

DEGENERATE NECKPINCHES IN RICCI FLOW

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ABSTRACT. In earlier work [2], we derived formal matched asymptotic profiles for families of Ricci flow solutions developing Type-II degenerate neckpinches. In the present work, we prove that there do exist Ricci flow solutions that develop singularities modeled on each such profile. In particular, we show that for each positive integer $k \geq 3$, there exist compact solutions in all dimensions $m \geq 3$ that become singular at the rate $(T - t)^{-2+2/k}$.

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1. INTRODUCTION

While the work of Gu and Zhu [10] establishes the existence of Ricci flow solutions that form Type-II singularities, it tells us very little regarding the details of the evolving geometries of such solutions. The numerical work of Garfinkle with one of the authors [8, 9], together with the formal matched asymptotics derived by all three authors [2], strongly suggest what some of these details might be, at least for those solutions $(\mathcal{S}^{n+1}, g(t))$ that are rotationally symmetric and involve a degenerate neckpinch. However, these works do not prove that solutions with the prescribed behavior exist. In the present work, we prove that indeed, for each of the prescribed models of the evolving geometry near the singularity discussed in our work on matched asymptotics [2], there is a Ricci flow solution that asymptotically approaches this model. Specifically, we prove the main conjecture stated in our earlier work [2, §7]. It follows from the results proved here and stated below that in every dimension $n + 1 \geq 3$ and for each integer $k \geq 3$, there is a Ricci flow solution that develops a degenerate neckpinch singularity with the characteristic profile (locally) of a Bryant soliton, and with the rate of curvature blowup given by

$$(1.1) \quad \sup_{x \in \mathcal{S}^{n+1}} |\text{Rm}(x, t)| \sim \frac{C}{(T - t)^{2-2/k}},$$

for t approaching the singularity time $T < \infty$, and for some constant C .

In determining that there exist Ricci flow solutions that form Type-II singularities, Gu and Zhu [10] show that there are solutions in which the curvature of the evolving metrics satisfies $\limsup_{t \nearrow T} \{(T - t) \sup_{x \in \mathcal{S}^{n+1}} |\text{Rm}(x, t)|\} = \infty$. Their work does not, however, determine the specific rate of curvature blowup in these solutions. Enders [7] has defined *Type-A* singularities as those with curvature blowing up at the rate $(T - t)^{-r}$, with $r \in [1, \frac{3}{2})$, and has questioned whether there are any compact solutions outside this class. Our work shows that indeed there are: as noted above, we prove that there exist solutions with curvature blowing up at the rate $(T - t)^{-2+2/k}$ for all positive integers $k \geq 3$; all but the $k = 3$ solutions fall outside Enders' Type-A class of solutions.

Noting the existence of solutions with these discrete curvature blowup rates, we are led to ask if there exist (compact) solutions — degenerate neckpinch or otherwise — that exhibit blowup rates other than these discrete values. Our work does not shed light on this question. We do note that for two-dimensional Ricci flow, in which case the flow is conformal and the conformal factor evolves by logarithmic fast diffusion, $u_t = \Delta \log u$, Daskalopoulos and Hamilton [6] prove that there exist complete noncompact solutions that form singularities at the rate $(T - t)^{-2}$.

The class of metrics we consider in this work are $\text{SO}(n + 1)$ -invariant Riemannian metrics on the sphere \mathcal{S}^{n+1} . We work with Ricci flow solutions $g(t)$ for such initial data, and focus on solutions that develop a singularity at one or both of the poles at finite time T (proving that such solutions do exist). To define what it means for a singularity to be a neckpinch, we recall that a sequence $\{(x_j, t_j)\}_{j=0}^{\infty}$ of points and times in a Ricci flow solution is called a *blow-up sequence* at time T if $t_j \nearrow T$ and if $|\text{Rm}(x_j, t_j)| \rightarrow \infty$ as $j \rightarrow \infty$; such a sequence has a corresponding *pointed singularity model* if the sequence of parabolic dilation metrics $g_j(x, t) := |\text{Rm}(x_j, t_j)| g(x, t_j + |\text{Rm}(x_j, t_j)|^{-1} t)$ has a complete smooth limit. We say that a Ricci flow solution develops a *neckpinch singularity* at time T if there is some blow-up sequence at T whose corresponding pointed singularity model exists

and is given by the self-similarly shrinking Ricci soliton on the cylinder $\mathbb{R} \times \mathbb{S}^n$. We call a neckpinch singularity *nondegenerate* if every pointed singularity model of any blowup sequence corresponding to T is a cylindrical solution, and we call it *degenerate* if there is at least one blowup sequence at T with a pointed singularity model that is not a cylindrical solution.

Rotationally symmetric nondegenerate neckpinches have been studied extensively by Simon [12] and by two of the authors [3, 4]. In the latter works, it is shown that there is an open set of (rotationally symmetric) compact initial manifolds whose Ricci flows develop nondegenerate neckpinch singularities, all of which are Type-I in the sense that $\limsup_{t \nearrow T} \{(T - t) \sup_{x \in \mathbb{S}^{n+1}} |\text{Rm}(x, t)|\} < \infty$. Further, in the presence of reflection symmetry, [4] provides a detailed set of models for the asymptotic behavior of the geometry near a developing nondegenerate neckpinch, with those models collectively serving as attractors for these flows.

Unlike the formation of nondegenerate neckpinches, the formation of degenerate neckpinches in Ricci flow is expected to be an unstable property. This is evident in [8, 9] as well as in [10]: in all three of these works, one studies flows that develop degenerate neckpinches by considering one-parameter families of initial data such that for all values of the parameter above a threshold value, the Ricci flow solutions develop nondegenerate neckpinches, while for all parameter values below that value, there is no neckpinch singularity. The flows with initial data *at* the threshold value of the parameter are the ones that develop degenerate neckpinches. This instability leads us to use the somewhat indirect *Ważewski retraction method* [13] to explore the asymptotic behavior of the geometry near these degenerate neckpinches as they form. We discuss this in detail below.

The matched asymptotics derived in [2] rely heavily on the imposition of a series of *Ansatz* conditions to characterize the formal solutions of interest. (It is through these *Ansatz* conditions that one builds the formation of degenerate neckpinches into the formal solutions.) By contrast, no such *a priori* assumptions are needed (or used) in the present work. Rather, having determined in [2] the nature and the explicit approximate forms (in regions near the degenerate neckpinch) for our formal solutions, we show here without imposing any further assumptions that there exist Ricci flow solutions that approach each of the formal solutions. It follows that degenerate neckpinches form in these Ricci flow solutions.

Each of the solutions we consider here is characterized by an integer $k \geq 3$. As our results show, there is at least a one-parameter family of solutions corresponding to each value of k . In fact, our construction in Section 7 below reveals that degenerate neckpinches form in solutions starting from a set of initial data of codimension- k in the space of $\text{SO}(n + 1)$ -invariant solutions. Besides determining the rate of curvature blowup, the integer k also characterizes to an extent the detailed asymptotic behavior of the solution in a neighborhood of the singularity. While we discuss the details of this characterization below (in Section 3), we note here one important feature that depends only on the parity of k . If k is even, then the solution is reflection symmetric across the equator, neckpinch singularities occur simultaneously at both poles, and the volume of $(\mathbb{S}^{n+1}, g(t))$ approaches zero at the time of the singularity. If on the other hand k is odd, then the neckpinch occurs at one pole only, and the volume of $(\mathbb{S}^{n+1}, g(t))$ remains positive at the time of the singularity. We note that in either case, for $t < T$, the curvature has local maxima both at one of the poles and at a nearby latitude sphere \mathbb{S}^n where the neck

is maximally pinched. As t approaches the singular time T , the distance between the neckpinch sphere and the pole approaches zero, and the curvature becomes infinite simultaneously (albeit at different rates) both at the neckpinch sphere and at the pole.

A detailed statement of our main results depends on the details of the asymptotic behavior of the formal model solutions. Referring to our discussion of this behavior below in Section 3, we can state our main theorem as follows:

Main Theorem. *For every integer $k \geq 3$ and every real number $b_k < 0$, there exist rotationally symmetric Ricci flow solutions $(S^{n+1}, g(t))$ in each dimension $n+1 \geq 3$ that develop degenerate neckpinch singularities at $T < \infty$. For each choice of n, k , and b_k , the corresponding solutions have distinct asymptotic behavior.*

In each case, the singularity is Type-II — slowly forming — with

$$\sup_{x \in S^{n+1}} |\text{Rm}(x, t)| = \frac{C}{(T-t)^{2-2/k}}$$

attained at a pole, where $C = C(n, k, b_k)$.

Rescaling a solution corresponding to $\{n, k, b_k\}$ so that the distance from the pole dilates at the Type-II rate $(T-t)^{-(1-1/k)}$, one finds that the metric converges uniformly on intervals of order $(T-t)^{1-1/k}$ to the steady Bryant soliton.

Rescaling any solution so that the distance from the smallest neck dilates at the parabolic rate $(T-t)^{-1/2}$, one finds that the metric converges uniformly on intervals of order $\sqrt{T-t}$ to the shrinking cylinder soliton.

Furthermore, the solutions exhibit the precise asymptotic behavior summarized in Section 3, and they satisfy the estimates summarized in Section 4.4.

Since the formal model solutions play a major role in this work, after setting up the needed coordinates and metric representations for our analysis in Section 2, we carefully review the nature of these formal solutions and their matched asymptotic expansions in considerable detail in Section 3. While the formal solutions we discuss here are the same as those analyzed in [2], we note that here we use somewhat altered coordinate representations in certain regions near the pole. In Section 4, we describe the general structure of the proof of the main theorem and outline its key steps. The technical work to carry out these steps is detailed in Sections 5–7.

2. COORDINATES FOR THE FOUR REGIONS

As noted above, in this paper we study $\text{SO}(n+1)$ -invariant metrics g evolving by Ricci flow on $S^{n+1} \times [0, T_0)$, where $T_0 = T_0(g_0) \in (0, \infty]$ for initial data g_0 . Each such metric may be identified with functions $\varphi, \psi : (-1, 1) \times [0, T_0) \rightarrow \mathbb{R}_+$ via

$$(2.1) \quad g(x, t) = \varphi^2(x, t) (dx)^2 + \psi^2(x, t) g_{\text{can}},$$

where g_{can} is the canonical round unit-radius metric on S^n . Smoothness at the poles requires that φ, ψ satisfy the boundary condition (2.4) given below. Under Ricci flow, the quantities φ and ψ evolve by

$$\begin{aligned} \varphi_t &= n \left(\frac{\psi_{xx}}{\varphi\psi} - \frac{\varphi_x\psi_x}{\varphi^2\psi} \right), \\ \psi_t &= \frac{\psi_{xx}}{\varphi^2} - \frac{\varphi_x\psi_x}{\varphi^3} + (n-1) \frac{\psi_x^2}{\varphi^2\psi} - \frac{n-1}{\psi}, \end{aligned}$$

respectively. This system is only weakly parabolic, reflecting its invariance under the full diffeomorphism group. Below, we remedy this by suitable choices of gauge.

We are interested in proving the existence of solutions that are close to the formal solutions constructed in [2]; accordingly, we follow the notation of [2] in large measure. That paper describes solutions in four regions, the *outer*, *parabolic*, *intermediate*, and *tip*, using either of two coordinate systems. We do the same here, although we choose the alternate coordinate system to describe the intermediate and outer regions. For clarity, we review both systems below.

2.1. Coordinates for the parabolic region. We call the $\text{SO}(n+1)$ -orbit $\{0\} \times S^n$ the “equator” and denote the signed metric distance from it by

$$s(x, t) := \int_0^x \varphi(\hat{x}, t) d\hat{x}.$$

Then the metric (2.1) may be written as

$$(2.2) \quad g = (ds)^2 + \psi^2(s(x, t), t) g_{\text{can}},$$

where d denotes the spatial differential. We write

$$\left. \frac{\partial}{\partial t} \right|_x \quad \text{and} \quad \left. \frac{\partial}{\partial t} \right|_s$$

to indicate time derivatives taken with x and s held fixed, respectively. With this convention, one has the commutator

$$\left[\left. \frac{\partial}{\partial t} \right|_x, \left. \frac{\partial}{\partial s} \right] = -n \frac{\psi_{ss}}{\psi} \frac{\partial}{\partial s}$$

and the relation

$$\left. \frac{\partial}{\partial t} \right|_x = \left. \frac{\partial}{\partial t} \right|_s + nI[\psi] \frac{\partial}{\partial s},$$

where $I[\psi]$ is the nonlocal term

$$I[\psi](s, t) := \int_0^s \frac{\psi_{\hat{s}\hat{s}}(\hat{s}, t)}{\psi(\hat{s}, t)} d\hat{s}.$$

In terms of these coordinates, the evolution of the metric (2.2) by Ricci flow is determined by the scalar equation

$$(2.3) \quad \left. \frac{\partial \psi}{\partial t} \right|_x = \psi_{ss} - (n-1) \frac{1 - \psi_s^2}{\psi}.$$

Smoothness at the poles requires that ψ satisfy the boundary conditions

$$(2.4) \quad \psi_s|_{x=\pm 1} = \mp 1.$$

The quantity φ , which is effectively suppressed in these coordinates, evolves by

$$\left. \frac{\partial(\log \varphi)}{\partial t} \right|_x = n \frac{\psi_{ss}}{\psi}.$$

2.2. Coordinates for the other regions. We employ a different coordinate system in the other regions. In any region where $\psi_s \neq 0$ (e.g., in the neighborhood of the north pole $x = 1$ where $\psi_s < 0$) one may use ψ as a coordinate, writing

$$(2.5) \quad g = z(\psi, t)^{-1} (d\psi)^2 + \psi^2 g_{\text{can}},$$

where

$$z(\psi(s, t), t) := \psi_s^2(s, t).$$

The evolution of the metric (2.5) is determined by

$$(2.6) \quad \left. \frac{\partial z}{\partial t} \right|_{\psi} = \mathcal{E}_{\psi}[z],$$

where \mathcal{E}_{ψ} is the quasilinear (but purely local) operator

$$(2.7) \quad \mathcal{E}_{\psi}[z] := z z_{\psi\psi} - \frac{1}{2} z_{\psi}^2 + (n-1-z) \frac{z_{\psi}}{\psi} + 2(n-1) \frac{(1-z)z}{\psi^2}.$$

We can split the operator \mathcal{E} into a linear and a quadratic component,

$$\mathcal{E}_{\psi}[z] = \mathcal{L}_{\psi}[z] + \mathcal{Q}_{\psi}[z],$$

where

$$(2.8a) \quad \mathcal{L}_{\psi}[z] := (n-1) \left\{ \frac{z_{\psi}}{\psi} + 2 \frac{z}{\psi^2} \right\},$$

$$(2.8b) \quad \mathcal{Q}_{\psi}[z] := z z_{\psi\psi} - \frac{1}{2} z_{\psi}^2 - \frac{z z_{\psi}}{\psi} - 2(n-1) \frac{z^2}{\psi^2},$$

respectively. The quadratic part also defines a symmetric bilinear operator,

$$(2.9) \quad \hat{\mathcal{Q}}_{\psi}[z_1, z_2] := \frac{1}{2} \left\{ z_1(z_2)_{\psi\psi} + z_2(z_1)_{\psi\psi} - (z_1)_{\psi}(z_2)_{\psi} \right\} \\ - \frac{1}{2} \frac{z_1(z_2)_{\psi} + z_2(z_1)_{\psi}}{\psi} - 2(n-1) \frac{z_1 z_2}{\psi^2}.$$

In terms of this notation, one has $\mathcal{Q}_{\psi}[z] = \hat{\mathcal{Q}}_{\psi}[z, z]$.

3. THE FORMAL SOLUTION REVISITED

In [2], we present a complete formal matched asymptotic treatment of a class of rotationally symmetric Ricci flow formal solutions that form degenerate neck-pinchings. These formal solutions serve as the approximate models which the solutions we discuss here asymptotically approach. Since we find it useful in our present analysis to work with different coordinate representations (namely $z(u, \tau)$) in the intermediate region than we use for the same region in [2] (effectively $u(\sigma, t)$), we now briefly review some of the analysis of [2].

3.1. Approximate solutions in the parabolic region. Roughly speaking, the parabolic region is that portion of the manifold, away from the tip, where the geometry approaches a shrinking cylinder, and where the diameter of the neck has at least one local minimum. (We give a precise definition of this region in

equation (4.6) in Section 4.4 below.) As discussed in [2], it is useful in this region to work with coordinates consistent with a parabolic cylindrical blowup:¹

$$(3.1) \quad u := \frac{\psi}{\sqrt{2(n-1)(T-t)}}, \quad \sigma := \frac{s}{\sqrt{T-t}}, \quad \tau := -\log(T-t).$$

In terms of these coordinates, the Ricci flow evolution equation becomes²

$$(3.2) \quad \frac{\partial u}{\partial \tau} \Big|_{\sigma} = u_{\sigma\sigma} - \left(\frac{\sigma}{2} + nI[u] \right) u_{\sigma} + \frac{1}{2} \left(u - \frac{1}{u} \right) + (n-1) \frac{u_{\sigma}^2}{u},$$

with

$$(3.3) \quad I[u](\sigma, \tau) = \int_0^{\sigma} \frac{u_{\hat{\sigma}\hat{\sigma}}(\hat{\sigma}, \tau)}{u(\hat{\sigma}, \tau)} d\hat{\sigma}.$$

Setting $u = 1 + v$ and linearizing at $u = 1$, we are led to

$$\frac{\partial v}{\partial \tau} \Big|_{\sigma} = v_{\sigma\sigma} - \frac{\sigma}{2} v_{\sigma} + v + \{ \text{nonlinear terms} \}.$$

This form suggests writing the solution to this equation by expanding v in Hermite polynomials h_m , which are eigenfunctions of the linear operator

$$(3.4) \quad \mathcal{A} := \frac{\partial^2}{\partial \sigma^2} - \frac{\sigma}{2} \frac{\partial}{\partial \sigma} + 1$$

satisfying $(\mathcal{A} + \mu_m)h_m = 0$, where

$$(3.5) \quad \mu_m := \frac{m}{2} - 1.$$

Writing v in this fashion leads to the approximation

$$(3.6) \quad u \approx 1 + \sum_{m=0}^{\infty} b_m e^{-\mu_m \tau} h_m(\sigma).$$

This expansion can at best be an approximation to the actual solution, if only because the variable σ is bounded. Note that one characterization of the parabolic region is that it is the (time-dependent) range of σ for which the series terms in equation (3.6) are sufficiently small.

We are interested in solutions for which the term with $m = k$ is dominant for some specified $k \geq 3$. (Compare *Ansatz Condition 2* of [2].) Thus the approximation above takes the form

$$u = 1 + b_k e^{-\mu_k \tau} h_k(\sigma) + \dots,$$

where the sign $b_k < 0$ reflects the fact that the singularity forms at the north pole.

We normalize so that the leading term in $h_k(\sigma)$ is σ^k . We show below that, as they evolve, solutions of interest become C^1 -close to the formal solution

$$(3.7) \quad u = 1 + b_k e^{-\mu_k \tau} \sigma^k + \dots$$

in that portion of the parabolic region with $|\sigma| \gg 1$. Assuming this for now, we compute an approximation of the formal solution in terms of the quantity z introduced above. To approximate z , we differentiate the expression for u with respect to σ , leading to an expression for z in terms of σ . Using the relation

¹In Section 2, $T_0 > 0$ denotes the maximal existence time of a solution with initial data g_0 . Here, $T > 0$ denotes the singularity time of the formal solution we construct below, following [2]. In what follows, T is fixed, albeit arbitrary and unspecified.

²See equation (3.4) of [2].

between u and σ , we then write z in terms of u . The details of the calculation are as follows.

Since $\psi = u\sqrt{2(n-1)(T-t)}$ and since $s = \sigma\sqrt{T-t}$, we have

$$z = \psi_s^2 = 2(n-1)u_\sigma^2 \approx 2(n-1)k^2b_k^2e^{-(k-2)\tau}\sigma^{2k-2}$$

so long as $|\sigma| \gg 1$. On the other hand, we also have

$$\sigma \approx \left\{ \frac{1-u}{-b_k} \right\}^{1/k} e^{\gamma_k \tau},$$

where

$$(3.8) \quad \gamma_k := \frac{\mu_k}{k} = \frac{1}{2} - \frac{1}{k}.$$

Thus for $|\sigma| \gg 1$, we get

$$\begin{aligned} z &\approx 2(n-1)k^2b_k^2e^{-(k-2)\tau}\sigma^{2k-2} \\ &\approx 2(n-1)k^2b_k^2e^{-(k-2)\tau} \left(\left\{ \frac{1-u}{-b_k} \right\}^{1/k} e^{\gamma_k \tau} \right)^{2k-2} \\ &= 2(n-1)k^2(-b_k)^{2/k} e^{-2\gamma_k \tau} (1-u)^{2-2/k}. \end{aligned}$$

Hence at the interface between the parabolic and intermediate regions, where u is slightly smaller than 1, one has

$$(3.9) \quad z \approx c_k e^{-2\gamma_k \tau} (1-u)^{2-2/k},$$

with c_k defined by

$$(3.10) \quad c_k := 2(n-1)k^2(-b_k)^{2/k}.$$

3.2. Approximate solutions in the intermediate region. The intermediate region is a time-dependent subset of the neighborhood of the north pole where $-1 < u_\sigma < 0$ and $0 < u < 1$. (We provide a precise definition in (5.12) below.) By equation (2.6), we know that

$$\left. \frac{\partial z}{\partial t} \right|_\psi = \mathcal{E}_\psi[z].$$

Since $\psi = \sqrt{2(n-1)(T-t)}u = \sqrt{2(n-1)}e^{-\tau/2}u$, we obtain

$$(3.11) \quad \left. \frac{\partial z}{\partial \tau} \right|_u = \frac{1}{2(n-1)}\mathcal{E}_u[z] - \frac{1}{2}uz_u.$$

Noting the expansion (3.9) for z near $u = 1$, our first impulse is to look for approximate solutions of the form $z \approx e^{-2\gamma_k \tau} Z_1(u)$. Wishing to refine this approximation, with the goal of constructing lower and upper barriers $z_- \leq z \leq z_+$ for the intermediate region, we are led by past experience to construct a formal Taylor expansion in time, namely

$$(3.12) \quad z = e^{-2\gamma_k \tau} Z_1(u) + e^{-4\gamma_k \tau} Z_2(u) + \dots = \sum_{m \geq 1} e^{-2m\gamma_k \tau} Z_m(u).$$

We substitute this expansion into (3.11) and split $\mathcal{E}_u[z]$ into linear and quadratic parts, as in (2.8). By comparing the coefficients of $e^{-2m\gamma_k \tau}$ in the resulting equation, we find that Z_m must satisfy

$$-2m\gamma_k Z_m = \frac{1}{2(n-1)} \left\{ \mathcal{L}_u[Z_m] + \sum_{i=1}^{m-1} \hat{\mathcal{Q}}_u[Z_i, Z_{m-i}] \right\} - \frac{1}{2}u \frac{dZ_m}{du}.$$

This leads to the family of ODE

$$(3.13) \quad \frac{1}{2}(u^{-1} - u) \frac{dZ_m}{du} + (u^{-2} + 2m\gamma_k)Z_m = -\frac{1}{2(n-1)} \sum_{i=1}^{m-1} \hat{Q}_u[Z_i, Z_{m-i}],$$

from which the Z_m can be computed recursively, up to a solution of the associated homogeneous equation (obtained by replacing the RHS of equation (3.13) with zero). The general solution of the m^{th} homogeneous equation is

$$(3.14) \quad Z_{m,\text{hom}}(u) = \hat{c}_m u^{-2}(1-u^2)^{1+2m\gamma_k},$$

where \hat{c}_m is arbitrary.

For $m = 1$, the RHS of equation (3.13) vanishes. Therefore Z_1 satisfies the homogeneous equation, whereupon setting $\hat{c}_1 = c_k$ yields

$$(3.15) \quad Z_1(u) = Z_{1,\text{hom}}(u) = c_k u^{-2}(1-u^2)^{1+2\gamma_k}.$$

The function Z_2 is harder to compute (we have not tried to find a solution in closed form). However, the expression (3.15) for Z_1 shows that $Z_1 \approx c_k 2^{1+2\gamma_k}(1-u)^{1+2\gamma_k}$, $Z_1' \approx -c_k(1+2\gamma_k)2^{1+2\gamma_k}(1-u)^{2\gamma_k}$, and $Z_1'' \approx c_k(\gamma_k + 2\gamma_k^2)2^{2+2\gamma_k}(1-u)^{2\gamma_k-1}$ as $u \nearrow 1$. Thus we calculate

$$\hat{Q}_u[Z_1, Z_1] = Q_u[Z_1] \approx -c_k^2 2^{1+4\gamma_k}(1-4\gamma_k^2)(1-u)^{4\gamma_k}.$$

It follows that for some constant \hat{c} , the solution Z_2 of equation (3.13) takes the form

$$(3.16) \quad Z_2(u) = (\hat{c} + o(1))(1-u)^{4\gamma_k} \quad \text{as } u \nearrow 1.$$

The solution $Z_{2,\text{hom}}$ of the homogeneous equation is dominated by $(1-u)^{1+4\gamma_k}$, hence is smaller than the RHS of (3.16) for $u \nearrow 1$. Therefore, all solutions Z_2 of equation (3.14) satisfy (3.16).

It follows from these calculations that, near the neck, where $u \approx 1$, the first two terms in our formal expansion of z predict that

$$(3.17) \quad z = c_k e^{-2\gamma_k \tau} u^{-2}(1-u^2)^{1+2\gamma_k} + \mathcal{O}(e^{-4\gamma_k \tau}(1-u)^{4\gamma_k}).$$

Since $\gamma_k < \frac{1}{2}$, we have $1 + 2\gamma_k > 4\gamma_k$. Thus for fixed τ , the remainder term becomes larger than the leading term as $u \nearrow 1$. In [2], we characterize the intermediate region using the coordinate system (2.2); here, we identify it in terms of the system (2.5) as the region where our formal solution $e^{-2\gamma_k \tau} Z_1$ is a good approximation to the true solution. The formal solution is a good approximation only so long as the remainder is smaller than the leading term. This is true where $e^{-4\gamma_k \tau}(1-u)^{4\gamma_k} \ll e^{-2\gamma_k \tau} u^{-2}(1-u^2)^{1+2\gamma_k}$, or equivalently where $e^{-2\gamma_k \tau} \ll (1-u)^{1-2\gamma_k} = (1-u)^{2/k}$. So our approximation in the intermediate region is valid where $1-u \gg e^{-k\gamma_k \tau}$. Expansion (3.7) in the parabolic region shows that $1-u \gg e^{-k\gamma_k \tau} = e^{-(k/2-1)\tau}$ precisely when $\sigma \gg 1$.

Going in the other direction, as $u \searrow 0$, one has the expansion

$$(3.18) \quad Q_u[Z_1] \approx -2c_k^2(n-4)u^{-6} + 4c_k^2(n-3)(1+2\gamma_k)u^{-4} \\ - 2c_k^2(1+2\gamma_k)[n(1+4\gamma_k) - 10\gamma_k - 1]u^{-2}.$$

Therefore, in general dimensions, the solution Z_2 of equation (3.13) satisfies

$$(3.19) \quad Z_2(u) = \mathcal{O}(u^{-4}) \quad \text{as } u \searrow 0.$$

Near the tip, where $u \approx 0$, the first two terms in our formal expansion of z predict that

$$(3.20) \quad z = c_k e^{-2\gamma_k \tau} u^{-2} + \mathcal{O}(e^{-4\gamma_k \tau} u^{-4}).$$

This shows that our approximation in the intermediate region is expected to be valid in general only for $u \gg e^{-\gamma_k \tau}$, or equivalently for $\psi \gg e^{-(1-1/k)\tau}$, which is in agreement with equation (6.11) of [2].

3.3. Approximate solutions at the tip. Since the solutions in the intermediate region are predicted to be valid only for $u \gg e^{-\gamma_k \tau}$, we introduce the variable

$$r = e^{\gamma_k \tau} u$$

to be used in the tip region, which is defined in equation (5.14) below. Then, in view of (3.11), one has

$$\left. \frac{\partial z}{\partial \tau} \right|_r = \left. \frac{\partial z}{\partial \tau} \right|_u - \gamma_k r z_r$$

and hence

$$\left. \frac{\partial z}{\partial \tau} \right|_r = \frac{1}{2(n-1)} \mathcal{E}_u[z] - \frac{k-1}{k} r z_r = \frac{e^{2\gamma_k \tau}}{2(n-1)} \mathcal{E}_r[z] - \frac{k-1}{k} r z_r.$$

We rewrite this as $\mathcal{T}_r[z] = 0$, where

$$(3.21) \quad \mathcal{T}_r[z] := e^{-2\gamma_k \tau} \left\{ \left. \frac{\partial z}{\partial \tau} \right|_r + \frac{k-1}{k} r z_r \right\} - \frac{1}{2(n-1)} \mathcal{E}_r[z].$$

For large τ , we assume for the formal argument here that we can ignore the terms containing $e^{-2\gamma_k \tau}$, which reduces the equation to

$$\mathcal{E}_r[z] = 0.$$

The solutions of this equation (see [1, 2]) are the Bryant solitons,

$$z(r) = \mathfrak{B}(a_k r),$$

with $a_k > 0$ an arbitrary constant (to be fixed below by matching considerations). To obtain more accurate approximate solutions of $\mathcal{T}_r[z] = 0$, one could try an expansion of the form

$$(3.22) \quad z = \mathfrak{B}(a_k r) + e^{-2\gamma_k \tau} \beta_1(r) + e^{-4\gamma_k \tau} \beta_2(r) + \dots$$

In Section 5.2, we construct sub- and supersolutions based on the first two terms of this expansion. If as in [1] and [2] we normalize the Bryant soliton by requiring

$$(3.23) \quad \mathfrak{B}(r) = \frac{1 + o(1)}{r^2}, \quad (r \rightarrow \infty),$$

then the *Ansatz* (3.22) leads to

$$z \sim a_k^{-2} r^{-2} \quad (r \rightarrow \infty),$$

and thus in terms of the u coordinate, to

$$z \sim a_k^{-2} e^{-2\gamma_k \tau} u^{-2} \quad (\tau \rightarrow \infty, u \text{ small}).$$

Matching this expression with that given in (3.20), we obtain $c_k = a_k^{-2}$, namely

$$(3.24) \quad a_k := \frac{1}{\sqrt{c_k}}.$$

3.4. General features of the formal solutions. In addition to recalling (with certain adaptations) the explicit coordinate representations of the formal solutions in the parabolic, intermediate, and tip regions, we wish to note a few general features of the formal solutions that play a role in the statement and proof of our main theorem. We first note that the matched asymptotics analysis of [2] holds for all space dimensions $n + 1 \geq 3$, for all Hermite indices $k \geq 3$, and for all negative real values of the constant b_k . We thus observe that the set of formal solutions is parameterized by these three numbers $\{n, k, b_k\}$.

Next, it follows from the form of the formal solutions in the parabolic region (and from the properties of the Hermite polynomials) that the formal solutions are reflection-symmetric across the ($s = 0$) equator if and only if k is even. These are the solutions that have degenerate neckpinches forming simultaneously at both poles. Also, asymptotically, they have only tip, intermediate, and parabolic regions. Hence, in studying (below) the convergence of full Ricci flow solutions to these formal approximations, it is only for the solutions with odd k that we must work in an outer region (beyond the parabolic region) as well as in the other regions.

Finally, we note that if we calculate the curvature for one of these formal solutions (see Section 6 of [2]), we find that the norm of the curvature tensor achieves its maximum value at the tip, where it takes the value

$$(3.25) \quad |\text{Rm}(\text{tip}, t)| = \frac{C_k}{(T - t)^{2-2/k}},$$

with C_k a constant³ depending only on n, k , and b_k . The curvature of a Ricci flow solution that asymptotically approaches this formal solution necessarily blows up at the same rate.

4. OUTLINE OF THE PROOF

The main result of this paper, as stated above in the main theorem, is that for each formal degenerate neckpinch solution described above (with specified values of the parameters $\{n, k, b_k\}$), there exist rotationally symmetric Ricci flow solutions that asymptotically approach this solution, and therefore share its asymptotic properties. This result verifies the conjecture in Section 7 of [2]. The rest of this paper is devoted to proving this result. In this section, we outline the overall strategy we use to carry out the proof. In the remaining sections, we provide the details.

Our strategy for proving that there exist solutions obeying the asymptotic profiles derived in [2] follows Ważewski's retraction principle [13], as used in [5]. We apply this to solutions of Ricci flow, regarded as trajectories

$$g : [0, T_0) \rightarrow C^\infty \left((T^*S^{n+1} \otimes_{\text{sym}} T^*S^{n+1})_+ \right)$$

satisfying the isometry condition $\Upsilon^*g(t) = g(t)$ for all $\Upsilon \in \text{SO}(n+1)$ and $t \in [0, T_0)$. As noted in Section 2, such solutions may be naturally identified with ordered pairs of maps

$$(\varphi, \psi) : [0, T_0) \rightarrow (\mathbb{R}_+ \times \mathbb{R}_+)$$

satisfying the boundary conditions (2.4) for all $t \in [0, T_0)$, where $T_0 = T_0(g_0)$ is the maximal existence time of the unique solution with initial data g_0 .

³At the tip, the sectional curvatures \mathcal{K}^\top of a 2-plane tangent to an S^n factor and \mathcal{K}^\perp of an orthogonal 2-plane both take the value $\mathcal{K} = \frac{\hat{C}_k}{(T-t)^{2-2/k}}$, with $\hat{C}_k := \frac{4}{k^2(n-1)^2} (2|b_k|)^{-\frac{2}{k}}$.

To implement this strategy, we define tubular neighborhoods Ξ_ε of the formal (approximate) solutions discussed above and in [2]. For a given fixed formal solution $\hat{g}_{\{n,k,b_k\}}(t)$ with singularity at time T , and for a given positive $\varepsilon \ll 1$, we choose a tubular neighborhood of $\hat{g}_{\{n,k,b_k\}}(t)$ by specifying time-dependent inequalities to be satisfied by φ and ψ , with these inequalities becoming more restrictive as $t \nearrow T$, narrowing to the formal solution at time $t = T$. Decomposing the boundary of the tubular neighborhood $\partial\Xi_\varepsilon = (\partial\Xi_\varepsilon)_- \cup (\partial\Xi_\varepsilon)_o \cup (\partial\Xi_\varepsilon)_+$, we construct barriers and prove *entrapment lemmas* which show that solutions $g \in \Xi_\varepsilon$ never contact $(\partial\Xi_\varepsilon)_+$. We also prove an *exit lemma* which shows that any solution $g \in \Xi_\varepsilon$ that contacts $(\partial\Xi_\varepsilon)_-$ must immediately exit. This ensures that the exit times of those solutions that never reach the neutral part $(\partial\Xi_\varepsilon)_o$ of the boundary depend continuously on the initial data. In Section 7, we construct (for each specified formal solution) a k -dimensional family of initial data $g_\alpha(t_0) \in (\Xi_\varepsilon \cap \{t = t_0\})$ parameterized by $\alpha \in \mathcal{B}^k$ (the closure of a k -dimensional topological ball) such that all $g_\alpha(t_0)$ with $\alpha \in \partial\mathcal{B}^k$ lie in the exit set $(\partial\Xi_\varepsilon)_-$, and such that the map $\phi : \alpha \in \partial\mathcal{B}^k \rightarrow g_\alpha(t_0) \in (\partial\Xi_\varepsilon)_-$ is not contractible. (As seen in Section 7, $t_0 \in [0, T)$ is chosen to be very close to the singularity time T of the formal solution.) We choose the initial data $g_\alpha(t_0)$ at a sufficiently large distance from the neutral boundary $(\partial\Xi_\varepsilon)_o$ to guarantee that as long as the corresponding solution $\{g_\alpha(t) : t \geq t_0\}$ stays within Ξ_ε , it never reaches $(\partial\Xi_\varepsilon)_o$. In particular, the $g_\alpha(t_0)$ are chosen so that the solutions $g_\alpha(t)$ can only leave Ξ_ε through $(\partial\Xi_\varepsilon)_-$. If for every $\alpha \in \mathcal{B}^k$ the solution $g_\alpha(t)$ were to leave Ξ_ε at some time $t_\alpha > t_0$, then the exit map $\alpha \mapsto g_\alpha(t_\alpha)$ would yield a continuous map $\Phi : \mathcal{B}^k \rightarrow (\partial\Xi_\varepsilon)_-$. Since $t_\alpha = t_0$ for all $\alpha \in \partial\mathcal{B}^k$, the map Φ would extend ϕ , and therefore ϕ would have to be contractible, in contradiction with our construction of ϕ . It follows that for at least one $\alpha \in \mathcal{B}^k$, the solution $g_\alpha(t)$ never leaves the tubular neighborhood Ξ_ε . Such a solution must then exhibit the asymptotic behavior derived in [2] (and summarized above in Section 3) for the formal model solution $\hat{g}_{\{n,k,b_k\}}(t)$.

We outline further the steps of this strategy in Section 4.4 below. Before doing so, we briefly describe how the tubular neighborhood Ξ_ε (hereafter called “the tube”) is defined in each of the four regions. In subsequent sections, we show that the tube, as defined here and (more explicitly) in Section 5, has all the properties needed to carry through the proof.

4.1. Defining the tube in the tip and intermediate regions. In both of these regions, we define the tubular neighborhood Ξ_ε using upper and lower barrier functions, $z_+(u, \tau)$ and $z_-(u, \tau)$, which are defined with respect to the coordinate system described in Section 2.2. By the parabolic maximum principle, an evolving solution cannot cross a barrier. Thus the barriers provide an *entrapment condition*, ensuring that solutions never exit through the portions of $(\partial\Xi_\varepsilon)_+$ they define. The barriers are constructed for a precise subset of $u \in [0, 1)$ using functions adapted to each region separately. To complete the construction, we “patch” these functions together in the transitional areas where the regions overlap, using the fact that the maximum (minimum) of two subsolutions (supersolutions) is a subsolution (supersolution). This work is done in Section 5.

4.2. Defining the tube in the parabolic region. If Ξ_ε could be defined by barriers everywhere, it would follow that solutions starting from an open set of rotationally symmetric initial data would all develop degenerate neckpinches at the

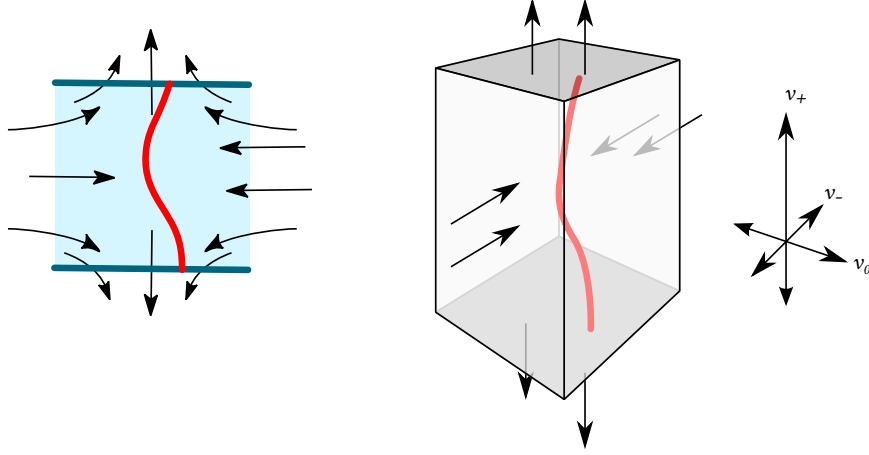
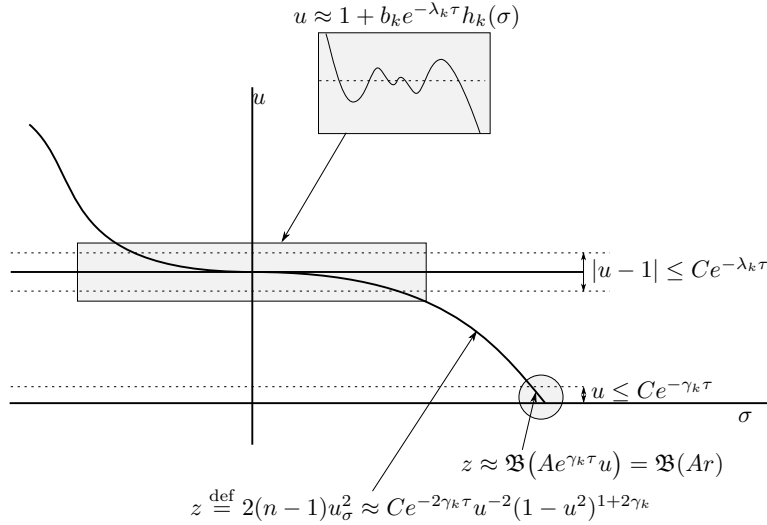

 FIGURE 1. A schematic view of a tubular neighborhood Ξ_ε .


FIGURE 2. An overview of the separate regions in the solution

same time $t = T$. This is of course impossible, since one can always change the singularity time of a solution by shifting it in time. So rather than constructing barriers in the parabolic region, we must adopt an alternate (necessarily more complicated) approach.

We define Ξ_ε in the parabolic region working with the coordinate system discussed in Section 2.1. In order to do so, we must introduce more notation. As discussed in Section 3.1 and in [2], the operator \mathcal{A} defined in (3.4) naturally arises

in the analysis of the parabolic region, and we are led to write the formal solution as an expansion in its Hermite polynomial eigenfunctions. To make this more precise, we note that \mathcal{A} is self-adjoint in the weighted Hilbert space $\mathfrak{h} := L^2(\mathbb{R}; e^{-\frac{\sigma^2}{4}} d\sigma)$. Denoting the inner product in \mathfrak{h} by $(f, g)_{\mathfrak{h}} := \int_{\mathbb{R}} f(\sigma)g(\sigma) e^{-\frac{\sigma^2}{4}} d\sigma$, and the norm by $\|f\|_{\mathfrak{h}} := \sqrt{(f, f)_{\mathfrak{h}}}$, we find that the Hermite polynomials $\{h_m(\cdot)\}_{m=0}^{\infty}$ constitute a complete orthogonal basis of \mathfrak{h} . We normalize so that $h_m(\sigma) = \sigma^m + \dots$.

Solutions $v = u - 1$ do not belong to \mathfrak{h} , because they are not defined for all σ . So we fix $\eta > 0$ to be a smooth even cutoff function such that $\eta(y) = 1$ for $0 \leq |y| \leq 1$ and $\eta(y) = 0$ for $|y| \geq \frac{6}{5}$. We then define

$$(4.1) \quad \check{v}(\sigma, \tau) := \begin{cases} \eta(e^{-\gamma_k \tau/4} \sigma) v(\sigma, \tau) & \text{for } |\sigma| \leq \frac{6}{5} e^{\gamma_k \tau/5}, \\ 0 & \text{for } |\sigma| > \frac{6}{5} e^{\gamma_k \tau/5}. \end{cases}$$

It follows that $\check{v} \in \mathfrak{h}$ is smooth and satisfies $\check{v}(\sigma, \tau) = v(\sigma, \tau)$ for $|\sigma| \leq e^{\gamma_k \tau/5}$. One computes from equation (3.2) that

$$(4.2) \quad \left. \frac{\partial \check{v}}{\partial \tau} \right|_{\sigma} = \mathcal{A}\check{v} + \eta N[v] + E[\eta, v],$$

where

$$\begin{aligned} N[v](\sigma, \tau) &:= \frac{2(n-1)v_{\sigma}^2 - v^2}{2(1+v)} - nI[v]v_{\sigma}, \\ E[\eta, v](\sigma, \tau) &:= \left(\eta_{\tau} - \eta_{\sigma\sigma} + \frac{\sigma}{2}\eta_{\sigma} \right) v - 2\eta_{\sigma}v_{\sigma}, \end{aligned}$$

with

$$\begin{aligned} \eta_{\sigma} &= \frac{\partial \eta(e^{-\gamma_k \tau/4} \sigma)}{\partial \sigma} = e^{-\gamma_k \tau/4} \eta'(e^{-\gamma_k \tau/4} \sigma), & \eta_{\sigma\sigma} &= e^{-\gamma_k \tau/2} \eta''(e^{-\gamma_k \tau/4} \sigma), \\ \eta_{\tau} &= -\frac{\gamma_k}{4} \sigma e^{-\gamma_k \tau/4} \eta'(e^{-\gamma_k \tau/4} \sigma), \end{aligned}$$

and where $I[v] = I[u]$ is defined in (3.3). We note that in calculating $\eta N[v]$ and $E[\eta, v]$, one may take v to be identically zero for any $|\sigma| > \frac{6}{5} e^{\gamma_k \tau/5}$ at which v is not defined.

Because $\check{v} \in \mathfrak{h}$, it makes sense to define the projections

$$(4.3a) \quad v_{k-} := \sum_{j=0}^{k-1} \frac{(\check{v}, h_j)_{\mathfrak{h}}}{\|h_j\|_{\mathfrak{h}}^2} h_j,$$

$$(4.3b) \quad v_k := \frac{(\check{v}, h_k)_{\mathfrak{h}}}{\|h_k\|_{\mathfrak{h}}^2} h_k,$$

$$(4.3c) \quad v_{k+} := \sum_{j=k+1}^{\infty} \frac{(\check{v}, h_j)_{\mathfrak{h}}}{\|h_j\|_{\mathfrak{h}}^2} h_j \quad (= \check{v} - (v_{k-} + v_k)).$$

With regard to the linear operator \mathcal{A} , the projections v_{k-} , v_k , and v_{k+} represent rapidly-growing, neutral, and rapidly-decaying perturbations, respectively, of the formal solution. We show below that v_{k-} is associated to the boundary $(\partial \Xi_{\varepsilon})_{-}$, v_k to the boundary $(\partial \Xi_{\varepsilon})_{\circ}$, and v_{k+} to the boundary $(\partial \Xi_{\varepsilon})_{+}$.

In the parabolic region, Ξ_ε is defined by four conditions: a solution $g(\cdot)$ belongs to Ξ_ε if it satisfies the three L^2 inequalities

$$(4.4a) \quad \|v_{k-}\|_{\mathfrak{h}} \leq \varepsilon e^{-\mu_k \tau},$$

$$(4.4b) \quad \|v_k - b_k e^{-\mu_k \tau} h_k\|_{\mathfrak{h}} \leq \varepsilon e^{-\mu_k \tau},$$

$$(4.4c) \quad \|v_{k+}\|_{\mathfrak{h}} \leq \varepsilon e^{-\mu_k \tau},$$

together with the pointwise inequality

$$(4.4d) \quad \sup_{|\sigma| \leq P} (|v| + |\partial_\sigma v|) \leq W e^{-\mu_k \tau},$$

where b_k is the parameter that selects a particular formal solution, P is a constant chosen below to define the range of the parabolic region, W is a large constant to be fixed below, and μ_k is defined in (3.5). We show below that the first inequality above is an *exit condition*, meaning that any solution that contacts $\partial\Xi_\varepsilon$ with $\|v_{k-}\|_{\mathfrak{h}} = \varepsilon e^{-\mu_k \tau}$ immediately exits Ξ_ε . We further show that the remaining three inequalities are *entrapment conditions*, ensuring that solutions of interest never contact the parts of $\partial\Xi_\varepsilon$ that they define. This work is done in Section 6.

4.3. Defining the tube in the outer region. If $k \geq 4$ is even, we do not have to deal with an outer region when defining Ξ_ε : in this case, the formal solution described in Section 3 is reflection symmetric. So we encase it (outside the central parabolic region) in barriers that are themselves reflection symmetric, placing such properly ordered and patched barriers on either side of the parabolic region.⁴

If $k \geq 3$ is odd, on the other hand, we define Ξ_ε outside the parabolic region with respect to the coordinate system described in Section 2.2, using barriers adapted to a precise subset of $u \in (1, e^{\tau/2})$. This work, which is highly analogous to the constructions for the tip and intermediate regions, is also done in Section 5.

4.4. Summary of the main estimates used in the proof. We presume now that we have fixed $n \geq 2$ and $k \geq 3$, and chosen $b_k < 0$, thereby selecting a formal solution $u(\sigma, \tau) = 1 + b_k e^{-\mu_k \tau} h_k(\sigma) + \dots$ (corresponding to $\hat{g}_{\{n, k, b_k\}}(\tau)$) to serve as a model in the parabolic region for the solutions whose existence we prove in this paper. Two parameters remain to be determined: the small constant ε appearing in (4.4a)–(4.4c), and the large constant W appearing in (4.4d). Carefully tracking which estimates depend on which choices, we show below that at the end of the proof, we can choose W sufficiently large, depending only on b_k , such that our exit and entrapment results, Lemmas 15–18, and hence the Main Theorem, hold for all sufficiently small ε , depending on $\{b_k, W\}$. Toward this end, we proceed as follows.

Using the relation $z = \psi_s^2 = 2(n-1)u_\sigma^2$, we employ properly ordered and patched barriers constructed in Section 5 to argue in Section 6 that there exist constants A^\pm , B , δ_1 , and τ_8 ,⁵ depending only on b_k , such that solutions $v = u - 1$ in the tube Ξ_ε satisfy

$$\frac{\sqrt{A^-}}{k} < |\partial_\sigma \{(e^{\mu_k \tau} |v|)^{\frac{1}{k}}\}| < \frac{\sqrt{A^+}}{k}$$

⁴If $k \geq 4$ is even, it is not necessary that the initial data we construct in Section 7 be reflection symmetric. For such data to belong to Ξ_ε , it suffices that they lie inside the reflection symmetric barriers defined in Section 5, and that they satisfy estimates (4.4a)–(4.4d).

⁵At several steps of the proofs in Sections 5–6, one has to restrict to sufficiently large times τ ; we address this in Section 7 by constructing initial data that become singular sufficiently quickly.

for $\tau \geq \tau_8$ and $|\sigma| \in \left(\frac{3}{5}P, \delta_1 e^{\gamma_k \tau}\right)$, where

$$(4.5) \quad P := 2 \left(\frac{B}{-b_k} \right)^{\frac{1}{k}}.$$

The inequality above provides a precise sense in which solutions in Ξ_ε resemble the formal solution for large $|\sigma|$, i.e., in the interface between the parabolic and intermediate regions. Accordingly, we define the parabolic region as

$$(4.6) \quad \mathcal{P} := \{\sigma : |\sigma| \leq P\}.$$

For later use, we note that P and the other constants introduced in Section 5 depend only on b_k and are in particular independent of W .

Further analyzing the parabolic–intermediate interface, we prove in Section 6 that there exist C_\pm depending on $\{b_k, W\}$ such that solutions in Ξ_ε satisfy

$$C_- |\sigma|^k \leq e^{\mu_k \tau} |v| \leq C_+ |\sigma|^k$$

for $|\sigma| \in \left(\frac{3}{5}P, \delta_1 e^{\gamma_k \tau}\right)$ and $\tau \geq \tau_8$. Combining this with the pointwise bound (4.4d), we then show that for solutions in Ξ_ε , one has a pointwise estimate

$$|(\partial_\tau - \mathcal{A})v| \leq C_1 W^2 e^{-2\mu_k \tau}$$

in \mathcal{P} for $\tau \geq \tau_9$, where C_1 depends only on b_k , and $\tau_9 \geq \tau_8$ depends on $\{b_k, W\}$. We next argue that there exists C_2 depending only on b_k such that one has an L^2 bound

$$\|(\partial_\tau - \mathcal{A})\check{v}\|_{\mathfrak{h}} \leq C_2 e^{-\frac{3}{2}\mu_k \tau}$$

for all $\tau \geq \tau_{10}$, if $\tau_{10} \geq \tau_9$ is chosen sufficiently large, depending on $\{b_k, W\}$.

To finish the proof, we apply our estimates for $\|(\partial_\tau - \mathcal{A})\check{v}\|_{\mathfrak{h}}$ to the three L^2 inequalities (4.4a)–(4.4c) defining Ξ_ε in the parabolic region to prove an exit lemma for the first inequality and entrapment lemmas for the remaining two. Then we use the analytic semigroup generated by \mathcal{A} to derive C^1 bounds

$$\sup_{|\sigma| \leq P} \{|\check{v}(\sigma, \tau)| + |\check{v}_\sigma(\sigma, \tau)|\} \leq C_3 e^{-\mu_k \tau}$$

that hold at all sufficiently large times $\tau \geq \tau_{11} \geq \tau_{10}$, where τ_{11} depends on $\{b_k, W\}$, but C_3 depends only on b_k . We use this estimate to prove an entrapment lemma that says that for W large enough and all small enough ε , solutions in Ξ_ε never contact $\partial\Xi_\varepsilon$ by achieving equality in (4.4d). Collectively, these results hold for all times $\tau \geq \bar{\tau} \geq \tau_{11}$, thereby establishing all necessary properties of the tubular neighborhood Ξ_ε . The details of the steps outlined here for the parabolic region are provided in Section 6.

The argument we have outlined above proves that for at least one initial datum belonging to Ξ_ε at the initial time, the solution that flows from this datum develops a degenerate neckpinch — provided that the set of such initial data is nonempty. The work to construct such data is done in Section 7, where we choose $t_0 \in [0, T)$ close enough to T so that $t = t_0$ corresponds to $\tau = \bar{\tau}$. This completes the outline of the proof of the main theorem, which validates the predictions we made in [2].⁶ We proceed in the sections below to supply the details of the proof.

⁶One may translate between the coordinate representations used in [2] and those used here by following the methods illustrated in Section 3 above.

5. CONSTRUCTING BARRIERS

5.1. Barriers in the intermediate region. In the intermediate region, one says that z is a subsolution (supersolution) of equation (3.11) if $\mathcal{D}_u z \leq 0$ (≥ 0), where

$$(5.1) \quad \mathcal{D}_u z := \frac{\partial z}{\partial \tau} \Big|_u - \frac{1}{2}(u^{-1} - u)z_u - u^{-2}z - \frac{\mathcal{Q}_u[z]}{2(n-1)}.$$

A lower (upper) barrier is a subsolution (supersolution) that lies below (above) a formal solution in an appropriate space-time region. These are obtained by first constructing parameterized families of sub- and supersolutions, and then making suitable parameter choices to ensure properly ordered barriers.

Lemma 1. *Let $Z_1(u) = A_1 u^{-2}(1 - u^2)^{1+2\gamma_k}$ for some $A_1 > 0$. There exist a function $\zeta : (0, 1) \rightarrow \mathbb{R}$, a constant $B_1 > 0$, and a constant $A_2^{\min} < \infty$ depending only on A_1 such that for any $A_2 \geq A_2^{\min}$, the function*

$$(5.2) \quad z_{\pm}(u, \tau) := e^{-2\gamma_k \tau} Z_1(u) \pm A_2 e^{-4\gamma_k \tau} \zeta(u)$$

is a supersolution (+) or a subsolution (−) in the space-time region

$$(5.3) \quad B_1 \sqrt{\frac{A_2}{A_1}} e^{-\gamma_k \tau} \leq u \leq 1 - B_1 \left(\frac{A_2}{A_1} \right)^{k/2} e^{-\mu_k \tau}, \quad \tau \geq \tau_1,$$

where τ_1 depends only on A_1 and A_2 .

We do not claim that $\zeta(u) > 0$, so the subsolutions and supersolutions provided by this lemma may not be ordered. In Lemma 2, below, we construct properly ordered barriers z_{\pm} by specifying a pair of constants A_1^+ and A_1^- , setting $A_1 = A_1^+$ in the expression (5.2) for z_+ and $A_1 = A_1^-$ in the expression (5.2) for z_- . The time-dependent domain (5.12) where these barriers are valid provides the precise characterization of the intermediate region, as promised in Section 3.2 above.

Proof. We verify here that z_+ is a supersolution. (The verification that z_- is a subsolution follows from the same argument, with the sign of A_2 changed.) Applying the operator \mathcal{D}_u defined in (5.1) to the expression (5.2), we obtain

$$\begin{aligned} \mathcal{D}_u[z_+] &= e^{-2\gamma_k \tau} \left\{ -\frac{1}{2}(u^{-1} - u)Z_1' - (u^{-2} + 2\gamma_k)Z_1 \right\} \\ &\quad + e^{-4\gamma_k \tau} \left\{ -\frac{1}{2}(u^{-1} - u)A_2\zeta' - (u^{-2} + 4\gamma_k)A_2\zeta - \frac{1}{2(n-1)}\mathcal{Q}_u[Z_1] \right. \\ &\quad \left. - \frac{A_2}{n-1}e^{-2\gamma_k \tau}\hat{\mathcal{Q}}_u[Z_1, \zeta] - \frac{A_2^2}{2(n-1)}e^{-4\gamma_k \tau}\mathcal{Q}_u[\zeta] \right\}. \end{aligned}$$

It follows from the definition of Z_1 that the terms multiplying $e^{-2\gamma_k \tau}$ vanish; thus we have

$$(5.4) \quad e^{4\gamma_k \tau} \mathcal{D}_u[z_+] = A_2 \left\{ -\frac{1}{2}(u^{-1} - u)\zeta' - (u^{-2} + 4\gamma_k)\zeta \right\} - \frac{1}{2(n-1)}\mathcal{Q}_u[Z_1] \\ - \frac{A_2}{n-1}e^{-2\gamma_k \tau}\hat{\mathcal{Q}}_u[Z_1, \zeta] - \frac{A_2^2}{2(n-1)}e^{-4\gamma_k \tau}\mathcal{Q}_u[\zeta].$$

We want to choose ζ and A_2 so that for sufficiently large values of τ , $\mathcal{D}_u[z_+] \geq 0$ ($\mathcal{D}_u[z_-] \leq 0$ for the subsolution). To achieve this, we first note that as $\tau \rightarrow \infty$, only the first two terms on the right in (5.4) remain. We therefore choose ζ so that those first two terms together are positive so long as A_2 is large enough. This then

implies that for any fixed $u \in (0, 1)$, one has $\mathcal{D}_u[z_+](u, \tau) > 0$ if τ is large enough. We conclude the proof by estimating how large τ must be for the constant terms in (5.4) to outweigh the remaining terms.

Estimate for $\mathcal{Q}_u[Z_1]$: For $0 < u < 1$ (a necessary condition for the intermediate region), it is easy to see that

$$|Z_1'| \leq \frac{C}{u(1-u^2)} Z_1 \quad \text{and} \quad |Z_1''| \leq \frac{C}{u^2(1-u^2)^2} Z_1.$$

Using $\mathcal{Q}_u[Z_1] = Z_1 Z_1'' - \frac{1}{2}(Z_1')^2 - u^{-1} Z_1 Z_1' - 2(n-1)u^{-2} Z_1^2$, we find that

$$(5.5) \quad |\mathcal{Q}_u[Z_1]| \leq C A_1^2 u^{-6} (1-u^2)^{4\gamma_k}.$$

Here and in what follows, unnumbered constants C are generic and may change from line to line.

Choice of ζ : Let $\zeta : (0, 1) \rightarrow \mathbb{R}$ be any solution of the inhomogeneous ODE

$$(5.6) \quad -\frac{1}{2}(u^{-1} - u) \frac{d\zeta}{du} - (4\gamma_k + u^{-2})\zeta = u^{-6}(1-u^2)^{4\gamma_k}.$$

Replacing u by u^2 as the independent variable, one can rewrite this equation as

$$(5.7) \quad \frac{d\zeta}{d(u^2)} + \left\{ \frac{1+4\gamma_k}{1-u^2} + \frac{1}{u^2} \right\} \zeta = -\frac{(1-u^2)^{4\gamma_k-1}}{u^6},$$

whereupon it is easy to see that the equation has regular singular points both at $u = 0$ and at $u = 1$. The solution of the corresponding homogeneous equation is given by (3.14) with $m = 2$, namely $\zeta_{\text{hom}} = \hat{c}_2 u^{-2} (1-u^2)^{1+4\gamma_k}$. One can either use the Frobenius method of power series or variation of constants to conclude that every choice of ζ solving (5.6) has the asymptotic behavior

$$(5.8) \quad \zeta(u) = \begin{cases} u^{-4} + \mathcal{O}(u^{-2} \log u) & (u \searrow 0), \\ -(1-u^2)^{4\gamma_k} + \mathcal{O}((1-u^2)^{4\gamma_k+1} \log(1-u^2)) & (u \nearrow 1) \end{cases}$$

at the endpoints of the interval $0 < u < 1$.

Constructing supersolutions: Estimate (5.5) and formula (5.4) together tell us that

$$e^{4\gamma_k \tau} \mathcal{D}_u[z_+] \geq \{A_2 - C A_1^2\} u^{-6} (1-u)^{4\gamma_k} - \frac{A_2}{n-1} e^{-2\gamma_k \tau} |\hat{\mathcal{Q}}_u[Z_1, \zeta]| - \frac{A_2^2}{2(n-1)} e^{-4\gamma_k \tau} |\mathcal{Q}_u[\zeta]|.$$

To ensure that the τ -independent term is positive, we must choose $A_2 > C A_1^2$. We set

$$A_2^{\min} = 2C A_1^2,$$

which then implies that for all $A_2 \geq A_2^{\min}$, one has

$$(5.9) \quad e^{4\gamma_k \tau} \mathcal{D}_u[z_+] \geq C A_1^2 u^{-6} (1-u)^{4\gamma_k} - \frac{A_2}{n-1} e^{-2\gamma_k \tau} |\hat{\mathcal{Q}}_u[Z_1, \zeta]| - \frac{A_2^2}{2(n-1)} e^{-4\gamma_k \tau} |\mathcal{Q}_u[\zeta]|.$$

Determining where z_+ is a supersolution: By definition, one has

$$2\hat{\mathcal{Q}}_u[Z_1, \zeta] = Z_1\zeta'' + Z_1''\zeta - Z_1'\zeta' - u^{-1}(Z_1\zeta' + Z_1'\zeta) - 4(n-1)u^{-2}Z_1\zeta.$$

For $0 < u \leq \frac{1}{2}$, we recall (5.8) to see that

$$|\hat{\mathcal{Q}}_u[Z_1, \zeta]| \leq CA_1u^{-8}$$

and

$$|\mathcal{Q}_u[\zeta]| \leq Cu^{-10}.$$

Combining these estimates, we find there is $c > 0$ such that

$$\begin{aligned} e^{4\gamma_k\tau}\mathcal{D}_u[z_+] &\geq cu^{-6}(A_1^2 - CA_1A_2e^{-2\gamma_k\tau}u^{-2} - CA_2^2e^{-4\gamma_k\tau}u^{-4}) \\ &= cu^{-6}A_1^2\left(1 - C\frac{A_2}{A_1(e^{\gamma_k\tau}u)^2} - C\frac{A_2^2}{A_1^2(e^{\gamma_k\tau}u)^4}\right) \end{aligned}$$

for $0 < u \leq \frac{1}{2}$. It follows from this inequality that there exists a constant $B_1 < \infty$ such that z_+ is a supersolution in the region $(e^{\gamma_k\tau}u)^2 \geq B_1^2A_2/A_1$, equivalently

$$B_1\sqrt{\frac{A_2}{A_1}}e^{-\gamma_k\tau} \leq u \leq \frac{1}{2}.$$

On the other hand, for the case $\frac{1}{2} \leq u < 1$, we have $Z_1 \sim A_1(1-u^2)^{1+2\gamma_k}$ and $\zeta \sim (1-u^2)^{4\gamma_k}$, so that

$$|\hat{\mathcal{Q}}_u[Z_1, \zeta]| \leq CA_1(1-u^2)^{-1+6\gamma_k}$$

and

$$|\mathcal{Q}_u[\zeta]| \leq C(1-u^2)^{-2+8\gamma_k}.$$

Thus in the interval $\frac{1}{2} \leq u < 1$, we have

$$\begin{aligned} e^{4\gamma_k\tau}\mathcal{D}_u[z_+] &\geq \\ c(1-u^2)^{4\gamma_k} &\{A_1^2 - CA_1A_2e^{-2\gamma_k\tau}(1-u^2)^{-1+2\gamma_k} - CA_2^2e^{-4\gamma_k\tau}(1-u^2)^{-2+4\gamma_k}\}. \end{aligned}$$

Hence, making B_1 larger if necessary, we see that z_+ is a supersolution if

$$1-u^2 \geq B_1\left(\frac{A_2}{A_1}e^{-2\gamma_k\tau}\right)^{1/(1-2\gamma_k)} = B_1\left(\frac{A_2}{A_1}\right)^{k/2}e^{-\mu_k\tau}.$$

Since $1-u^2 = (1-u)(1+u)$, we have $1-u < 1-u^2 < 2(1-u)$, which proves the lemma for z_+ . The same arguments apply to z_- , so the proof is complete. \square

We now construct a pair of properly ordered barriers, based on the sub- and supersolutions of Lemma 1. To do this, we first specify

$$(5.10) \quad A_1^\pm := (1 \pm \delta)c_k$$

where c_k is defined in (3.10), and $0 < \delta \ll \frac{1}{2}$ is a small constant fixed here once and for all. We then define $A_3 := \max\{A_2(A_1^-), A_2(A_1^+)\}$, where the dependence of A_2 on A_1^\pm and A_1^\pm is that which is discussed in Lemma 1, with the added requirement that $A_2(A_1^\pm) \geq A_2^{\min}(A_1^\pm)$ are large enough to satisfy condition (5.23) in Lemma 5 below. (This is necessary so that the ordered barriers we construct here are compatible with those we construct in Section 5.2 for the tip region.) We further fix $\tau_2 := \max\{\tau_1(A_1^-), \tau_1(A_1^+)\}$ and $B_2 := \max\{B_1(A_1^-), B_1(A_1^+)\}$. We then have the following:

Lemma 2. *There exist $\tau_3 \geq \tau_2$ and $B_3 \geq B_2$ depending only on b_k , such that if ζ is the function appearing in Lemma 1, then*

$$(5.11) \quad z_{\pm}(u, \tau) = (1 \pm \delta)c_k e^{-2\gamma_k \tau} u^{-2} (1 - u^2)^{1+2\gamma_k} \pm A_3 e^{-4\gamma_k \tau} \zeta(u)$$

are a pair of properly ordered barriers in the space-time region

$$(5.12) \quad B_2 \sqrt{\frac{2A_3}{c_k}} e^{-\gamma_k \tau} \leq u \leq 1 - B_3 \left(\frac{2A_3}{c_k} \right)^{k/2} e^{-\mu_k \tau}, \quad \tau \geq \tau_3.$$

Proof. Using the asymptotic expansion (5.8) for ζ , we easily determine that $z_- < z_+$ as $u \searrow 0$. It is also straightforward to verify that on any interval $c \leq u \leq 1 - c$, there exists $\tau_3 \geq \tau_2$ such that $z_- < z_+$ if $c \leq u \leq 1 - c$ and $\tau \geq \tau_3$.

To study the rest of the intermediate region, we define $w := 1 - u^2$. Then as $u \nearrow 1$, the asymptotic expansion (5.8) implies that

$$w^{-1-2\gamma_k} (z_+ - z_-) \geq 2\delta c_k u^{-2} - 2A_3 e^{-4\gamma_k \tau} w^{2\gamma_k - 1} (1 + w \log w).$$

Using the facts that $2\gamma_k - 1 = -2/k$ and that $|w \log w| \leq 1$ for $0 < w \leq 1$, one then estimates for $u \leq 1 - B_3(2A_3/c_k)^{k/2} e^{-\mu_k \tau}$ and $\tau \geq \tau_3$ that

$$\begin{aligned} w^{-1-2\gamma_k} (z_+ - z_-) &\geq 2\delta c_k u^{-2} - 4A_3 e^{-4\gamma_k \tau} w^{-2/k} \\ &\geq 2c_k \{ \delta u^{-2} - B_3^{-2/k} (1 + u)^{-2/k} e^{-2\gamma_k \tau_3} \} \\ &> 0 \end{aligned}$$

provided that $B_3 \geq B_2$ is chosen sufficiently large. \square

5.2. Barriers in the tip region. In Section 3.3, we observe that the evolution of the metric is governed by the parabolic PDE $\mathcal{T}_r[z] = 0$, where $r = e^{\gamma_k \tau} u$, and where \mathcal{T}_r is defined in equation (3.21). This motivates us to look for sub- and supersolutions — and then properly ordered upper and lower barriers — of the form $z_{\pm} = \mathfrak{B}(A_4 r) \pm e^{-2\gamma_k \tau} \beta(r)$, where $\beta(r)$ is to be chosen to solve a suitable ODE, where A_4 is a parameter to be determined, and where we recall that \mathfrak{B} denotes the functional expression for the Bryant soliton.

Lemma 3. *For any $A_4 > 0$, there exist a bounded function $\beta : (0, \infty) \rightarrow \mathbb{R}$, a sufficiently small $B_4 > 0$, and a sufficiently large $\tau_4 < \infty$, all depending only on A_4 , such that*

$$(5.13) \quad z_{\pm} := \mathfrak{B}(A_4 r) \pm e^{-2\gamma_k \tau} \beta(r)$$

are sub- (z_-) and super- (z_+) solutions in the region

$$(5.14) \quad 0 \leq r \leq B_4 e^{\gamma_k \tau}$$

for all $\tau \geq \tau_4$.

We note that inequality (5.14) provides a working definition of the tip region, which overlaps the intermediate region (5.12) for all sufficiently large times τ .

As with the intermediate region, we do not claim that β has a sign; to get a pair of properly ordered barriers for the tip region, one makes suitable choices of the parameter A_4 (namely A_4^- and A_4^+ as defined in (5.20) below), relying on the fact that \mathfrak{B} is monotone decreasing.

Proof. Using the abbreviation $\hat{B}(r) := \mathfrak{B}(A_4 r)$, we see that for the function $z = \hat{B}(r) + e^{-2\gamma_k \tau} \beta(r)$ to be a supersolution, it suffices that $\mathcal{T}_r[z] \geq 0$. Using the definition of \mathcal{T}_r in (3.21) and the definitions of \mathcal{L}_r and $\hat{\mathcal{Q}}_r$ from (2.8) and (2.9), respectively, we find that

$$(5.15) \quad \mathcal{T}_r[\hat{B}(r) + e^{-2\gamma_k \tau} \beta(r)] = e^{-2\gamma_k \tau} \left\{ \frac{k-1}{k} r \hat{B}'(r) - \frac{1}{2(n-1)} (\mathcal{L}_r[\beta(r)] + 2\hat{\mathcal{Q}}_r[\hat{B}(r), \beta(r)]) \right\} + e^{-4\gamma_k \tau} \left\{ -2\gamma_k \beta(r) + \frac{k-1}{k} r \beta'(r) - \frac{1}{2(n-1)} \hat{\mathcal{Q}}_r[\beta(r), \beta(r)] \right\}.$$

Here we use the fact that \hat{B} satisfies $\mathcal{E}_r[\hat{B}] = 0$. For large τ , the sign of the RHS is determined by the sign of the coefficient of $e^{-2\gamma_k \tau}$. This coefficient is linear inhomogeneous in β and its derivatives. To ensure that $\hat{B}(r) + e^{-2\gamma_k \tau} \beta(r)$ is a supersolution for large enough τ , we choose $\beta(r)$ in such a way that the factor multiplying $e^{-2\gamma_k \tau}$ is positive. More precisely, we recall that $r\hat{B}'(r) < 0$ for all $r > 0$, which leads us to choose β to be a solution of the linear inhomogeneous differential equation

$$(5.16) \quad \mathcal{L}_r[\beta(r)] + 2\hat{\mathcal{Q}}_r[\hat{B}(r), \beta(r)] = 2(n-1)\hat{A}r\hat{B}'(r),$$

with the constant \hat{A} to be chosen below. Applying the definitions of \mathcal{L}_r and $\hat{\mathcal{Q}}_r$, we find that (5.16) takes the following form:

$$(5.17) \quad \hat{B}(r) \frac{d^2 \beta(r)}{dr^2} + \left\{ \frac{n-1}{r} - \hat{B}'(r) - \frac{\hat{B}(r)}{r} \right\} \frac{d\beta(r)}{dr} + \left\{ \hat{B}''(r) - \frac{\hat{B}'(r)}{r} + 2(n-1) \frac{1-2\hat{B}(r)}{r^2} \right\} \beta(r) = 2(n-1)\hat{A}r\hat{B}'(r).$$

This equation has a two-parameter family of solutions. We seek a specific solution from this family that leads to the positivity of the RHS of (5.15) for sufficiently large τ . To determine the region on which $\hat{B}(r) + e^{-2\gamma_k \tau} \beta(r)$ is a supersolution, we need to know the asymptotic behavior of our solution $\beta(r)$ both as $r \searrow 0$ and as $r \nearrow \infty$. We now show that this asymptotic behavior is the same for all solutions of (5.17), because near both $r = 0$ and $r = \infty$, we find that the solutions to the homogeneous equation are much smaller than any particular solution.

Asymptotic behavior at $r = 0$: At $r = 0$, we have $\hat{B}(0) = 1$, $\hat{B}'(0) = 0$ and $\hat{B}''(0) = -A_4^2 \mathfrak{B}''(0)$. Since $\mathfrak{B}(r)$ is an analytic function at $r = 0$, the ODE (5.17) has a regular singular point at $r = 0$, and to leading order can be written as

$$\frac{d^2 \beta(r)}{dr^2} + \frac{n-2}{r} \frac{d\beta(r)}{dr} - \frac{2(n-1)}{r^2} \beta(r) = -Cr^2$$

for some constant C . The general solution of this equation is

$$\beta_0(r) = a_1 r^{1-n} + a_2 r^2 - \tilde{C} r^4,$$

for some \tilde{C} . Discarding the unbounded solution and choosing $a_2 = 1$, we conclude that there exists a solution $\beta_p(r)$ of the true ODE (5.17) that satisfies $\beta_p(r) = r^2 + o(r^2)$ as $r \searrow 0$.

Asymptotic behavior at $r = \infty$: We have normalized \mathfrak{B} so that $\mathfrak{B}(r) = r^{-2} + \mathcal{O}(r^{-4})$ as $r \rightarrow \infty$. This choice yields $\hat{B}(r) = \mathfrak{B}(A_4 r) = A_4^{-2} r^{-2} + \mathcal{O}(A_4^{-4} r^{-4})$ near $r = \infty$. Keeping only the lowest-order terms in r^{-1} , we see that for r large, the ODE (5.17) is a small perturbation of the ODE

$$(5.18) \quad \frac{1}{A_4^2 r^2} \beta''_\infty(r) + \frac{n-1}{r} \beta'_\infty(r) + \frac{2(n-1)}{r^2} \beta_\infty(r) = -\frac{4(n-1)\hat{A}}{A_4^2 r^2},$$

whose general solution is

$$(5.19) \quad \beta_\infty(r) = \tilde{a}_1 r e^{-\alpha r^2} + \tilde{a}_2 r \int_1^r \rho^{-2} e^{-\hat{\alpha}(r^2 - \rho^2)} d\rho - \frac{2\hat{A}}{A_4^2},$$

where $\hat{\alpha} := \frac{n-1}{2} A_4^2$. The first two terms in (5.19), which solve the homogeneous equation associated to (5.18), both vanish as $r \nearrow \infty$: the first has Gaussian decay; the second is $\mathcal{O}(r^{-2})$. Hence every solution of the true ODE (5.17), including the choice of $\beta_p(r)$ made above, has the property that $\beta(r) = (1 + o(1))(-2\hat{A}/A_4^2)$ as $r \rightarrow \infty$.

To determine how to choose \hat{A} , we now estimate

$$\begin{aligned} \mathcal{J}_r[\hat{B}(r) + e^{-2\gamma_k \tau} \beta_p(r)] &= e^{-2\gamma_k \tau} \left(\frac{k-1}{k} - \hat{A} \right) r \hat{B}'(r) \\ &\quad + e^{-4\gamma_k \tau} \left(-2\gamma_k \beta_p(r) + \frac{k-1}{k} r \beta'_p(r) - \frac{\hat{Q}_r[\beta_p(r), \beta_p(r)]}{2(n-1)} \right). \end{aligned}$$

Since $\hat{B}'(r) < 0$, the first term on the RHS is positive if we choose $\hat{A} > \frac{k-1}{k}$. To make a definite choice, we set

$$\hat{A} := \frac{k-1}{k} + 1 = 2 - \frac{1}{k}.$$

Near $r = 0, \infty$, the asymptotics of \mathfrak{B} then imply that

$$-r \hat{B}'(r) = \begin{cases} \tilde{c}_1 r^2 + o(r^2), & (r \searrow 0) \\ \tilde{c}_2 r^2 + o(1/r^2) & (r \nearrow \infty) \end{cases}$$

for certain positive constants \tilde{c}_1 and \tilde{c}_2 , and hence that

$$-r \hat{B}'(r) \geq \tilde{c} \min\{r^2, r^{-2}\},$$

for some constant \tilde{c} . We also know for β_p that

$$\beta_p(r) = \mathcal{O}(r^2) \quad (r \searrow 0), \quad \beta_p(r) = -\frac{2\hat{A}}{A_4^2} + \mathcal{O}(r^{-2}) \quad (r \nearrow \infty).$$

These expansions can be differentiated. Hence we have for $0 < r \leq 1$,

$$\left| -2\gamma_k \beta_p(r) + \frac{k-1}{k} r \beta'_p(r) - \frac{\hat{Q}_r[\beta_p(r), \beta_p(r)]}{2(n-1)} \right| \leq C r^2,$$

and thus

$$\begin{aligned} \mathcal{J}_r[\hat{B}(r) + e^{-2\gamma_k \tau} \beta_p(r)] &\geq -e^{-2\gamma_k \tau} r \hat{B}'(r) - e^{-4\gamma_k \tau} C r^2 \\ &\geq e^{-2\gamma_k \tau} r^2 (\tilde{c} - e^{-2\gamma_k \tau} C) \\ &> 0, \end{aligned}$$

if τ is large enough. For $r \geq 1$, we have

$$\left| -2\gamma_k \beta_p(r) + \frac{k-1}{k} r \beta_p'(r) - \frac{\hat{Q}_r[\beta_p(r), \beta_p(r)]}{2(n-1)} \right| \leq C,$$

so that

$$\begin{aligned} \mathcal{J}_r[\hat{B}(r) + e^{-2\gamma_k \tau} \beta_p(r)] &\geq -e^{-2\gamma_k \tau} r \hat{B}'(r) - e^{-4\gamma_k \tau} C \\ &\geq e^{-2\gamma_k \tau} (\tilde{c} r^{-2} - C e^{-2\gamma_k \tau}) \\ &> 0, \end{aligned}$$

provided that $r < \tilde{\delta} e^{\gamma_k \tau}$, where $\tilde{\delta} = \sqrt{\tilde{c}/C}$.

This completes the construction of the supersolution. A subsolution can be constructed along the same lines, but one could also just set $\beta(r) = 0$ and observe that (5.15) then takes the form

$$\mathcal{J}_r[\hat{B}(r) + e^{-2\gamma_k \tau} \beta(r)] = \frac{k-1}{k} e^{-2\gamma_k \tau} r \hat{B}'(r).$$

Because $\hat{B}'(r) < 0$ for all r , the RHS of this equation is negative; hence we see that $\hat{B}(r) = \mathfrak{B}(A_4 r)$ is a subsolution. \square

We now use the sub- and supersolutions from Lemma 3 to construct properly ordered upper and lower barriers for the tip region. To do this, we specify the constants A_4^\pm as follows, using the small constant $0 < \delta \ll 1$ fixed above:

$$(5.20a) \quad (A_4^-)^{-2} = (1 + \delta) \left(1 + \frac{3}{8} B_3^{-2} \right) (1 - \delta) c_k,$$

$$(5.20b) \quad (A_4^+)^{-2} = (1 - \delta) \left(1 + \frac{1}{2} B_3^{-2} \right) (1 + \delta) c_k.$$

We note several consequences of these choices. First, they ensure that both barriers are close to the formal solution $\mathfrak{B}(c_k^{-1/2} r)$. Second, they ensure that $A_4^+ < A_4^-$; because \mathfrak{B} is monotone decreasing, this implies that $\mathfrak{B}(A_4^- r) < \mathfrak{B}(A_4^+ r)$ for all $r > 0$, which is useful in establishing the desired ordering of the barriers. Third, the choices above, and in particular the leading $1 \pm \delta$ factors, ensure that A_4^\pm satisfy the strict inequalities in condition (5.22) of Lemma 5 below. We use this fact to guarantee that the barriers constructed here are compatible with those constructed in Lemma 2 within the intersection of the intermediate and tip regions. These regions overlap for all sufficiently large τ , depending on B_2 in Lemma 2 and B_4 in Lemma 3.

Lemma 4. *If the parameters A_4^\pm take the values specified by (5.20), and if β is the function appearing in Lemma 3, then there exists $\tau_5 \geq \tau_4$ such that*

$$(5.21) \quad z_\pm = \mathfrak{B}(A_4^\pm r) \pm e^{-2\gamma_k \tau} \beta(r)$$

form a pair of properly ordered barriers that are valid for all points $0 \leq r \leq B_4 e^{\gamma_k \tau}$ and all times $\tau \geq \tau_5$.

Proof. Near $r = 0$, one has $\mathfrak{B}(A_4^\pm r) = 1 - \mathfrak{b}(A_4^\pm r)^2 + \mathcal{O}(r^4)$ for some constant $\mathfrak{b} > 0$, and $\beta(r) = [1 + o(1)]r^2$. Thus as $r \searrow 0$, we see that

$$z_+ - z_- = \{ \mathfrak{b}[(A_4^-)^2 - (A_4^+)^2] + 2[1 + o(1)]e^{-\gamma_k \tau} \} r^2 + \mathcal{O}(r^4) > 0.$$

Near $r = \infty$, one has the expansions $\mathfrak{B}(A_4^\pm r) = 1 + (A_4^\pm r)^{-2} + \mathcal{O}(r^{-4})$ and $\beta = [1 + o(1)]\{-2(1 + 1/k)(A_4^\pm)^{-2}\}$. It then follows from the specifications of A_4^\pm given in (5.20) that

$$z_+ - z_- = \frac{(1 - \delta^2)B_3^{-2}c_k}{8} \left\{ 1 - 2[1 + o(1)]\frac{k+1}{k}e^{-2\gamma_k\tau} \right\} + \mathcal{O}(r^{-4}),$$

which is positive for all sufficiently large τ and r .

It is straightforward to verify that $z_+ > z_-$ on any bounded interval $c < r < C$, for all sufficiently large τ . The result follows. \square

5.3. The intermediate-tip interface. Let z_\pm^{int} denote the upper and lower barriers constructed in Lemma 2 for the intermediate region, and let z_\pm^{tip} denote the upper and lower barriers constructed in Lemma 4 for the tip region. The domains of definition of these two pairs of functions intersect for sufficiently large times. To obtain barriers that work throughout the union of these regions, we must verify that z_\pm^{int} and z_\pm^{tip} are properly ordered in their intersection. In what follows, we focus on upper barriers, omitting the entirely analogous argument for lower barriers. To avoid notational prolixity, we write A_1 and A_4 for A_1^+ and A_4^+ , respectively, in the statement of the lemma and its proof.

Lemma 5. *Suppose that A_1 and A_4 satisfy strict inequalities*

$$(5.22) \quad \left(1 + \frac{3}{8}B_3^{-2}\right)A_1 < A_4^{-2} < \left(1 + \frac{1}{2}B_3^{-2}\right)A_1,$$

where B_3 is the constant from Lemma 2.

Then there exists a constant \hat{C} such that if

$$(5.23) \quad A_3 \geq \hat{C} \left\{ A_1^{1/2} + \left(1 + \frac{1}{2}B_3^{-2}\right)^2 A_1^2 \right\},$$

then one has $A_3 \geq \hat{C}(A_1^{1/2} + A_4^{-4})$, and consequently

$$(5.24a) \quad z_+^{\text{tip}} \leq z_+^{\text{int}} \text{ at } r = B_3\sqrt{A_3/A_1},$$

$$(5.24b) \quad z_+^{\text{tip}} \geq z_+^{\text{int}} \text{ at } r = 2B_3\sqrt{A_3/A_1},$$

for all $\tau \geq \tau_6$, where $\tau_6 \geq \max\{\tau_3, \tau_5\}$ is sufficiently large.

Proof. Near infinity, the Bryant soliton has an expansion of the form (see Lemma 18 of [1])

$$\mathfrak{B}(\rho) = \frac{1}{\rho^2} + \frac{\mathfrak{b}_2}{\rho^4} + \frac{\mathfrak{b}_3}{\rho^6} + \dots$$

Written in terms of the r coordinate, the supersolutions satisfy the expansion

$$\begin{aligned} z_+^{\text{tip}} &= \mathfrak{B}(A_4 r) + e^{-2\gamma_k\tau}\beta(r) \\ &= A_4^{-2}r^{-2} + \mathfrak{b}_2 A_4^{-4}r^{-4} + \mathcal{O}(r^{-6}) + \mathcal{O}(e^{-2\gamma_k\tau}) \end{aligned}$$

as $r \rightarrow \infty$. The $\mathcal{O}(r^{-6})$ term comes from the asymptotic expansion of $\mathfrak{B}(A_2 r)$ and is therefore uniform in time (in fact, independent of time).

Using our asymptotic description (5.8) of ζ , we find that as $u \searrow 0$, one has

$$\begin{aligned} z_+^{\text{int}} &= e^{-2\gamma_k\tau} Z_1(u) + e^{-4\gamma_k\tau} A_3 \zeta(u) \\ &= A_1 e^{-2\gamma_k\tau} u^{-2} (1 - u^2)^{1+2\gamma_k} + A_3 e^{-4\gamma_k\tau} u^{-4} (1 + \mathcal{O}(u^2 \log u)) \\ &= A_1 r^{-2} (1 - e^{-2\gamma_k\tau} r^2)^{1+2\gamma_k} + A_3 r^{-4} (1 + \mathcal{O}(u^2 \log u)) \\ &= A_1 r^{-2} + A_3 r^{-4} + \mathcal{O}(\tau e^{-2\gamma_k\tau}), \end{aligned}$$

where $r = e^{\gamma_k\tau} u$, and where $\mathcal{O}(\tau e^{-2\gamma_k\tau})$ is uniform on any compact r interval. Hence on bounded r -intervals, one has

$$\begin{aligned} r^2(z_+^{\text{tip}} - z_+^{\text{int}}) &= (A_4^{-2} - A_1) + (\mathfrak{b}_2 A_4^{-4} - A_3) r^{-2} + \mathcal{O}(r^{-4}) + \mathcal{O}(\tau e^{-2\gamma_k\tau}) \\ &= (A_4^{-2} - A_1) + (\mathfrak{b}_2 A_4^{-4} + \mathcal{O}(r^{-2}) - A_3) r^{-2} + \mathcal{O}(\tau e^{-2\gamma_k\tau}). \end{aligned}$$

We want this quantity to change from negative to positive as r increases from $B_3 \sqrt{A_3/A_1}$ to $2B_3 \sqrt{A_3/A_1}$. At $r = B_3 \sqrt{A_3/A_1}$, we have

$$r^2(z_+^{\text{tip}} - z_+^{\text{int}}) = A_4^{-2} - A_1 + \left\{ \frac{\mathfrak{b}_2 A_1}{A_3 A_4^4} + \mathcal{O}\left(\left(\frac{A_1}{A_3}\right)^2\right) - A_1 \right\} B_3^{-2} + \mathcal{O}(\tau e^{-2\gamma_k\tau}),$$

and at $r = 2B_3 \sqrt{A_3/A_1}$, we have

$$r^2(z_+^{\text{tip}} - z_+^{\text{int}}) = A_4^{-2} - A_1 + \frac{1}{4} \left\{ \frac{\mathfrak{b}_2 A_1}{A_3 A_4^4} + \mathcal{O}\left(\left(\frac{A_1}{A_3}\right)^2\right) - A_1 \right\} B_3^{-2} + \mathcal{O}(\tau e^{-2\gamma_k\tau}).$$

If $A_3 \geq \hat{C} A_4^{-4}$ and $A_3 \geq \hat{C} \sqrt{A_1}$ both hold for sufficiently large \hat{C} , then

$$\left| \frac{\mathfrak{b}_2 A_1}{A_3 A_4^4} + \mathcal{O}\left(\frac{A_1^2}{A_3^2}\right) \right| \leq \frac{A_1}{2},$$

and we find that

$$\begin{aligned} r^2(z_+^{\text{tip}} - z_+^{\text{int}}) &\leq A_4^{-2} - \left(1 + \frac{1}{2} B_3^{-2}\right) A_1 + \mathcal{O}(\tau e^{-2\gamma_k\tau}) \text{ at } r = B_3 \sqrt{A_3/A_1}, \\ r^2(z_+^{\text{tip}} - z_+^{\text{int}}) &\geq A_4^{-2} - \left(1 + \frac{3}{8} B_3^{-2}\right) A_1 + \mathcal{O}(\tau e^{-2\gamma_k\tau}) \text{ at } r = 2B_3 \sqrt{A_3/A_1}. \end{aligned}$$

From this it is clear that if the strict inequalities (5.22) hold, then the upper barriers z_+^{tip} and z_+^{int} satisfy the patching condition (5.24) for all sufficiently large τ . \square

5.4. Barriers in the outer region. If $k \geq 4$ is even, then as explained in Section 4.3, solutions in Ξ_ε are controlled by the barriers constructed above. If $k \geq 3$ is odd, we need additional barriers to define $(\partial\Xi_\varepsilon)_+$ in a suitable subset of $u \in (1, e^{\tau/2})$.

Lemma 6. *For any $A_5 > 0$, there exist a function $\tilde{\zeta} : (1, \infty) \rightarrow \mathbb{R}$ and constants A_6 and $B_5 > 0$ depending only on A_5 such that*

$$(5.25) \quad \tilde{z}_\pm = A_5 e^{-2\gamma_k\tau} (u^2 - 1)^{1+2\gamma_k} u^{-2} \pm A_6 e^{-4\gamma_k\tau} \tilde{\zeta}(u)$$

are sub- (\tilde{z}_-) and super- (\tilde{z}_+) solutions in the region

$$(5.26) \quad 1 + B_5 e^{-\mu_k\tau} \leq u \leq B_5^{-1} e^{\tau/2}.$$

Proof. We sketch the proof, because it is very similar to that of Lemma 1.

Let $\tilde{z}_1(u) = (u^2 - 1)^{1+2\gamma_k} u^{-2}$. Then for $1 \leq u < \infty$, one estimates $|\tilde{z}_1| \leq C(u - 1)^{2\gamma_k+1} u^{2\gamma_k-1}$, $|\tilde{z}'_1| \leq C(u - 1)^{2\gamma_k} u^{2\gamma_k-1}$, and $|\tilde{z}''_1| \leq C(u - 1)^{2\gamma_k-1} u^{2\gamma_k-1}$. Hence

$$|\mathcal{Q}_u[\tilde{z}_1]| \leq C(u - 1)^{4\gamma_k} u^{4\gamma_k-2}.$$

This motivates us to choose the function $\tilde{\zeta}$ to be a solution of the ODE

$$\frac{1}{2}(u - u^{-1})\frac{d\tilde{\zeta}}{du} - (4\gamma_k + u^{-2})\tilde{\zeta} = (u - 1)^{4\gamma_k}u^{4\gamma_k-2}.$$

Its solutions are $\tilde{\zeta} = \tilde{\zeta}_p + Z_{2,\text{hom}}$, where $\tilde{\zeta}_p$ is any particular solution, and $Z_{2,\text{hom}}$ solves the associated homogeneous ODE. We may thus choose a solution $\tilde{\zeta}$ such that $\tilde{\zeta} \sim (u - 1)^{4\gamma_k}$ (the behavior of $\tilde{\zeta}_p$) as $u \searrow 1$ and $\tilde{\zeta} \sim u^{8\gamma_k}$ (the behavior of $Z_{2,\text{hom}}$) as $u \rightarrow \infty$. Estimating that $|\tilde{\zeta}| \leq C(u - 1)^{4\gamma_k}u^{4\gamma_k}$, $|\tilde{\zeta}'| \leq C(u - 1)^{4\gamma_k-1}u^{4\gamma_k}$, and $|\tilde{\zeta}''| \leq C(u - 1)^{4\gamma_k-2}u^{4\gamma_k}$, we observe that

$$|\mathcal{Q}_u[\tilde{\zeta}]| \leq C(u - 1)^{8\gamma_k-2}u^{8\gamma_k},$$

and

$$|\hat{\mathcal{Q}}_u[\tilde{z}_1, \tilde{\zeta}]| \leq C(u - 1)^{6\gamma_k-1}u^{6\gamma_k-1}.$$

With these estimates for $\tilde{z}_1(u)$ established, we write

$$\tilde{z}_\pm = A_5 e^{-2\gamma_k\tau} \tilde{z}_1(u) + A_6 e^{-4\gamma_k\tau} \tilde{\zeta}(u),$$

as above, and seek to determine A_6 (depending only on A_5) so that \tilde{z}_+ is a supersolution. Based on the estimates above, we have

$$e^{4\gamma_k\tau} \mathcal{D}_u[\tilde{z}_+] \geq \left\{ A_6 - \frac{C}{2(n-1)} A_5^2 \right\} (u - 1)^{4\gamma_k} u^{4\gamma_k-2} + \mathcal{O}(e^{-2\gamma_k\tau}).$$

This leads us to choose $A_6 := 1 + \frac{C}{2(n-1)} A_5^2$. Then one obtains

$$e^{4\gamma_k\tau} \mathcal{D}_u[\tilde{z}_+] \geq (u - 1)^{4\gamma_k} u^{4\gamma_k-2} \{1 - C A_5 A_6 X - C (A_6 X)^2\},$$

where $X := e^{-2\gamma_k\tau} (u - 1)^{2\gamma_k-1} u^{2\gamma_k+1}$. It follows that \tilde{z}_+ is a subsolution in any spacetime region where $C A_5 (A_6 X) + C (A_6 X)^2 < 1$.

For $u > 1$ near 1, we observe that there is a constant B' depending only A_5 such that \tilde{z}_+ is a supersolution if $(u - 1)^{1-2\gamma_k} \geq B' e^{-2\gamma_k\tau}$, hence if $v^{2/k} \geq B' e^{-2\gamma_k\tau}$. It is easy to see that this is implied by the constraint $u \geq 1 + B_5 e^{-\mu_k\tau}$ if B_5 is large enough.

For $u \gg 1$, we observe that there is a constant B'' depending only on A_5 such that \tilde{z}_+ is a supersolution if $B'' u \leq e^{\tau/2}$, hence if $B'' \psi \leq 1$.

Similar considerations show that \tilde{z}_- is a subsolution. \square

As in the other regions, by making suitable modifications to the constants, one can ensure that the sub- and supersolutions of Lemma 6 are properly ordered upper and lower barriers in the outer region. To do so, we specify constants $A_5^\pm = (1 \pm \delta)c_k$, where c_k is defined in (3.10), and $0 < \delta \ll 1$ is the small constant fixed above. We define $A_7 := \max\{A_6(A_5^-), A_6(A_5^+)\}$, and we set $B_6 := \max\{B_5(A_5^-), B_5(A_5^+)\}$. Then we have the following:

Lemma 7. *There exist $B_7 \geq B_6$ depending only on b_k such that if $\tilde{\zeta}$ is the function appearing in Lemma 6, then*

$$(5.27) \quad \tilde{z}_\pm = (1 \pm \delta)c_k e^{-2\gamma_k\tau} (u^2 - 1)^{1+2\gamma_k} u^{-2} \pm A_7 e^{-4\gamma_k\tau} \tilde{\zeta}(u)$$

are a pair of properly ordered barriers in the region

$$(5.28) \quad 1 + B_5 e^{-\mu_k\tau} \leq u \leq (B_7)^{-1} e^{\tau/2}.$$

Proof. We sketch the proof, which is similar to that of Lemma 2, but somewhat simpler.

As $u \searrow 1$, one calculates that $\tilde{\zeta} \sim (u-1)^{4\gamma_k} > 0$. And as $u \rightarrow \infty$, one has $e^{-2\gamma_k\tau}(u^2-1)^{1+2\gamma_k}u^{-2} \sim e^{-2\gamma_k\tau}u^{4\gamma_k}$ and $e^{-4\gamma_k\tau}\tilde{\zeta} \sim (e^{-2\gamma_k\tau}u^{4\gamma_k})^2$. It follows that the barriers are properly ordered provided that $B_7e^{-\tau/2}u \leq 1$ for some $B_7 \geq B_6$ sufficiently large. \square

6. ANALYSIS OF THE PARABOLIC REGION

Ricci flow solutions cannot escape from the portion of $\partial\Xi_\varepsilon$ that is associated with the tip, intermediate, and outer regions, and is explicitly constructed in Section 5 using upper and lower barriers. Solutions can, however, escape from the portion of $\partial\Xi_\varepsilon$ that is associated with the parabolic region, and is defined by the inequalities listed in (4.4a)–(4.4d). In this section, after establishing a collection of key estimates in Lemmas 8–14, we derive the exit and entrapment results, Lemmas 15–18, that relate to these inequalities and control Ξ_ε in the parabolic region. As noted in Section 4.4, these results play a crucial role in our proof of our main theorem.

6.1. The parabolic-intermediate interface. We recall from equation (3.9) that in the parabolic region, the formal solution indexed by n , k , and b_k satisfies

$$z \approx 2(n-1)k^2(-b_k)^{2/k}e^{-2\gamma_k\tau}(1-u)^{1+2\gamma_k}$$

for $|\sigma| \gg 1$; here we use the identity $2 - 2/k = 1 + 2\gamma_k$. As shown in Lemmas 1–5, we can construct properly ordered and patched barriers that encase this formal solution and are valid for $\tau \geq \tau_6$ and $u \leq 1 - B_8e^{-\mu_k\tau}$; i.e., for $e^{\mu_k\tau}|v| \geq B_8$. Here τ_6 and $B_8 := B_3(2A_3/c_k)^{k/2}$ depend only on b_k . These barriers define Ξ_ε in the intermediate and tip regions. We now use them to derive information about the parabolic-intermediate interface for solutions belonging to Ξ_ε . This interface corresponds to $|\sigma|$ sufficiently large, where “sufficiently large” is made precise below.

Lemma 8. *There exist constants A_8^\pm , B_8 and $\tau_7 \geq \tau_6$ depending only on b_k such that every solution in Ξ_ε satisfies*

$$(6.1) \quad \frac{\sqrt{A_8^-}}{k} \leq \partial_\sigma \{(-e^{\mu_k\tau}v)^{\frac{1}{k}}\} \leq \frac{\sqrt{A_8^+}}{k}$$

for $B_8 \leq e^{\mu_k\tau}|v| \leq \frac{1}{2}e^{\mu_k\tau}$ and $\tau \geq \tau_7$.

Proof. By using (5.8) to bound $|\zeta| \leq C(1-u)^{4\gamma_k}$ for $\frac{1}{2} \leq u \leq 1$, we see from Lemmas 1–2 and the fact that $2\gamma_k = \mu_k(1-2\gamma_k)$ that there exist constants \hat{A}^- , \hat{A}^+ such that any solution in Ξ_ε satisfies

$$\hat{A}^-e^{-2\gamma_k\tau}u^{-2}(1-u^2)^{1+2\gamma_k} \leq z \leq \hat{A}^+e^{-2\gamma_k\tau}u^{-2}(1-u^2)^{1+2\gamma_k}$$

for $\frac{1}{2} \leq u \leq 1 - B_8e^{-\mu_k\tau}$ and $\tau \geq \tau_7 \geq \tau_6$, where \hat{A}^\pm , B_8 , and τ_7 depend only on b_k . This implies that there exist constants A_8^\pm depending only on b_k such that one can bound $(\partial_\sigma(v^{\frac{1}{k}}))^2 = k^{-2}(1-u)^{\frac{2}{k}-2}z$ in the same space-time region as follows:

$$\frac{A_8^-}{k^2}e^{-2\gamma_k\tau} \leq (\partial_\sigma(v^{\frac{1}{k}}))^2 \leq \frac{A_8^+}{k^2}e^{-2\gamma_k\tau}.$$

Recalling that $-v = 1 - u > 0$ and $\mu_k/k = \gamma_k$, we obtain (6.1). \square

We now show that Lemma 8 gives pointwise control of $e^{\mu_k \tau} |v|$ for large $|\sigma|$, specifically in the parabolic-intermediate intersection. Below, we use properties of the operator \mathcal{A} to provide pointwise control of this quantity for smaller $|\sigma|$. Both of these arguments are facilitated by a judicious choice of the parameter $P = 2(B/|b_k|)^{1/k}$ defined in equation (4.5). Noting that there is a universal constant C_k with $|h_k(\sigma) - \sigma^k| \leq C_k |\sigma|^{k-2}$ for $|\sigma| \geq 1$, we choose $B \geq B_8$ large enough, depending only on b_k , so that

$$(6.2) \quad P \geq \frac{10}{3} \sqrt{C_k}.$$

Lemma 9. *If $P = 2(B/|b_k|)^{1/k}$ is chosen as in (6.2), depending only on b_k , then for all sufficiently small ε , depending on $\{b_k, W\}$, every solution in Ξ_ε satisfies*

$$e^{\mu_k \tau} |v(\frac{2}{5}P, \tau)| < B, \quad \text{and} \quad e^{\mu_k \tau} |v(\sigma, \tau)| > B \quad \text{if} \quad \frac{3}{5}P \leq |\sigma| \leq P.$$

Proof. We first show that the bounds are true for the formal solution. Our choice of P ensures that

$$(6.3) \quad |b_k| h_k(\frac{2}{5}P) \leq \left(\frac{4}{5}\right)^k B \left\{ 1 + C_k \left(\frac{5}{4}\right)^2 \left(\frac{2}{P}\right)^2 \right\} \leq \left(\frac{4}{5}\right)^{k-2} B \leq \frac{4}{5}B.$$

On the other hand, if $|\sigma| \geq \frac{3}{5}P$, then

$$|b_k| h_k(\sigma) \geq |b_k| |\sigma|^k \{1 - C_k |\sigma|^{-2}\} \geq \frac{3}{4} |b_k| |\sigma|^k.$$

In particular,

$$|b_k| h_k(\frac{3}{5}P) > \left(\frac{6}{5}\right)^{k-2} B \geq \frac{6}{5}B.$$

Now because $\|\cdot\|_{L^2(\mathcal{P}; e^{-\frac{\sigma^2}{4} d\sigma})} \leq \|\cdot\|_{\mathfrak{h}}$, one may apply a Gagliardo–Nirenberg interpolation inequality (e.g., see p. 125 of [11]) in the compact parabolic region \mathcal{P} to see that

$$\begin{aligned} & \|e^{\mu_k \tau} v - b_k h_k\|_{L^\infty(\mathcal{P})} \\ & \leq C \left\{ \|\partial_\sigma(e^{\mu_k \tau} v - b_k h_k)\|_{L^\infty(\mathcal{P})}^{1/3} \|e^{\mu_k \tau} v - b_k h_k\|_{\mathfrak{h}}^{2/3} + \|e^{\mu_k \tau} v - b_k h_k\|_{\mathfrak{h}} \right\}. \end{aligned}$$

Conditions (4.4a)–(4.4c) imply that $\|e^{\mu_k \tau} v - b_k h_k\|_{\mathfrak{h}} \leq 3\varepsilon$ for solutions in Ξ_ε . Hence by condition (4.4d), one has

$$\|e^{\mu_k \tau} v - b_k h_k\|_{L^\infty(\mathcal{P})} \leq C(1 + W^{1/3})\varepsilon^{2/3}.$$

So the result holds for all sufficiently small ε , depending on $\{b_k, W\}$, where the constant W is chosen in Lemma 18 below. \square

Remark. Because v is continuous, it follows that there exists $\sigma_B(\tau) \in (\frac{2}{5}P, \frac{3}{5}P)$ such that $e^{\mu_k \tau} v(\sigma_B(\tau), \tau) = -B$.

Hereafter, we assume that $\varepsilon = \varepsilon(b_k, W)$ is sufficiently small, as indicated in Lemma 9. One then has the following result, anticipated in Section 4.4:

Lemma 10. *There exist $A^\pm = A_8^\pm$, δ_1 , and $\tau_8 \geq \tau_7$ depending only on b_k such that for $\tau \geq \tau_8$, every solution in Ξ_ε satisfies*

$$\frac{\sqrt{A^-}}{k} \leq \partial_\sigma \left\{ (-e^{\mu_k \tau} v)^{\frac{1}{k}} \right\} \leq \frac{\sqrt{A^+}}{k}$$

in the parabolic-intermediate interface $|\sigma| \in (\frac{3}{5}P, \delta_1 e^{\gamma k \tau})$, where $P = 2(B/|b_k|)^{1/k}$.

Proof. Lemma 9 implies that any solution $v \in \Xi_\varepsilon$ obeys the derivative estimate of Lemma 8 for all $|\sigma| \geq \frac{3}{5}P$ such that $|v| \leq \frac{1}{2}$. Further, so long as Lemma 8 applies, one has

$$(e^{\mu_k \tau} |v|)^{\frac{1}{k}} \leq B^{\frac{1}{k}} + \frac{\sqrt{A^+}}{k} \left(|\sigma| - \frac{3}{5}P \right).$$

The RHS is bounded above by $(\frac{1}{2}e^{\mu_k \tau})^{\frac{1}{k}}$ provided that $|\sigma| \leq \delta_1 e^{\gamma k \tau}$ and $\tau \geq \tau_8$, where we choose $\tau_8 \geq \tau_7$ large enough so that $\delta_1 e^{\gamma k \tau_8} \geq 2P$. \square

We next establish the following bound, also anticipated in Section 4.4:

Lemma 11. *There exist C_\pm depending on $\{b_k, W\}$ such that for $|\sigma| \in (\frac{3}{5}P, \delta_1 e^{\gamma k \tau})$ and $\tau \geq \tau_8$, every solution in Ξ_ε satisfies*

$$C_- |\sigma|^k \leq e^{\mu_k \tau} |v| \leq C_+ |\sigma|^k.$$

Proof. With $C_- := (\min\{B^{1/k}/(\frac{3}{5}P), \sqrt{A^-}/k\})^k$, we apply Lemmas 9–10 to see that $(e^{\mu_k \tau} |v|)^{1/k} \geq \int_0^\sigma C_-^{1/k} d\sigma$ for $\frac{3}{5}P \leq \sigma \leq \delta_1 e^{\gamma k \tau_1}$. An analogous estimate holds for $-\delta_1 e^{\gamma k \tau_1} \leq \sigma \leq -\frac{3}{5}P$ in the case that k is even. This proves the lower bound.

For the upper bound, note that $(e^{\mu_k \tau} |v(\frac{2}{5}P, \tau)|)^{1/k} \leq B^{1/k}$ holds by Lemma 9. In the region $\frac{2}{5}P \leq |\sigma| \leq \frac{3}{5}P$, one has $|b_k| h_k(\sigma) \geq \frac{7}{16} (\frac{4}{5})^k B$ for the formal solution, and hence $e^{\mu_k \tau} |v| \geq (\frac{4}{5})^{k+3} B$, as in the proof of Lemma 9. So for $\frac{2}{5}P \leq |\sigma| \leq \frac{3}{5}P$, condition (4.4d) implies that $\partial_\sigma (e^{\mu_k \tau} |v|)^{\frac{1}{k}} \leq \vartheta(b_k)$, where

$$\vartheta(b_k) := \frac{W}{k} \left(\left(\frac{4}{5} \right)^{k+3} B \right)^{\frac{1}{k}-1}.$$

If we now choose $C_+ := (\max\{B^{1/k}/(\frac{2}{5}P), \vartheta(b_k), \sqrt{A^+}/k\})^k$, then the upper bound follows. \square

Remark. Combining Lemma 10 with Lemma 11, one obtains

$$(6.4) \quad C'_- |\sigma|^{k-1} \leq e^{\mu_k \tau} |v_\sigma| \leq C'_+ |\sigma|^{k-1}$$

for $|\sigma| \in (\frac{3}{5}P, \delta_1 e^{\gamma k \tau})$, where $C'_\pm := C_\pm^{1-1/k} \sqrt{A^\pm}$ depend on $\{b_k, W\}$.

6.2. Estimating the “error terms”. The results in the previous section prepare us to derive useful bounds for $|(\partial_\tau - \mathcal{A})v|$ and $\|(\partial_\tau - \mathcal{A})\check{v}\|_b$.

Lemma 12. *There exist C_1 depending only on b_k , and $\tau_9 \geq \tau_8$ depending on $\{b_k, W\}$, such that for every solution in Ξ_ε , the pointwise bound*

$$|(\partial_\tau - \mathcal{A})v| \leq C_1 W^2 e^{-2\mu_k \tau}$$

holds in the parabolic region \mathcal{P} for all $\tau \geq \tau_9$.

Proof. Recalling that $\check{v} = v$ in the parabolic region, we determine from (4.2) that $(\partial_\tau - \mathcal{A})v = N_{\text{loc}} - nv_\sigma I$ in \mathcal{P} , where

$$N_{\text{loc}} = \frac{2(n-1)v_\sigma^2 - v^2}{2(1+v)}$$

and

$$I = \int_0^\sigma \frac{v_{\hat{\sigma}\hat{\sigma}}(\hat{\sigma}, \tau)}{1+v(\hat{\sigma}, \tau)} d\hat{\sigma}.$$

If we choose $\tau_9 \geq \tau_8$, depending on $\{b_k, W\}$, so that $We^{-\mu_k\tau_9} \leq \frac{1}{2}$, then it follows from (4.4d) that one has $1 + v \geq 1 - We^{-\mu_k\tau} \geq \frac{1}{2}$ for $|\sigma| \leq P$ and $\tau \geq \tau_9$. Inequality (4.4d) also implies that there exists C_0 (depending only on b_k) such that $|N_{\text{loc}}| \leq C_0W^2e^{-2\mu_k\tau}$ holds in \mathcal{P} for $\tau \geq \tau_9$.

Integration by parts shows that

$$(6.5) \quad I = \frac{v_\sigma}{1+v} \Big|_0^\sigma + \int_0^\sigma \frac{v_\sigma^2}{1+v} d\hat{\sigma}.$$

In \mathcal{P} , the first term on the RHS of (6.5) is bounded by $4We^{-\mu_k\tau}$, while the second is bounded by $2PW^2e^{-2\mu_k\tau} \leq 4PWe^{-\mu_k\tau}$. Noting that $|v_\sigma| \leq We^{-\mu_k\tau}$ in \mathcal{P} , we obtain a suitable bound for $|v_\sigma I|$ and thus complete the proof. \square

Lemma 13. *There exist C_2 depending only on b_k , and $\tau_{10} \geq \tau_9$ depending on $\{b_k, W\}$, such that for all $\tau \geq \tau_{10}$, every solution in Ξ_ε satisfies*

$$\|(\partial_\tau - \mathcal{A})\check{v}\|_{\mathfrak{h}} \leq C_2e^{-\frac{3}{2}\mu_k\tau}.$$

Proof. It follows from Lemma 12 that there exists \hat{C}_0 depending only on b_k such that a pointwise bound $|(\partial_\tau - \mathcal{A})v| \leq \hat{C}_0e^{-\frac{3}{2}\mu_k\tau}$ holds for $|\sigma| \leq P$, and for all sufficiently large τ , depending on $\{b_k, W\}$. Thus to complete the proof, we concentrate on the rest of the support of $(\partial_\tau - \mathcal{A})v$, namely $P \leq |\sigma| \leq \frac{6}{5}e^{\gamma_k\tau/5}$. We assume that τ is large enough so that $e^{\gamma_k\tau/5} \leq \delta_1e^{\gamma_k\tau}$.

It follows from equation (4.2) that $(\partial_\tau - \mathcal{A})\check{v} = \eta(N_{\text{loc}} - nv_\sigma I) + E$ for $|\sigma| \in (P, \frac{6}{5}e^{\gamma_k\tau/5})$, where $E = (\eta_\tau - \eta_\sigma\sigma + \frac{\sigma}{2}\eta_\sigma)v - 2\eta_\sigma v_\sigma$.

For ε chosen as in Lemma 9, it follows from Lemma 10 that $1 + v \geq \frac{1}{2}$ in the interval $(\frac{3}{5}P, \delta_1e^{\gamma_k\tau})$. In the same region, one has $C_-|\sigma|^k \leq e^{\mu_k\tau}|v| \leq C_+|\sigma|^k$ and $C'_-|\sigma|^{k-1} \leq e^{\mu_k\tau}|v_\sigma| \leq C'_+|\sigma|^{k-1}$ as consequences of Lemma 11 and estimate (6.4), respectively.

Combining these inequalities, and again integrating by parts to evaluate I , we obtain \hat{C}_1 depending on $\{b_k, W\}$ and \hat{C}_2 depending only on b_k such that one has

$$|N_{\text{loc}}| + |v_\sigma I| \leq \hat{C}_1e^{-2\mu_k\tau}|\sigma|^{2k} \leq \hat{C}_2e^{-\frac{3}{2}\mu_k\tau}$$

in the interval $P \leq |\sigma| \leq \frac{6}{5}e^{\gamma_k\tau/5}$, at all τ sufficiently large, depending on $\{b_k, W\}$. Here we use the fact that $2k\gamma_k/5 < \mu_k/2$.

Similarly, the pointwise estimates for v and v_σ above further imply that there exist constants \hat{C}_3, \hat{C}_4 depending on $\{b_k, W\}$, and \hat{C}_5 depending only on b_k , such that the estimates

$$\begin{aligned} \|E\|_{\mathfrak{h}}^2 &\leq \hat{C}_3e^{-2\mu_k\tau} \int_{e^{\gamma_k\tau/5}}^\infty |\sigma|^{2k+2} e^{-\sigma^2/4} d\sigma \\ &\leq \hat{C}_4e^{-2\mu_k\tau} \exp\left(-\frac{e^{2\gamma_k\tau/5}}{4}\right) e^{(2k+1)\gamma_k\tau/5} \\ &\leq \hat{C}_5e^{-3\mu_k\tau} \end{aligned}$$

hold for all τ sufficiently large, depending on $\{b_k, W\}$. The result follows. \square

Finally, we derive pointwise bounds for \check{v} that are independent of W — bounds that apply to solutions originating from initial data we construct in Section 7 below.

Given a smooth function $f(\sigma)$ and a constant $R > 0$, we define

$$\|f\|_{C^1(R)} := \sup_{|\sigma| \leq R} (|f(\sigma)| + |f_\sigma(\sigma)|).$$

Lemma 14. *If at time $\tau_{11} \geq \tau_{10}$, one has*

$$\|\check{v}(\cdot, \tau_{11})\|_{C^1(2P)} \leq M e^{-\mu_k \tau_{11}}$$

for some constant $M > 0$, then for all $\tau \geq \tau_{11}$, one has

$$(6.6) \quad \|v(\cdot, \tau)\|_{C^1(P)} \leq C_6(1 + M)e^{-\mu_k \tau},$$

where $C_6 \geq 1$ depends only on b_k (and is in particular independent of W).

Proof. By (4.2), we have $\check{v}_\tau - \mathcal{A}\check{v} = F(\sigma, \tau)$, where $F = \eta N[v] + E[\eta, v]$. Thus for $\tau \geq \tau_{11} + 1$, the variation of constants formula lets us write

$$(6.7) \quad \check{v}(\cdot, \tau) = e^{\mathcal{A}}\check{v}(\cdot, \tau - 1) + \int_{\tau-1}^{\tau} e^{(\tau-\tau')\mathcal{A}} F(\cdot, \tau') d\tau'.$$

Standard regularizing estimates for the operator \mathcal{A} imply that

$$\|e^{\mathcal{A}}\check{v}(\cdot, \tau - 1)\|_{C^1(P)} \leq \check{C}_1 \|\check{v}(\cdot, \tau - 1)\|_{\mathfrak{h}},$$

and for $0 < \tau - \tau' < 1$, that

$$\|e^{(\tau-\tau')\mathcal{A}} F(\cdot, \tau')\|_{C^1(P)} \leq \check{C}_1 (\tau - \tau')^{-3/4} \|F(\cdot, \tau')\|_{\mathfrak{h}},$$

where \check{C}_1 depends only on P , hence by (4.5), only on b_k . Conditions (4.4a)–(4.4c) imply that there exists \check{C}_2 depending only on b_k such that

$$\|\check{v}(\cdot, \tau - 1)\|_{\mathfrak{h}} \leq \check{C}_2 e^{-\mu_k \tau}.$$

Furthermore, because $\tau_{11} \geq \tau_{10} = \tau_{10}(b_k, W)$, Lemma 13 provides C_2 depending only on b_k such that

$$\|F(\cdot, \tau')\|_{\mathfrak{h}} \leq C_2 e^{-\frac{3}{2}\mu_k \tau'}.$$

Combining these estimates and integrating (6.7) gives (6.6) for times $\tau \geq \tau_{11} + 1$.

To prove (6.6) for $\tau \in (\tau_{11}, \tau_{11} + 1)$, we again use variation of constants, writing

$$(6.8) \quad \check{v}(\cdot, \tau) = e^{(\tau-\tau_{11})\mathcal{A}}\check{v}(\cdot, \tau_{11}) + \int_{\tau_{11}}^{\tau} e^{(\tau-\tau')\mathcal{A}} F(\cdot, \tau') d\tau'.$$

As above, we can use the estimate for $\|F(\cdot, \tau')\|_{\mathfrak{h}}$ given by Lemma 13 to obtain a satisfactory C^1 estimate for the integral term. However, we cannot use the smoothing properties of the operator $e^{\theta\mathcal{A}}$ to estimate the first term on the RHS, since the time delay $\theta = \tau - \tau_{11}$ may now be arbitrarily short. But since \mathcal{A} is a nondegenerate parabolic operator, there exists \check{C}_3 depending only on P , hence only on b_k , such that for $\theta \in (0, 1)$, one has

$$\|e^{\theta\mathcal{A}}\check{v}(\cdot, \tau_{11})\|_{C^1(P)} \leq \check{C}_3 \|\check{v}(\cdot, \tau_{11})\|_{C^1(2P)}.$$

Since $\|\check{v}(\cdot, \tau_{11})\|_{C^1(2P)} \leq M e^{-\mu_k \tau_{11}}$ by hypothesis, this lets us estimate the first term on the RHS of (6.8). Thus we obtain (6.6) for $\tau_{11} < \tau < \tau_{11} + 1$, which completes the proof. \square

6.3. Exit and entrapment results. We now use the estimates derived above to prove the exit and entrapment results corresponding to various portions of the boundary of Ξ_ε in the parabolic region. Our first result shows that solutions immediately exit if they contact $(\partial\Xi_\varepsilon)_-$.

Lemma 15. *There exists $\bar{\tau} \geq \tau_{10}$ depending on $\{b_k, W\}$ such that if a solution $v \in \Xi_\varepsilon$ contacts $\partial\Xi_\varepsilon$ by achieving equality in (4.4a) at $\tau \geq \bar{\tau}$, then it immediately exits Ξ_ε .*

Proof. We recall from Section 4 that the projection v_{k-} represents the rapidly-growing perturbations of a formal solution. Since the projection from \check{v} to v_{k-} commutes with the operator \mathcal{A} , and since the projections $\{v_{k-}, v_k, v_{k+}\}$ are pairwise orthogonal, one has

$$\frac{1}{2} \frac{d}{d\tau} \|v_{k-}\|_{\mathfrak{h}}^2 = \left(v_{k-}, \frac{\partial \check{v}}{\partial \tau} \Big|_{\sigma} \right)_{\mathfrak{h}} = (v_{k-}, \mathcal{A}\check{v} + (\partial_{\tau} - \mathcal{A})\check{v})_{\mathfrak{h}}.$$

We readily verify the inequality $(v_{k-}, \mathcal{A}\check{v})_{\mathfrak{h}} \geq -\mu_{k-1} \|v_{k-}\|_{\mathfrak{h}}^2$, from which it follows that

$$\frac{1}{2} \frac{d}{d\tau} \|e^{\mu_k \tau} v_{k-}\|_{\mathfrak{h}}^2 \geq (\mu_k - \mu_{k-1}) \|e^{\mu_k \tau} v_{k-}\|_{\mathfrak{h}}^2 - |(e^{\mu_k \tau} v_{k-}, e^{\mu_k \tau} (\partial_{\tau} - \mathcal{A})\check{v})_{\mathfrak{h}}|.$$

Hence using Lemma 13 and Cauchy–Schwarz, we obtain

$$\frac{1}{2} \frac{d}{d\tau} \|e^{\mu_k \tau} v_{k-}\|_{\mathfrak{h}}^2 \Big|_{\|e^{\mu_k \tau} v_{k-}\|_{\mathfrak{h}} = \varepsilon} \geq \varepsilon \left(\frac{\varepsilon}{2} - C_2 e^{-\frac{1}{2} \mu_k \tau} \right) > 0$$

at all times $\tau \geq \bar{\tau}$, for $\bar{\tau} \geq \tau_{10}$ chosen sufficiently large, depending on $\{b_k, W\}$. \square

Our next three results are entrapment lemmas.

Lemma 16. *There exists $\bar{\tau} \geq \tau_{10}$ depending on $\{b_k, W\}$ such that solutions $v \in \Xi_{\varepsilon}$ cannot contact $\partial \Xi_{\varepsilon}$ by achieving equality in condition (4.4c) at any $\tau \geq \bar{\tau}$.*

Proof. Arguing as in the proof of Lemma 15, one obtains

$$\frac{1}{2} \frac{d}{d\tau} \|v_{k+}\|_{\mathfrak{h}}^2 = (v_{k+}, \mathcal{A}\check{v} + (\partial_{\tau} - \mathcal{A})\check{v})_{\mathfrak{h}}.$$

Here, one has $(v_{k+}, \mathcal{A}\check{v})_{\mathfrak{h}} \leq -\mu_{k+1} \|v_{k+}\|_{\mathfrak{h}}^2$, from which it follows that

$$\frac{1}{2} \frac{d}{d\tau} \|e^{\mu_k \tau} v_{k+}\|_{\mathfrak{h}}^2 \leq (\mu_k - \mu_{k+1}) \|e^{\mu_k \tau} v_{k+}\|_{\mathfrak{h}}^2 + |(e^{\mu_k \tau} v_{k+}, e^{\mu_k \tau} (\partial_{\tau} - \mathcal{A})\check{v})_{\mathfrak{h}}|.$$

Combining this inequality with Lemma 13, we see that if a solution were to contact $\partial \Xi_{\varepsilon}$, then one would have

$$\frac{1}{2} \frac{d}{d\tau} \|e^{\mu_k \tau} v_{k+}\|_{\mathfrak{h}}^2 \Big|_{\|e^{\mu_k \tau} v_{k+}\|_{\mathfrak{h}} = \varepsilon} \leq \varepsilon \left(\frac{\varepsilon}{2} + C_2 e^{-\frac{1}{2} \mu_k \tau} \right) < 0$$

for all $\tau \geq \bar{\tau} \geq \tau_{10}$. By continuity, in a neighborhood of $\{\|e^{\mu_k \tau} v_{k+}\|_{\mathfrak{h}} = \varepsilon\}$, one has $\frac{d}{d\tau} \|e^{\mu_k \tau} v_{k+}\|_{\mathfrak{h}}^2 < 0$ for $\tau \geq \bar{\tau}$; this ensures that equality is never achieved. \square

Our final two entrapment results apply to solutions originating in the set $D_{\varepsilon}^{\bar{\tau}}$ defined by

$$(6.9) \quad D_{\varepsilon}^{\bar{\tau}} := \left\{ v : \begin{array}{l} \|v_k(\cdot, \bar{\tau}) - b_k e^{-\mu_k \bar{\tau}} h_k\|_{\mathfrak{h}} \leq \frac{\varepsilon}{2} e^{-\mu_k \bar{\tau}} \\ \|v_k(\cdot, \bar{\tau}) - b_k e^{-\mu_k \bar{\tau}} h_k\|_{C^1(2P)} \leq 100 e^{-\mu_k \bar{\tau}} \end{array} \right\}.$$

This set is designed so that solutions satisfying $v(\cdot, \bar{\tau}) \in \Xi_{\varepsilon} \cap D_{\varepsilon}^{\bar{\tau}}$ cannot exit Ξ_{ε} by violating conditions (4.4b) or (4.4d).

Lemma 17. *There exists $\bar{\tau} \geq \tau_{10}$ large enough, depending on $\{b_k, W\}$, such that solutions satisfying $v(\cdot, \bar{\tau}) \in \Xi_{\varepsilon} \cap D_{\varepsilon}^{\bar{\tau}}$ cannot contact $\partial \Xi_{\varepsilon}$ by achieving equality in condition (4.4b) at any $\tau \geq \bar{\tau}$.*

Proof. We first note that one can write $v_k(\sigma, \tau) = \phi(\tau)\eta(P^{-1}\sigma)h_k(\sigma)$, where the quantity

$$\phi(\tau) := \|h_k\|_{\mathfrak{h}}^{-2} (\check{v}, h_k)_{\mathfrak{h}}$$

can be shown to evolve by

$$\frac{d}{d\tau}\phi = -\mu_k\phi + \|h_k\|_{\mathfrak{h}}^{-2} ((\partial_\tau - \mathcal{A})\check{v}, h_k)_{\mathfrak{h}}.$$

Observing that

$$\|e^{\mu_k\tau}v_k - b_k h_k\|_{\mathfrak{h}} = (e^{\mu_k\tau}\eta\phi - b_k)\|h_k\|_{\mathfrak{h}},$$

one computes that

$$\begin{aligned} \frac{1}{2} \frac{d}{d\tau} \|e^{\mu_k\tau}v_k - b_k h_k\|_{\mathfrak{h}}^2 &= \|h_k\|_{\mathfrak{h}}^2 (e^{\mu_k\tau}\phi - b_k) \frac{d}{d\tau} (e^{\mu_k\tau}\phi - b_k) \\ &\leq \eta \|e^{\mu_k\tau}v_k - b_k h_k\|_{\mathfrak{h}} |((\partial_\tau - \mathcal{A})\check{v}, h_k)_{\mathfrak{h}}|. \end{aligned}$$

It then follows from Lemma 13 that one has

$$\frac{d}{d\tau} \|e^{\mu_k\tau}v_k - b_k h_k\|_{\mathfrak{h}} \leq C_2 \|h_k\|_{\mathfrak{h}} e^{-\frac{3}{2}\mu_k\tau}.$$

So if $v \in \Xi_\varepsilon \cap D_\varepsilon^{\bar{\tau}}$, then for all $\tau \geq \bar{\tau}$, one has

$$\|e^{\mu_k\tau}v_k - b_k h_k\|_{\mathfrak{h}} \leq \frac{\varepsilon}{2} + \frac{2C_2 \|h_k\|_{\mathfrak{h}}}{3\mu_k} e^{-\frac{3}{2}\mu_k\bar{\tau}} < \varepsilon$$

provided that $\bar{\tau}$ is chosen sufficiently large, depending on b_k and ε (which depends on $\{b_k, W\}$ via Lemma 9). We conclude that equality in (4.4b) cannot be attained. \square

Lemma 18. *There exists W sufficiently large, depending only on b_k , and there exists $\bar{\tau} \geq \tau_{10}$ sufficiently large, depending on $\{b_k, W\}$, such that solutions satisfying $v(\cdot, \bar{\tau}) \in \Xi_\varepsilon \cap D_\varepsilon^{\bar{\tau}}$ cannot contact $\partial\Xi_\varepsilon$ by achieving equality in (4.4d) at any $\tau \geq \bar{\tau}$.*

Proof. Applying Lemma 14 with $\tau_{11} = \bar{\tau} \geq \tau_{10}$ and $M = 100$ yields

$$\sup_{|\sigma| \leq P} \{|v(\sigma, \tau)| + |v_\sigma(\sigma, \tau)|\} \leq C_3 e^{-\mu_k\tau}$$

for all $\tau \geq \bar{\tau}$, where $C_3 := 101C_6$ depends only on b_k . Hence the result holds for any $W > C_3$. \square

7. CONSTRUCTING SUITABLE INITIAL DATA

With the properties of the tubular neighborhood Ξ_ε surrounding a given formal solution $\hat{g}_{\{n,k,b_k\}}(t)$ established (primarily in Sections 5–6), including the exit and entrapment properties of portions of the boundary of Ξ_ε , the remaining work needed in order to complete the proof of our main theorem is to show that there exist parameterized sets of initial data yielding Ricci flow solutions that have the properties discussed in the introduction to Section 4. In particular, presuming that the choices of $\hat{g}_{\{n,k,b_k\}}(t)$ have been fixed, we seek a continuous bijective map Φ from a closed topological k -ball \mathcal{B}^k to the set of rotationally symmetric metrics on \mathcal{S}^{n+1} with the following properties, for some choice of time $t_0 \in [0, T)$: (i) the image of Φ is contained in $\Xi_\varepsilon \cap \{t = t_0\}$; (ii) the image of Φ restricted to $\partial\mathcal{B}^k$ is contained in the exit set $(\partial\Xi_\varepsilon)_-$; (iii) $\Phi|_{\partial\mathcal{B}^k}$ is not contractible; and (iv) the image of Φ is far enough away from the neutral part of the boundary $(\partial\Xi_\varepsilon)_o$ to guarantee that so long as a Ricci flow solution that starts from initial data in $\Phi(\mathcal{B}^k)$ stays in Ξ_ε , it never reaches $(\partial\Xi_\varepsilon)_o$.

We show in this section that for each choice of $\hat{g}_{\{n,k,b_k\}}(t)$ and corresponding Ξ_ε , sets of initial data satisfying these conditions can be found, if ε is small enough.

We first fix the initial time t_0 at which we choose our sets of initial data. Noting that the results stated in the exit and entrapment lemmas discussed in Section 6.3 are only guaranteed to work for $\tau \geq \bar{\tau}$ (corresponding to $t \geq T - e^{-\bar{\tau}}$), we set $t_0 := T - e^{-\bar{\tau}}$, where $\bar{\tau}$ may be presumed to be large enough so that one has $t_0 > 0$ and $\frac{6}{5}e^{\gamma k \bar{\tau}/5} \geq 2P$.

We next define initial data locally using coordinates adapted to the parabolic region of the formal solution, since it is in this region where the restrictions stated above are most critical. Working with the metric variables and coordinates discussed in Section 2.1, and recalling that $k \geq 3$ and $b_k < 0$ are fixed, we see that for every set of k constants

$$b_j \in [-\underline{B}_j, \bar{B}_j], \quad 0 \leq j \leq k-1,$$

(with \underline{B}_j and \bar{B}_j small positive constants to be determined below), if we write

$$u = 1 + \sum_{j=0}^k b_j e^{-\mu_k \bar{\tau}} h_k(\sigma), \quad \left(|\sigma| \leq \frac{7}{5} e^{\gamma k \bar{\tau}/5} \right),$$

and if in addition we set $\varphi(\cdot, \bar{\tau}) \equiv 1$, then a metric $g = (d\sigma)^2 + u^2 g_{\text{can}}$ is determined locally. It is clear that one can choose $\underline{B}_j, \bar{B}_j > 0$ small enough such that the corresponding metrics are as close as desired in the C^1 topology to the approximate solution $1 + b_k e^{-\mu_k \bar{\tau}} h_k(\sigma)$ in the region $|\sigma| \leq \frac{7}{5} e^{\gamma k \bar{\tau}/5}$. Closeness to $\hat{g}_{\{n,k,b_k\}}$ in that same region then follows.

With an eye toward conditions (4.4a)–(4.4c) and (6.9), we choose $\underline{B}_j, \bar{B}_j > 0$ so that the sharp inequalities

$$\begin{aligned} \|v_{k-}\|_{\mathfrak{h}} &< \varepsilon e^{-\mu_k \bar{\tau}}, \\ \|v_k - b_k e^{-\mu_k \bar{\tau}} h_k\|_{\mathfrak{h}} &< \frac{\varepsilon}{2} e^{-\mu_k \bar{\tau}}, \\ \|v_{k+}\|_{\mathfrak{h}} &< \varepsilon e^{-\mu_k \bar{\tau}}, \end{aligned}$$

hold for $(b_0, \dots, b_{k-1}) \in \prod_{j=0}^{k-1} (-\underline{B}_j, \bar{B}_j)$, with

$$\|v_{k-}\|_{\mathfrak{h}} = \varepsilon e^{-\mu_k \bar{\tau}}$$

holding on the boundary, which ensures by Lemma 15 that solutions originating from initial data on the boundary of the topological ball $\prod_{j=0}^{k-1} [-\underline{B}_j, \bar{B}_j]$ immediately exit Ξ_ε . We verify that in choosing \underline{B}_j and \bar{B}_j as indicated, the conditions for these initial data to lie within the tube (at least for the portion of the metrics pertaining to the parabolic region) are satisfied. We may assume that ε is small enough, hence that $\underline{B}_j, \bar{B}_j > 0$ are small enough, so that the C^1 condition in (6.9) is satisfied. Combined with the second inequality above, this ensures that the initial data constructed here belong to $D_\varepsilon^{\bar{\tau}}$. In turn, by Lemma 18, this ensures that the data satisfy (4.4d).

We next obtain globally defined initial data in $(\Xi_\varepsilon \cap \{\tau = \bar{\tau}\}) \cap D_\varepsilon^{\bar{\tau}}$ by using cutoff functions to smoothly glue the metric $g = (d\sigma)^2 + u^2 g_{\text{can}}$ defined locally above to the formal solution in the intermediate and tip regions, and (for k odd) to that portion of the manifold which remains nonsingular. More specifically, we proceed as follows.

For $\frac{6}{5}e^{\gamma_k\bar{\tau}/5} \leq \sigma \leq \frac{7}{5}e^{\gamma_k\bar{\tau}/5}$, the construction above ensures that $z = \psi_s^2$ is as close as desired to its value in the formal solution (3.9), namely

$$\hat{z}_{\{n,k,b_k\}}(u) = c_k e^{-2\gamma_k\tau} (1-u)^{1+2\gamma_k} \quad (1 > u \gg e^{-\gamma_k\tau}).$$

In particular, making ε smaller if necessary, we can ensure that each initial datum lies between the barriers $z_- < z_+$ constructed in Lemma 2 for all u where both it and the barriers are defined. One may thus extend the solution by means of a cutoff function, obtaining \tilde{z} with the properties that $\tilde{z} = z$ at $u(\frac{6}{5}e^{\gamma_k\bar{\tau}/5}\sigma)$ and $\tilde{z} = \hat{z}_{\{n,k,b_k\}}$ for all $0 \leq u \leq u(\frac{7}{5}e^{\gamma_k\bar{\tau}/5}\sigma)$. Here we use the fact that $\hat{z}_{\{n,k,b_k\}}$ smoothly extends to $u = 0$ using the construction in Section 3.3. Moreover, $\hat{z}_{\{n,k,b_k\}}$ lies between the barriers constructed in Lemma 4.

If k is even, we repeat this step for $\frac{6}{5}e^{\gamma_k\bar{\tau}/5} \leq -\sigma \leq \frac{7}{5}e^{\gamma_k\bar{\tau}/5}$, and we are done.

If k is odd and $\frac{6}{5}e^{\gamma_k\bar{\tau}/5} \leq -\sigma \leq \frac{7}{5}e^{\gamma_k\bar{\tau}/5}$, we again find that each initial datum \tilde{z} is as close as desired to the formal solution

$$\hat{z}_{\{n,k,b_k\}}(u) = \hat{c}_k e^{-2\gamma_k\tau} (u^2 - 1)u^{-2} \quad (1 < u \ll e^{\tau/2}),$$

hence lies between the barriers $\tilde{z}_- < \tilde{z}_+$ constructed in Lemma 7. So we again extend \tilde{z} by means of a cutoff function until $u = \hat{\delta}e^{\tau/2}$, namely $\psi = \hat{\delta}$. There we smoothly glue on a smooth punctured sphere as in Section 8.2 of [3]. We omit further details.

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