} \rightarrow \mathcal{S}_2^+$ given by

$$(\psi^*g, h) \mapsto (\chi(\psi^*g))^*h$$

is a diffeomorphism onto a neighborhood of g in \mathcal{S}_2 . Infinitesimally, there is an orthogonal decomposition $T_g\mathcal{S}_2 = \mathcal{V}_g \oplus \mathcal{H}_g$, where \mathcal{V}_g is the image of the Lie derivative map (i.e., infinitesimal diffeomorphisms) and \mathcal{H}_g is the kernel of the divergence map.

4.1. The first step. Let Ξ be a compact set of strictly $\frac{1}{4}$ -pinched metrics on \mathbb{S}^n , ($n \geq 3$), and let $\omega_n = 2\pi^{(n+1)/2}/\Gamma((n+1)/2)$ denote the volume of the canonical round metric σ induced by the embedding $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. Noting that the map

$$\nu : g \mapsto \frac{\omega_n}{\int_{\mathbb{S}^n} d\mu[g]} g$$

depends continuously on g , we consider the compact set $\Xi_* := \nu(\Xi)$ of volume-normalized metrics. We evolve each metric $g_0 \in \Xi_*$ by normalized Ricci flow (NRF),

$$(6a) \quad \frac{\partial}{\partial t} g = -2 \text{Rc} + \frac{2}{n} \bar{R} g,$$

$$(6b) \quad g(0) = g_0,$$

where

$$\bar{R}(t) := \int_{\mathbb{S}^n} R d\mu := \frac{\int_{\mathbb{S}^n} R d\mu}{\int_{\mathbb{S}^n} d\mu}$$

denotes the average scalar curvature of the evolving metric.

Let $\mathcal{O}_\sigma := \mathcal{D}(\sigma)$ denote the orbit of the standard metric. Work of Brendle and Schoen [2] implies that every solution of NRF originating in Ξ_* converges in the C^∞ topology to an element of \mathcal{O}_σ . Given constants $\alpha_0, \dots, \alpha_\ell > 0$, we define a neighborhood of \mathcal{O}_σ in the space of metrics on \mathbb{S}^n by

$$\Sigma_{\alpha_0, \dots, \alpha_\ell} := \left\{ \psi^*g : \psi \in \mathcal{D} \text{ and } \|\nabla^j(\text{Rm}[g] - \text{Rm}[\sigma])\|_g \leq \alpha_j \text{ for all } 0 \leq j \leq \ell \right\}.$$

The norms above are in L^∞ measured with respect to g . It follows from [2], using standard facts about continuous dependence of solutions of parabolic equations on their initial data, that there exists a time T_* depending only on Ξ_* and $\alpha_0, \dots, \alpha_\ell$ such that every solution $g(t)$ of NRF originating in Ξ_* belongs to $\Sigma_{\alpha_0, \dots, \alpha_\ell}$ for all times $t \geq T_*$.

We now derive a uniform bound on how far solutions originating in Ξ_* can move in the direction of the \mathcal{D} -action up to the finite time T_* . Indeed, because [2] implies that solutions $g(t)$ originating in Ξ_* exist for all positive time, it follows from standard short-time existence results for Ricci flow, again using continuous dependence on initial data, that there exists C_* such that $\|\text{Rm}(\cdot, t)\|_{g(t)} \leq C_*$ for all solutions $g(t)$ originating in Ξ_* and all times $0 \leq t \leq T_*$. This gives a uniform bound for the RHS of (6a). It follows, again using the slice theorem, that all metrics $g(T_*)$ belong to the set

$$\Sigma_{\alpha_0, \dots, \alpha_\ell}^C := \left\{ \psi^*g : \psi \in \mathcal{D}, \|\nabla^j(\text{Rm}[g] - \text{Rm}[\sigma])\|_g \leq \alpha_j, \|\psi - \text{id}\|_{L^\infty} \leq C \right\},$$

and that $g(T_*)$ depends continuously on $g(0) = g_0 \in \Xi_*$.

4.2. The second step. To control how far solutions originating in Ξ_* can move in the direction of the \mathcal{D} -action for all time, we craft a modified linearization argument. Note that while $\Sigma_{\alpha_0, \dots, \alpha_k}$ is a neighborhood of the set \mathcal{O}_σ of stationary solutions of NRf , that set is infinite-dimensional, and standard stability theorems do not function in the presence of an infinite-dimensional center manifold.

The fact that Ricci flow is not strictly parabolic, again due to its invariance under the \mathcal{D} -action, is another obstacle to using standard linearization techniques. One obtains parabolicity by fixing a gauge, using the well-known DeTurck trick. While this remedies one problem, it creates another: the gauge-fixed flow introduces instabilities in the (formerly invisible) direction of the \mathcal{D} -action. We describe below how we remedy this.

If \hat{g} is a chosen element of \mathcal{O}_σ with Levi-Civita connection $\hat{\Gamma}$, one defines a 1-parameter family of vector fields \hat{W} on (\mathbb{S}^n, g) by

$$\hat{W}^k = g^{ij} \hat{A}_{ij}^k,$$

where \hat{A} is the tensor field $\Gamma - \hat{\Gamma}$. The decorations indicate that \hat{W} and \hat{A} depend on our choice of gauge $\hat{g} \in \mathcal{O}_\sigma$. The normalized Ricci–DeTurck flow (NRDF) is the strictly parabolic system

$$(7) \quad \frac{\partial}{\partial t} g = -2 \text{Rc} + \mathcal{L}_{\hat{W}}(g) + \frac{2}{n} \bar{R}g,$$

where \mathcal{L} denotes the Lie derivative.

We henceforth consider NRDF with initial data $g(T_*) \in \Sigma_{\alpha, \beta_0, \dots, \beta_k}^C$. By a slight abuse of notation, we continue to denote the evolving metrics by $g(t)$.

Standard variational formulas imply that the linearization of (7) at \hat{g} is the autonomous, self-adjoint, strictly parabolic system

$$(8) \quad \frac{\partial}{\partial t} h = \Delta_\ell h + 2(n-1) \left\{ h - \frac{1}{n} \bar{H}g \right\},$$

where $H = g^{ij} h_{ij}$ is the trace of the perturbation h , and $\bar{H} = \int_{\mathbb{S}^n} H d\mu$ is its average over the manifold. The operator Δ_ℓ denotes the Lichnerowicz Laplacian acting on symmetric $(2, 0)$ -tensor fields. In coordinates,

$$\Delta_\ell h_{ij} = \Delta_2 h_{ij} + 2\hat{R}_{ipqj} h^{pq} - \hat{R}_i^k h_{kj} - \hat{R}_j^k h_{ik},$$

where Δ_2 is the rough Laplacian acting on $(2, 0)$ -tensors, $\Delta_2 h_{ij} = g^{pq} \nabla_p \nabla_q h_{ij}$.

Let h° denote the trace-free part of h , defined by

$$h^\circ = h - \frac{1}{n} Hg.$$

Then, using the structure of the curvature operator $\hat{\text{Rm}}$, one can rewrite (8) as

$$(9) \quad \frac{\partial}{\partial t} h = Lh := \Delta_2 h - 2h^\circ + 2 \frac{n-1}{n} (H - \bar{H})g.$$

To proceed, we decompose the operator L defined in (9) as

$$Lh = \frac{1}{n} L_0 H + L_2 h^\circ,$$

where

$$L_0 H = \Delta_0 H + 2(n-1)(H - \bar{H}),$$

$$L_2 h^\circ = \Delta_2 h^\circ - 2h^\circ.$$

Here Δ_0 is the Laplace–Beltrami operator acting on scalar functions.

It is evident that L_2 is strictly stable. Using the L^2 inner product shows that

$$(L_0H, H) = -\|\nabla H\|^2 + 2(n-1)(\|H\|^2 - \omega_n \bar{H}^2).$$

So the spectrum of L_0 is bounded from above by that of $\Delta_0 + 2(n-1)$. Recall that the spectrum of $-\Delta_0$ is $\{j(n+j-1)\}_{j \geq 0}$. The eigenspace Φ_j consists of the restriction to $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ of j -homogeneous polynomials in the coordinate functions (x^1, \dots, x^{n+1}) . Hence Φ_0 consists of constant functions. And if H belongs to Φ_1 , then one has $\bar{H} = 0$ and hence

$$(L_0H, H) = \{-n + 2(n-1)\}\|H\|^2 = (n-2)\|H\|^2.$$

Because $\|h\|^2 = \frac{1}{n}\|H\|^2 + \|h^\circ\|^2$, we have proved the following. (Compare [9].)

Lemma 2. *For all $n \geq 3$, the operator $-L$ defined in (9) is unstable.*

It has a single unstable eigenvalue $2-n$ with $(n+1)$ -dimensional eigenspace consisting of infinitesimal conformal diffeomorphisms $\varphi \hat{g}$.

Its 1-dimensional null eigenspace is $\{c \hat{g} : c \in \mathbb{R}\}$.

Its remaining eigenspaces are strictly stable, with eigenvalues ≥ 4 .

For times $t \geq T_0 = T_*$, we decompose solutions $g(t)$ of NRDF as

$$(10) \quad g(t) = \left\{ a(t) + \sum_{i=1}^{n+1} b_i(t) \varphi_i \right\} \hat{g} + \check{g},$$

where φ_i are the infinitesimal conformal diffeomorphisms, and \check{g} belongs to the strictly stable spectrum of $-L$. Note that $a(t)$ is a redundant parameter: the fact that $g(t)$ has volume ω_n means it can be computed from the other elements of the decomposition. Indeed, as we now show, the parameter a is determined by \check{g} and b_k ($k = 1, 2, \dots, n+1$).

Lemma 3. *If $\|\check{g}\|$, $|a-1|$, and $\sum_k |b_k|$ are sufficiently small, then a is uniquely determined, and¹*

$$(11) \quad |a-1| \lesssim \sum_k |b_k|^2 + \|\check{g}\|_\infty^2.$$

Proof. Recall that we normalized $\text{Vol}(g) = \text{Vol}(\sigma) = \omega_n$, and that in (10), we decomposed g as

$$(12) \quad g = (a + b_i x^i) \hat{g} + \check{g},$$

where \check{g} is a chosen element of \mathcal{O}_σ , and $\check{g} \perp \{\hat{g}, x^1 \hat{g}, \dots, x^{n+1} \hat{g}\}$. Using the standard fact that $\frac{d}{d\varepsilon} \{\log \det(\hat{g} + \varepsilon h)\} = \hat{g}^{ij} h_{ij}$, we rewrite $\sqrt{\det g}$, obtaining

$$(13) \quad \sqrt{\det g} = a^{\frac{n}{2}} \sqrt{\det \hat{g}} \left\{ 1 + \frac{1}{2} \hat{g}^{-1} h + \mathcal{O}(\|h\|_\infty^2) \right\},$$

where

$$h := \frac{1}{a} \{b_i(t) x^i \hat{g} + \check{g}\}.$$

Consequently, writing $d\mu \equiv d\mu[\sigma]$, one has

$$\int \sqrt{\det g} \, d\mu = a^{\frac{n}{2}} \left\{ \int \sqrt{\det \hat{g}} \, d\mu + \frac{1}{2} \int \hat{g}^{-1} h \, d\mu + \int \mathcal{O}(\|h\|_\infty^2) \, d\mu \right\}.$$

Using the orthogonality condition $\check{g} \perp \hat{g}$ and volume normalization, this becomes

$$\omega_n = a^{\frac{n}{2}} \omega_n \{1 + \|h\|_\infty^2\},$$

which implies estimate (11).

¹We write $|F| \lesssim |G|$ if there exists a uniform constant C such that $|F| \leq C|G|$.

To see that a is unique, we regard a in (12) as an independent parameter, and prove that $|\partial_a \int \sqrt{\det g} d\mu| \geq c$ for a constant $c > 0$ provided that $|a - 1|$ is small. Specifically, computing as above, we find that

$$\frac{\partial}{\partial a} \int \sqrt{\det g} d\mu = \omega_n + \mathcal{O}\left(\sum_i |b_i| + |a - 1| + |(\hat{g})^{-1} \check{g}|\right).$$

So $\int \sqrt{\det g} d\mu$ depends almost linearly on a , provided that $\sum_i |b_i| + |a - 1| + |(\hat{g})^{-1} \check{g}|$ is small. This and the normalization $\int \sqrt{\det g} d\mu = \omega_n$ imply that a is unique. \square

Our modified linearization argument follows ideas introduced in [6] and [7]. At a sequence of times $T_\gamma \nearrow \infty$, with $T_0 = T_*$, we construct DeTurck background metrics \hat{g}_γ such that all $b_i(T_\gamma) = 0$, and all $b_i(t)$ remain small for $T_\gamma \leq t \leq T_{\gamma+1}$. We obtain estimates that prove that (i) the sequence \hat{g}_γ converges, and (ii) each solution $g(t)$ of NRF-NRDF converges to an element of \mathcal{O}_σ that depends continuously on $g_0 \in \Xi_*$. The details follow.

5. RICCI-DETURCK FLOW FOR A SEQUENCE OF BACKGROUND METRICS

In the remainder of this paper, the norms $\|\cdot\|_2$, $\|\cdot\|_\infty$, the inner product $\langle \cdot, \cdot \rangle$, and the definition of orthogonality \perp should be understood with respect to the metric σ induced on the unit sphere in \mathbb{R}^{n+1} .

In this section, we study NRF defined in (7),

$$\frac{\partial}{\partial t} g = -2 \text{Rc} + \mathcal{L}_{\hat{W}}(g) + \frac{2}{n} \bar{R}g.$$

As is well known, this equation is related to normalized Ricci flow,

$$\frac{\partial}{\partial t} \tilde{g} = -2 \text{Rc}[\tilde{g}] + \frac{2}{n} \bar{R}[\tilde{g}] \tilde{g}$$

by the relation $\phi_t^* g(t) = \tilde{g}(t)$, where ϕ_t is the one-parameter family of diffeomorphisms $\phi_t: \mathbb{S}^n \rightarrow \mathbb{S}^n$ defined by

$$(14) \quad \frac{\partial}{\partial t} \phi_t(p) = -\hat{W}(\phi_t(p), t),$$

$$(15) \quad \phi_0 = \text{Id}.$$

It is well known that $\text{Rc}[g]$ is diffeomorphism invariant. Taking advantage of this fact, we will study solutions of the system

$$(16a) \quad \frac{\partial}{\partial t} g = -2 \text{Rc} + \mathcal{L}_{\hat{W}}(g) + \frac{2}{n} \bar{R}g,$$

$$(16b) \quad g(T_0) = \phi^* g_0,$$

noting its evolution is identical to (7). Here ϕ is a diffeomorphism to be chosen. That choice will be the central topic in what follows.

We choose ϕ from an admissible class of diffeomorphisms. Intuitively, a diffeomorphism is admissible if it is ‘‘almost affine.’’ More precisely, for any vector $\vec{\epsilon} = (\epsilon_1, \dots, \epsilon_{n+1}) \in \mathbb{R}^{n+1}$, with $\|\vec{\epsilon}\|$ sufficiently small, we say a diffeomorphism $\phi(\vec{\epsilon}): \mathbb{S}^n \rightarrow \mathbb{S}^n$ is *admissible* if

$$(17) \quad \phi^*(\vec{\epsilon})g = \left(1 + \sum_{k=1}^{n+1} \epsilon_k x^k\right)g + \mathcal{O}(\|\epsilon\|^2).$$

Existence of admissible diffeomorphisms follows easily from the implicit function theorem.

Now we are ready to state the main theorem of this section. Recall that

$$(18) \quad g_0 = \left\{ a(0) + \sum_k b_k(0)x^k \right\} \hat{g} + \check{g}(0),$$

with $\check{g}(0) \perp \hat{g}$, $x^k \hat{g}$ for $k = 1, \dots, n+1$. Define a constant δ_0 by

$$(19) \quad \delta_0 := |1 - a(0)| + \sum_{k=1}^{n+1} |b_k(0)| + \sum_{|\alpha|=0}^{2n} \|\nabla^\alpha \check{g}(0)\|_2.$$

We shall prove:

Theorem 2. *There exist an element $\hat{g} \in \mathcal{O}_\sigma$ and an admissible diffeomorphism $\phi_\infty = \phi(\vec{\epsilon}_\infty)$, both depending continuously on g_0 , with $\vec{\epsilon}_\infty \in \mathbb{R}^{n+1}$ and $\|\vec{\epsilon}_\infty\| \ll 1$, such the the solution $g_\infty(t)$ of the system*

$$(20a) \quad \frac{\partial}{\partial t} g_\infty(t) = -2 \operatorname{Rc}[g_\infty(t)] + \mathcal{L}_{\hat{W}}[g_\infty(t)] + \frac{2}{n} \bar{R}[g_\infty(t)] g_\infty(t),$$

$$(20b) \quad g_\infty(0) = \phi_\infty^* g(T_0),$$

satisfies the estimate

$$(21) \quad \|g_\infty(t) - \hat{g}\|_\infty \leq C \delta_0 e^{-\frac{3}{2}t}$$

for some $C < \infty$, and thus has the property that

$$(22) \quad \hat{g} = \lim_{t \rightarrow \infty} g_\infty(t) \quad \text{in } L^\infty.$$

We will divide the proof into several steps that appear below. We note that Theorem 1 follows easily from Theorem 2 and the estimates obtained in its proof.

The general strategy is to approximate $\vec{\epsilon}_\infty$ by finding an ‘‘optimal’’ admissible diffeomorphism, associated with $\vec{\epsilon}_{T_\gamma}$, for each time T_γ , and proving that at least a subsequence of $\{\vec{\epsilon}_{T_\gamma}\}$ is Cauchy. Then we define $\vec{\epsilon}_\infty$ as the limit of that subsequence.

For simplicity of notation and where no confusion will result, we henceforth ignore the index γ , making the identifications $0 = T_\gamma$, $T = T_{\gamma+1}$, and writing $g(t, T) = g_T(t) = g_{T_\gamma}(t)$. Thus we consider the system

$$(23a) \quad \frac{\partial}{\partial t} g(t, T) = -2 \operatorname{Rc}[g(t, T)] + \mathcal{L}_{\hat{W}}[g(t, T)] + \frac{2}{n} \bar{R}[g(t, T)] g(t, T),$$

$$(23b) \quad g(0, T) = \phi_T^* g_0,$$

in a time interval $t \in [0, T]$. In Section 6, we prove the following result.

Lemma 4. *There exists a (possibly small) constant $M \in (0, \infty]$ such that for any $T \in [0, M]$ (or $[0, \infty)$ if $M = \infty$), there exists a unique admissible diffeomorphism $\phi_T = \phi(\vec{\epsilon}_T): \mathbb{S}^n \rightarrow \mathbb{S}^n$ such that for any time $t \in [0, T]$, the solution of (23a) with initial condition (23b) decomposes as*

$$(24) \quad g(t, T) = \left\{ a(t, T) + \sum_k b_k(t, T)x^k \right\} \hat{g} + \check{g}(t, T),$$

with

$$b_k(T, T) = 0 \quad \text{for all } k = 1, \dots, n+1,$$

and for all $t \in [0, T]$,

$$(25) \quad \check{g}(t, T) \perp \hat{g}, \quad x^k \hat{g}.$$

Henceforth, we call the admissible diffeomorphism $\phi_T = \phi(\vec{\epsilon}_T)$ constructed in Lemma 4 the *optimal admissible diffeomorphism* at time T . In what follows, we use bootstrap arguments to derive estimates for the solution $g(t, T)$ for $t \in [0, M]$, where M is the constant introduced in Lemma 4.

By the construction of the diffeomorphism $\phi_0 = \phi(\vec{\epsilon}_0)$ at time $T = 0$, we have

$$\phi_0^* g_0 = a(0, 0)\hat{g} + \check{g}(0, 0)$$

By (17), this implies that

$$\vec{\epsilon}_0 = (b_1(0), b_2(0), \dots, b_{n+1}(0)) + \mathcal{O}(\sum_{k=1}^{n+1} b_k^2(0)).$$

Furthermore, because $\check{g}(0, 0) \in \Sigma_{\alpha_0, \dots, \alpha_t}^C$, we may assume its Sobolev norms satisfy

$$(26) \quad \sum_{|\alpha|=0}^{2n} \|\nabla^\alpha \check{g}(0, 0)\|_2 \leq 2\delta_0,$$

where δ_0 is the constant defined in (19).

Next we make certain observations about the optimal diffeomorphisms $\phi_T = \phi(\vec{\epsilon}_T)$ whose existence is proved in Lemma 4, at least for T sufficiently small. By construction of ϕ_0 and continuity of the solution, we know that there exists a time $M_1 > 0$ such that for any time $T \in [0, M_1]$, the vector $\vec{\epsilon}_T$ satisfies

$$(27) \quad \|\vec{\epsilon}_T\| \leq \delta_0^{\frac{3}{4}},$$

where δ_0 is the constant in (19). This gives some control on the initial condition,

$$\phi_T^* g_0 = \left\{ a(0, T) + \sum_{k=1}^{n+1} b_k(0, T)x^k \right\} \hat{g} + \check{g}(0, T),$$

where $\check{g}(0, T) \perp \hat{g}$, $x^k \hat{g}$, and the Sobolev norms of $\check{g}(0, T)$ satisfy

$$(28) \quad \sum_{|\alpha|=0}^{2n} \|\nabla^\alpha \check{g}(0, T)\|_2 \leq 2\delta_0.$$

For the evolving solution

$$g(t, T) = \left\{ a(t, T) + \sum_{k=1}^{n+1} b_k(t, T)x^k \right\} \hat{g} + \check{g}(t, T),$$

we have the following estimates.

Lemma 5. *In the time interval $t \in [0, T]$, with $T \leq M_1$, if we have*

$$(29) \quad \sum_{|\alpha| \leq 2n} \|\nabla^\alpha \check{g}(t, T)\|_2 \leq \delta_0^{\frac{2}{3}} e^{-\frac{2}{3}t},$$

then there exists a constant c_1 , independent of t and T , such that

$$(30a) \quad |b_k(t, T)| \leq c_1 e^{-t} \delta_0^2, \quad (k = 1, \dots, n+1);$$

$$(30b) \quad \sum_{|\alpha| \leq 2n} \|\nabla^\alpha \check{g}(t, T)\|_2 \leq c_1 e^{-t} \delta_0;$$

$$(30c) \quad \|\vec{\epsilon}_T\| \leq \delta_0^{\frac{9}{10}}.$$

Note that (26) implies that (29) holds at $t = 0$. The lemma is proved in Section 7.

For our bootstrap argument to work, it is vital that estimate (30b) improves estimates (27) and (29). Hence by iterating the argument above, we can conclude that Lemma 5 holds for the maximal time interval in which an optimal admissible diffeomorphism exists, which is $[0, M]$, where M appears in Lemma 4.

Recall that at a sequence of times $T \leq M$, we choose optimal diffeomorphisms ϕ_T and study the Ricci–DeTurck flow (23a) with initial condition (23b). Now we compare various $\phi_T^* g_0$ for $T \leq M$. Our conclusion is that if $M = \infty$, then a subsequence of $\{\phi_{T_\gamma}^* g_0\}_{\gamma=0}^\infty$ is Cauchy.

Lemma 6. *For any $T_1, T_2 \geq 0$ with $T_2 \geq T_1$, one has*

$$\|\check{g}(t, T_2) - \check{g}(t, T_1)\|_\infty + \|\phi_{T_2}^* g_0 - \phi_{T_1}^* g_0\|_\infty \lesssim \delta_0^2 e^{-T_1}.$$

The lemma is proved in Section 8.

Next we show that the results above, specifically Lemmas 4–6, hold for $t \in [0, \infty)$. This follows from our next result, which implies that the maximal value of M is ∞ .

Lemma 7. *If there exists a constant $M > 0$, such that for any $T \in [0, M]$, there exists an optimal admissible diffeomorphism $\phi_T = \phi(\vec{\epsilon}_T) : \mathbb{S}^n \rightarrow \mathbb{S}^n$, then there exists a constant $\delta = \delta(M) > 0$ such that for each $T_1 \in [M, M + \delta]$, there exists a unique $\vec{\epsilon}_{T_1} \in \mathbb{R}^{n+1}$ such that $\phi(\vec{\epsilon}_{T_1})$ is an optimal admissible diffeomorphism, and $\check{g}(t, T_1)$ satisfies estimate (29).*

The lemma is proved in Section 9.

Now we are ready to prove the main result of this section.

Proof of Theorem 2. The fact that the maximal value of M is ∞ enables us to use Lemma 6 to see that the sequences $\{\phi_{T_\gamma}^* g_0\}_{\gamma=0}^\infty$ and $\{g(t, T_\gamma)\}_{\gamma=0}^\infty$, for any fixed t , are Cauchy. We denote their limits by $g(0, \infty)$ and $g(t, \infty)$, respectively. By the results stated above, it is not hard to see that

$$(31) \quad \|g(t, \infty) - \hat{g}\|_\infty \leq \|g(t, \infty) - g(t, T)\|_\infty + \|g(t, T) - \hat{g}\| \lesssim \delta_0 e^{-T}.$$

By well-posedness of NRDF , $g(t, \infty)$ is the solution of

$$\frac{\partial}{\partial t} g(t, \infty) = -2 \text{Rc}[g(t, \infty)] + \mathcal{L}_{\hat{W}}[g(t, \infty)] + \frac{2}{n} \bar{R}[g(t, \infty)] g(t, \infty).$$

Finally, we prove that there exists $\vec{\epsilon}_\infty \in \mathbb{R}^{n+1}$ such that $g(0, \infty) = \phi^*(\vec{\epsilon}_\infty)g_0$. We proved that the vectors $\{\vec{\epsilon}_{T_\gamma}\}_{\gamma=0}^\infty$, associated with ϕ_{T_γ} , are uniformly bounded. Hence there exists a Cauchy subsequence, again denoted by $\{\vec{\epsilon}_{T_\gamma}\}_{\gamma=1}^\infty$, such that $\lim_{\gamma \rightarrow \infty} T_\gamma = \infty$ and $\lim_{\gamma \rightarrow \infty} \vec{\epsilon}_{T_\gamma} = \vec{\epsilon}_\infty$. This together with the fact that $\{\phi_{T_\gamma}^* g_0\}_{T=0}^\infty$ is Cauchy implies the desired result, which is

$$\phi_\infty^*(\vec{\epsilon}_\infty)g_0 = \lim_{\gamma \rightarrow \infty} \phi_{T_\gamma}^* g_0 = g(0, \infty).$$

□

6. EXISTENCE AND UNIQUENESS OF ADMISSIBLE DIFFEOMORPHISMS

Proof of Lemma 4. We first show there is a unique admissible diffeomorphism at the initial time. That is, we seek a unique ϕ_0 such that

$$\phi_0^* g_0 = a(0, 0)\hat{g} + \check{g}(0, 0),$$

with $\check{g}(0, 0) \perp \{\hat{g}, x^k \hat{g}\}$ for $k = 1, \dots, n + 1$.

For this purpose, we recall that g_0 is of the form

$$(32) \quad g_0 = \{a(0) + \sum_{k=1}^{n+1} b_k(0)x^k\}\hat{g} + \check{g}(0),$$

with $|1 - a(0)| + \sum_k |b_k(0)| \ll 1$ and $\check{g}(0) \perp \hat{g}$, $x^k \hat{g}$. As observed in Section 5, for any vector $\vec{\epsilon} \in \mathbb{R}^{n+1}$ sufficiently small, there is an admissible diffeomorphism satisfying

$$\phi^*(\vec{\epsilon})g_0 = a(0)\hat{g} + \sum_{k=1}^{n+1} \{a(0)\epsilon_k + b_k\}x^k\hat{g} + \phi^*\check{g}(0) + \mathcal{O}(\|\vec{\epsilon}\|^2 + \|\vec{\epsilon}\| \sum_k |b_k|).$$

So we may apply the implicit function theorem to choose a unique $\vec{\epsilon} \in \mathbb{R}^{n+1}$ that makes $\phi^*(\vec{\epsilon})g_0 \perp x^k \hat{g}$ for $k = 1, \dots, n+1$, which yields condition (24).

Now we show that there is a unique admissible diffeomorphism ϕ_T for each sufficiently small time T . To do so, we rewrite (23a)–(23b) in the form

$$\begin{aligned} \frac{\partial}{\partial t} g(t, T) &= -2 \operatorname{Rc}[g(t, T)] + \mathcal{L}_{\hat{w}}[g(t, T)] + \frac{2}{n} \bar{R}[g(t, T)] g(t, T), \\ g(0, T) &= \phi^*(\vec{\epsilon})g_0 = \phi_0^*g_0 + \{\phi^*(\vec{\epsilon})g_0 - \phi_0^*g_0\}. \end{aligned}$$

By local well-posedness, we have that, in the interval $t \in [0, T]$,

$$g(t, T) = g(t, 0) + [\phi^*(\vec{\epsilon})g_0 - \phi_0^*g_0] + \rho(t)$$

where $g(t, 0)$ is the solution of the Cauchy problem

$$\begin{aligned} \frac{\partial}{\partial t} g(t, 0) &= -2 \operatorname{Rc}[g(t, 0)] + \mathcal{L}_{\hat{w}}[g(t, 0)] + \frac{2}{n} \bar{R}[g(t, 0)] g(t, 0), \\ g(0, 0) &= \phi_0^*g_0, \end{aligned}$$

and where the remainder term $\rho(t)$ satisfies the estimate

$$\sum_{|\alpha|=0,1} \|\nabla^\alpha \rho(t)\|_\infty \lesssim Ct \sum_{|\alpha|=0,1,2,3} \|\nabla^\alpha [\phi^*(\vec{\epsilon})g_0 - \phi_0^*g_0]\|_\infty.$$

Hence to make $g(T, T)$ be of the desired form,

$$g(T, T) = a(T)\hat{g} + \check{g}(T, T),$$

with $\check{g} \perp \hat{g}$, $x^k \hat{g}$, we again apply the implicit function theorem. \square

7. ESTIMATES FOR THE DIFFEOMORPHISMS

Proof of Lemma 5. To derive estimates for the components b_k , $k = 1, \dots, n+1$, and \check{g} , we begin by linearizing the equation for $g(t, T)$ around \hat{g} . Define $h(t, T)$ by

$$g(t, T) = \hat{g} + h(t, T).$$

By the evolution equation (23a) satisfied by $g(t, T)$, one sees that

$$\frac{\partial}{\partial t} h(t, T) = L[h(t, T)] + N_L[h(t, T)],$$

where L is the linearized operator defined in (9) and analyzed in Lemma 2, and $N_L[h(t, T)]$ is at least second-order in h . We note that the quadratic terms in $N_L[h(t, T)]$ are contractions of the form $h * h$, $h * \nabla h$, $\nabla h * \nabla h$, and $h * \nabla \nabla h$. (For explicit formulas, see Lemma 29 of [1].)

We decompose $h(t, T)$ according to the spectrum of L , writing

$$(33) \quad h(t, T) := \left\{ [a(t, T) - 1] + \sum_{k=1}^{n+1} b_k(t, T) x^k \right\} \hat{g} + \check{g}(t, T).$$

Below, we control the various components of $h(t, T)$ in (33).

It is easy to handle the component $a(t, T) - 1$ of $h(t, T)$ in (33). Indeed, in Lemma 3, we observed that it can be controlled by estimate (11), namely

$$|a(t, T) - 1| = \mathcal{O}(|b_k(t, T)|^2 + \|\check{g}\|_\infty^2).$$

So the desired estimate for $|a(t, T) - 1|$ follows from those derived below for the remaining components.

Now we turn to the terms $b_k(t, T)$, which are given by²

$$b_k(t, T) = \frac{\langle x^k \hat{g}, h(t, T) \rangle}{\langle x^k \hat{g}, x^k \hat{g} \rangle}.$$

At $s \in [t, T]$, we apply the inner product $\langle x^k \hat{g}, \cdot \rangle$ to the evolution equation (23a), obtaining

$$\dot{b}_k(s, T) \langle x^k \hat{g}, x^k \hat{g} \rangle = (n-2)b_k(s, T) \langle x^k \hat{g}, x^k \hat{g} \rangle + \langle x^k \hat{g}, N_L[h(s, T)] \rangle$$

where the term $\langle x^k \hat{g}, N_L[h(s, T)] \rangle$ satisfies the estimate

$$(34) \quad |\langle x^k \hat{g}, N_L[h(s, T)] \rangle| \lesssim \sum_k |b_k(s, T)|^2 + \sum_{|\alpha|=0,1} \|\nabla^\alpha \check{g}(s, T)\|_2^2.$$

Observe that the right-hand side of (34) does not depend on $\nabla^\alpha \check{g}$ for $|\alpha| = 2$, even though $N_L[h]$ does. This is because when computing the inner product, we integrate by parts to move one derivative off h . Using (34) together with the fact that $b_k(T, T) = 0$, we see that for any time $t \in [0, T]$, we have

$$b_k(t, T) = - \int_t^T e^{-(n-2)(s-t)} \frac{\langle x^k \hat{g}, N_L[h(s, T)] \rangle}{\langle x^k \hat{g}, x^k \hat{g} \rangle} ds.$$

Consequently, we have

$$(35) \quad |b_k(t, T)| \lesssim \int_t^T e^{-(n-2)(s-t)} \left\{ \sum_{k=1}^{n+1} |b_k(s, T)|^2 + \sum_{|\alpha|=0,1} \|\nabla^\alpha \check{g}(s, T)\|_2^2 \|\check{g}(s, T)\|_\infty \right\} ds.$$

Next we use assumption (29) in Lemma 5, which is $\sum_{|\alpha| \leq 2n} \|\nabla^\alpha \check{g}\|_2 \leq \delta_0^{\frac{2}{3}} e^{-\frac{2}{3}t}$, and Sobolev embedding, $\|\check{g}\|_\infty \lesssim \sum_{|\alpha| \leq n} \|\nabla^\alpha \check{g}\|_2$, to obtain

$$|b_k(t, T)| \lesssim \int_t^T e^{-(n-2)(s-t)} \left\{ \sum_{k=1}^{n+1} |b_k(s, T)|^2 + \delta_0^2 e^{-2s} \right\} ds.$$

Recall that $n-2 \geq 1$. By a standard fixed point argument we find there is $c > 0$ such that

$$(36) \quad |b_k(t, T)| \leq ce^{-t} \delta_0^2,$$

which gives estimate (30a).

²Recall that the L^2 inner product $\langle \cdot, \cdot \rangle$ is computed with respect to the standard metric σ induced on the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$.

To prove estimate (30b), we analyze the component \check{g} of $h(t, T)$ in (33). By Lemma 2 and (36) above, one has

$$\begin{aligned} \frac{d}{dt} \langle \check{g}, \check{g} \rangle &= 2 \langle \check{g}, L \check{g} \rangle + 2 \langle \check{g}, N_L[h] \rangle \\ &\leq -8 \langle \check{g}, \check{g} \rangle + \mathcal{O} \left(\sum_{k=1}^{n+1} |b_k|^2 + \sum_{|\alpha|=0,1} \|\nabla^\alpha \check{g}\|_2^3 \right) \\ &\leq -8 \langle \check{g}, \check{g} \rangle + \mathcal{O} \left(\delta_0^4 e^{-2t} + \sum_{|\alpha|=0,1} \|\nabla^\alpha \check{g}\|_2^3 \right) \end{aligned}$$

Therefore,

$$\frac{d}{dt} \langle \check{g}, \check{g} \rangle \leq e^{-8t} \langle \check{g}(0), \check{g}(0) \rangle + \int_0^t e^{-8(t-s)} \mathcal{O} \left(\delta_0^4 e^{-2s} + \sum_{|\alpha|=0,1} \|\nabla^\alpha \check{g}\|_2^3 \right) ds.$$

In similar fashion, one computes that, for any $j \leq n$,

$$(37) \quad \frac{d}{dt} \langle L^j \check{g}, L^j \check{g} \rangle = 2 \langle L^j \check{g}, L^{j+1} \check{g} \rangle + 2 \langle L^j \check{g}, L^j N_L(h) \rangle.$$

For the first term on the right-hand side, we claim that for a $\kappa > 0$ to be chosen below, one has

$$(38) \quad 2 \langle L^j \check{g}, L^{j+1} \check{g} \rangle \leq -\kappa \sum_{|\ell| \leq 2j+1} \|\nabla^\ell \check{g}\|_2^2 - 3 \langle L^j \check{g}, L^j \check{g} \rangle.$$

To see this, we decompose $2 \langle L^j \check{g}, L^{j+1} \check{g} \rangle$ into two terms,

$$2 \langle L^j \check{g}, L^{j+1} \check{g} \rangle = 2\kappa \langle L^j \check{g}, L^{j+1} \check{g} \rangle + 2(1-\kappa) \langle L^j \check{g}, L^{j+1} \check{g} \rangle.$$

For the first term, we use the derivatives in L^j to get $C_1, C_2 > 0$ such that

$$2\kappa \langle L^j \check{g}, L^{j+1} \check{g} \rangle \leq -C_1 \kappa \sum_{|\alpha| \leq 2j+1} \|\nabla^\alpha \check{g}\|_2^2 + C_2 \kappa \|\check{g}\|_2^2.$$

This, together with the estimate for the second term

$$\langle L^j \check{g}, L^{j+1} \check{g} \rangle \leq -4 \langle L^j \check{g}, L^j \check{g} \rangle$$

implies the claimed estimate (38).

For the last term on the right-hand side of (37), we have to be careful about the number of derivatives on \check{g} . Integrating by parts and using estimate (36), we obtain

$$(39) \quad \begin{aligned} |\langle L^j \check{g}, L^j N_L(h) \rangle| &\lesssim \sum_{k=1}^{n+1} |b_k|^2 + \sum_{|\alpha| \leq 2j+1} \|\nabla^\alpha \check{g}\|_2^2 \left\{ \sum_{|\beta| \leq j} \|\nabla^\beta \check{g}\|_\infty + \sum_{k=1}^{n+1} |b_k| \right\} \\ &\lesssim \delta_0^4 e^{-2t} + \sum_{|\alpha| \leq 2j+1} \|\nabla^\alpha \check{g}\|_2^2 \left\{ \sum_{|\beta| \leq j} \|\nabla^\beta \check{g}\|_\infty + \delta_0^2 e^{-t} \right\}. \end{aligned}$$

To control $\sum_{|\beta| \leq j} \|\nabla^\beta \check{g}\|_\infty$ for $j \leq n$, we use Sobolev embedding along with assumption (29) in Lemma 5 to obtain

$$(40) \quad \sum_{|\beta| \leq n} \|\nabla^\beta \check{g}\|_\infty \lesssim \sum_{|\beta| \leq 2n} \|\nabla^\beta \check{g}\|_2 \leq \delta_0^{\frac{3}{2}} \ll 1.$$

Combining this with (38), and choosing $\kappa \gg \delta_0^{\frac{2}{3}}$, one gets

$$\frac{d}{dt} \sum_{j=1}^n \langle L^j \check{g}, L^j \check{g} \rangle \leq -6 \sum_{j=1}^n \langle L^j \check{g}, L^j \check{g} \rangle + \mathcal{O}(\delta_0^4 e^{-2t}).$$

This estimate immediately implies that

$$(41) \quad \sum_{j=1}^n \langle L^j \check{g}(t, T), L^j \check{g}(t, T) \rangle \leq e^{-6t} \sum_{j=1}^n \langle L^j \check{g}(0, T), L^j \check{g}(0, T) \rangle + \int_0^t e^{-6(t-s)} \mathcal{O}(\delta_0^4 e^{-2s}) ds.$$

By estimate (28) for $\check{g}(0, T)$, we find there exists $c > 0$ such that

$$\sum_{|\alpha| \leq 2n} \|\nabla^\alpha \check{g}(t, T)\|_2 \lesssim \sum_{j=1}^n \langle L^j \check{g}(t, T), L^j \check{g}(t, T) \rangle \leq c \delta_0^2 e^{-2t},$$

which yields estimate (30b).

Finally, we prove the last estimate in Lemma 5. By the construction of $b_k(0, T)$ and our estimates above for $b_k(0, T)$ and $b_k(0, 0)$, we observe that

$$\check{\epsilon}_T = \mathcal{O} \left(\sum_k |b_k(0, T) + b_k(0, 0)| \right) = \mathcal{O}(\delta_0),$$

which yields (30c). □

8. EXISTENCE OF A CONVERGENT SUBSEQUENCE

Proof of Lemma 6. The basic idea is to compare solutions $g(t, T_2)$ and $g(t, T_1)$ of the DeTurck-Ricci flow (23a) in the time interval $[0, T_1]$.

Recall that at time $t = T_1$, g_{T_2} and g_{T_1} are of the form

$$(42) \quad g(T_1, T_2) = a(T_1, T_2) + \sum_{k=1}^{n+1} b(T_1, T_2) x^k \hat{g} + \check{g}(T_1, T_2),$$

$$(43) \quad g(T_1, T_1) = a(T_1, T_1) \hat{g} + \check{g}(T_1, T_1),$$

where $\check{g}(T_1, T_1)$ and $\check{g}(T_1, T_2)$ satisfy appropriate orthogonality conditions.

We begin by deriving an evolution equation for

$$\Psi(t) := g(t, T_2) - g(t, T_1)$$

from those for $g(t, T_2)$ and $g(t, T_1)$ given in (23a), obtaining

$$(44a) \quad \frac{\partial}{\partial t} \Psi = L\Psi + R_L(\Psi),$$

$$(44b) \quad \Psi(0) = g(0, T_2) - g(0, T_1) = \phi_{T_2}^* g_0 - \phi_{T_1}^* g_0,$$

where L is the linear operator in (8),

$$L\Psi = \Delta_t \Psi + 2(n-1) \left\{ \Psi - \frac{1}{n} \bar{H}_\Psi \hat{g} \right\},$$

and R_L denotes the nonlinear “remainder term.” In the time interval $[0, T_1]$, the remainder term satisfies

$$(45) \quad \|\mathcal{R}_L(\Psi)\|_\infty \lesssim \sum_{|\alpha|=0,1} \|\nabla^\alpha \Psi\|_\infty \times \left\{ \sum_{|\beta|=0,1,2} \|\nabla^\beta (g(t, T_1) - \hat{g})\|_\infty + \sum_{|\beta|=0,1,2} \|\nabla^\beta (g(t, T_2) - \hat{g})\|_\infty \right\}.$$

Wishing to exploit the spectrum of L using Lemma 2, we decompose Ψ as

$$(46) \quad \Psi(t) = \left\{ \delta(t) + \sum_{k=1}^{n+1} \beta_k(t) x^k \right\} \hat{g} + \eta(t),$$

with

$$\eta \perp \hat{g}, x^k \hat{g} \quad \text{for } k = 1, \dots, n+1.$$

To control the various components of Ψ in (46), the key steps are to prove that the components corresponding to $\beta_k(t)$ dominate, and that their evolutions are almost linear. In what follows, we outline the proof, omitting some details that are easily established by standard methods.

To begin, we recall that $\phi_{T_m} = \phi(\vec{e}_{T_m})$ for some $\vec{e}_{T_m} \in \mathbb{R}^{n+1}$ ($m = 1, 2$). Then we observe that

$$\begin{aligned} \Psi(0) &= \phi_{T_2}^* g_0 - \phi_{T_1}^* g_0 \\ &= a_0 \{ \phi_{T_2}^* \hat{g} - \phi_{T_1}^* \hat{g} \} + \sum_{k=1}^{n+1} b_k(0) \{ \phi_{T_2}^* x^k \hat{g} - \phi_{T_1}^* x^k \hat{g} \} + \phi_{T_2}^* \check{g} - \phi_{T_1}^* \check{g} \\ &= \left\{ \omega + \sum_{k=1}^{n+1} [\epsilon_k(T_1) - \epsilon_k(T_2) + \mu_k] \right\} x^k \hat{g} + \Phi. \end{aligned}$$

Here $\omega, \mu_k \in \mathbb{R}$, and Φ satisfy the estimates

$$|\omega| + \sum_{k=1}^{n+1} |\mu_k| + \|\Phi\|_2 \lesssim \|\vec{e}(T_1) - \vec{e}(T_2)\|_2 \delta_0,$$

where $\delta_0 \in \mathbb{R}^+$ is defined in (19) as

$$\delta_0 := |1 - a(0)| + \sum_{k=1}^{n+1} |b_k(0)| + \sum_{|\alpha|=0}^{2n} \|\nabla^\alpha \check{g}(0)\|_2.$$

We recall too that g_0 decomposes as

$$g_0 = \left\{ a(0) + \sum_{k=1}^{n+1} b_k(0) x^k \right\} \hat{g} + \check{g}(0).$$

These observations show that the initial value $\Psi(0)$ is dominated by its components in the directions $x^k \hat{g}$. That is,

$$\Psi(0) \approx \sum_{k=1}^{n+1} \{ \epsilon_k(T_1) - \epsilon_k(T_2) \} x^k \hat{g}.$$

Comparing this to the decomposition of $\Psi(t)$ in (46), one sees that

$$(\beta_1(0), \dots, \beta_{n+1}(0)) \approx \vec{\epsilon}_k(T_1) - \vec{\epsilon}_k(T_2).$$

For any later time $t \in [0, T_1]$, we claim that the terms $\sum_{k=1}^{n+1} \beta_k x^k \hat{g}$ continue to dominate $\Psi(t)$. The claim follows easily from two facts. (i) The smallness of $R_L(\Psi)$ in estimate (45) means that equation (44a) is dominated by its linear term. And (ii) the spectral decomposition of L in Lemma 2 shows that the fastest growing eigenspace is spanned by β_k , $k = 1, 2, \dots, n+1$, with eigenvalue $n-2 \geq 1$. All other eigenvalues are non-positive. By standard arguments, these facts imply that in the time interval $t \in [0, T_1]$, one has

$$(\beta_1(t), \beta_2(t), \dots, \beta_{n+1}(t)) \approx e^{(n-2)t} (\beta_1(0), \beta_2(0), \dots, \beta_{n+1}(0)).$$

Combining this with the estimates for $b_k(t, T_1)$ and $b_k(t, T_2)$ proved in Lemma 5 gives the estimate in the statement of Lemma 6. \square

9. LONG-TIME EXISTENCE

Proof of Lemma 7. To extend the existence of optimal admissible diffeomorphisms after $M < \infty$, we use our estimates for the solution $g(t, M)$ of NRDF with initial condition $\phi^*(\vec{\epsilon}_M)g_0$ to find an $\vec{\epsilon}_{M+\delta} \in \mathbb{R}^{n+1}$ determining an optimal admissible diffeomorphism at time $M + \delta$, for some $\delta > 0$ sufficiently small.

As in equation (24), we decompose the solution $g(t, M)$ as

$$g(t, M) = \left\{ a(t, M) + \sum_k b_k(t, M) x^k \right\} \hat{g} + \check{g}(t, M),$$

with

$$\check{g}(t, M) \perp \hat{g}, x^k \hat{g}.$$

The estimates in Lemma 5 for $g(M, M)$ show that $g(t, M)$ can be continued past M , at least for a short time. So by the consequence of Lemma 4 that all $b_k(M, M) = 0$, and local well-posedness of NRDF , we can choose $\delta > 0$ sufficiently small such that in the time interval $[M, M + \delta]$, the quantities $\sum_k |b_k(t, M)|$ ($k = 1, \dots, n+1$) and $\|\nabla^\alpha [\hat{g}(t, M) - \hat{g}(M, M)]\|_2$ ($|\alpha| \leq 2n$) are as small as one likes. Because of this, the construction proceeds exactly as in the proof of Lemma 4. Hence we omit the details. \square

REFERENCES

- [1] **Angenent, Sigurd B.; Knopf, Dan.** Precise asymptotics of the Ricci flow neckpinch. *Comm. Anal. Geom.* **15** (2007), no. 4, 773–844.
- [2] **Brendle, Simon; Schoen, Richard.** Manifolds with 1/4-pinched curvature are space forms. *J. Amer. Math. Soc.* **22** (2009), no. 1, 287–307.
- [3] **Ebin, David G.** The manifold of Riemannian metrics. *Proc. Sympos. Pure Math. (Global Analysis)*, **15** (1968) 11–40.
- [4] **Farrell, Thomas; Hsiang, W.C.** On the rational homotopy groups of the diffeomorphism groups of discs, spheres and aspherical manifolds. *Proc. Sympos. Pure Math. XXXII* **32**, Amer. Math. Soc., Providence, R.I., 1978, 325–337.
- [5] **Farrell, Thomas; Ontaneda, Pedro.** The Teichmüller space of pinched negatively curved metrics on a hyperbolic manifold is not contractible. *Ann. of Math. (2)* **170** (2009), 45–65.
- [6] **Gang, Zhou; Knopf, Dan; Sigal, Israel Michael.** Neckpinch dynamics of asymmetric surfaces evolving by mean curvature flow. *Mem. Amer. Math. Soc.* In press. (arXiv:1109.0939v2)
- [7] **Gang, Zhou; Knopf, Dan.** Universality in mean curvature flow neckpinches. *Duke Math. J.* **164** (2015), no. 12, 2341–2406.
- [8] **Hatcher, Allen E.** A proof of the Smale conjecture, $\text{DIFF}(\mathbb{S}^3) \simeq \text{O}(4)$. *Ann. of Math. (2)* **117** (1983), no. 3, 553–607.
- [9] **Knopf, Dan.** Convergence and stability of locally \mathbb{R}^N -invariant solutions of Ricci flow. *J. Geom. Anal.* **19** (2009), no. 4, 817–846.
- [10] **Smale, Stephen.** Diffeomorphisms of the 2-sphere. *Proc. Amer. Math. Soc.* **10** (1959) 621–626.

(Thomas Farrell) TSINGHUA UNIVERSITY
E-mail address: farrell@math.tsinghua.edu.cn

(Zhou Gang) CALIFORNIA INSTITUTE OF TECHNOLOGY
E-mail address: gzhou@caltech.edu

(Dan Knopf) UNIVERSITY OF TEXAS AT AUSTIN
E-mail address: danknopf@math.utexas.edu
URL: www.ma.utexas.edu/users/danknopf

(Pedro Ontaneda) BINGHAMTON UNIVERSITY
E-mail address: pedro@math.binghamton.edu
URL: www.math.binghamton.edu/pedro