SINGULARITY FORMATION OF COMPLETE RICCI FLOW SOLUTIONS

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Abstract. We study singularity formation of complete Ricci flow solutions, motivated by two applications: (a) improving the understanding of the behavior of the *essential blowup sequences* of Enders–Müller–Topping [EMT11] on noncompact manifolds, and (b) obtaining further evidence in favor of the conjectured stability of generalized cylinders as Ricci flow singularity models.

Contents

1. Introduction

1.1. **Motivations.** Much is known about Ricci flow in dimensions $n = 2, 3$ and on compact manifolds. Much less is known about solutions on higher-dimensional or noncompact manifolds. In this paper, using multiply-warped products, we investigate various phenomena that occur in singularity formation on complete noncompact solutions $\big(\mathcal{M}, g(t)\big)$ of Ricci flow in arbitrary dimensions. We are most interested in singularities for which noncompactness plays an essential role in the precise sense that the metric on any compact subset $K \subset \mathcal{M}$ remains nonsingular. Our most significant results for solutions of this type are found in Theorem 5, Theorem 6, and Corollary 7 below.

An application of those results, which is our main motivation for writing this paper, is as follows: we show that standard sequences of parabolic dilations at a singularity, which produce predictable subsequential limits on compact solutions, as shown by Enders–Müller–Topping [EMT11], can yield unexpected limits for noncompact solutions unless additional criteria are imposed. We state this application above the discussion of our main theorems. In a second application, introduced below that discussion, we prove a weak stability result for generalized cylinders

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evolving by Ricci flow, one that is motivated by well-known and much stronger results of Colding–Minicozzi [CM12, CM15] for mean curvature flow.

1.2. **Application: essential blowup sequences on noncompact manifolds.** The main application of Theorem 6 and Corollary 7 that we have in mind in this paper is to obtain new insights into blowup limits of singularities on complete noncompact manifolds. We rigorously explore the phenomena that occur if finite-time singularities form at spatial infinity on noncompact manifolds. More precisely, we construct complete Ricci flow solutions for which Type-I singularities occur at spatial infinity and which do not have any Type-I singular points. The existence of such (singly-warped) examples has been conjectured in [EMT11]. We show that for each of our (doubly-warped) examples, taking a blowup limit along some essential blow up sequence (see Section 4 for precise definitions) yields a gradient shrinking Ricci soliton in the subsequential limit, whereas taking a subsequential limit along some other essential blow up sequence yields a complete ancient solution that is not a soliton. We summarize these results in the following theorem:

1. **Theorem.** *For any $p \geq 2$, there exist complete, noncompact, κ-noncollapsed Ricci flow solutions $(\mathcal{M}, g(t))$, with $\mathcal{M} := \mathbb{R} \times \mathcal{S}^p \times \mathcal{S}^p$, that develop Type-I singularities at spatial infinity. For each of these solutions, the set of Type-I singular points is empty.*

Furthermore, on each of these solutions, there exist essential blowup sequences along which a blowup limit yields a nontrivial gradient shrinking Ricci soliton, and there exist essential blowup sequences along which no blowup limit can be a gradient shrinking Ricci soliton.

The key idea is that on noncompact Ricci flow solutions, there can be essential blowup sequences with no Type-I singular point limit, and these sequences may or may not have nontrivial gradient Ricci soliton limits.[1] However, one can obtain soliton limits by imposing another condition. Indeed, we show the following in the proof of Theorem 1:

2. **Corollary.** *Under the conditions of Theorem 1, a blowup limit of the flow along a sequence (x_j, t_j) with $|x_j| \to \infty$ and $t_j \to a_* < \infty$ is a nontrivial gradient soliton if and only if*

$$\lim_{j\to\infty} \frac{|\operatorname{Rm}(x_j, t_j)|}{\sup_{\mathcal{M}} |\operatorname{Rm}(\cdot, t_j)|} = 1. \tag{1}$$

In other words, to obtain a nontrivial gradient shrinking soliton limit, it is both necessary and sufficient that $|\operatorname{Rm}(x_j, t_j)| \to \sup_{\mathcal{M}} |\operatorname{Rm}(\cdot, t_j)|$ as $|x_j| \to \infty$. Clearly, the subsequences we construct in Theorem 1 that fail to have soliton limits do not satisfy this condition. For noncompact solutions on which the set of Type-I singular points is empty, we conjecture that (1) is sufficient to obtain gradient soliton limits for all singularities that form, not just those satisfying the multiply-warped *Ansatz* we employ in this paper.

We obtain a related result for solutions on $\mathcal{M} = \mathbb{R} \times \mathcal{S}^1 \times \mathcal{S}^p$. These are not κ-noncollapsed, hence do not have blowup limits except as étale groupoids, in the sense considered by Lott [Lott10].

[1] While singly-warped examples may also exhibit singularities that form at spatial infinity, we use the doubly-warped hypothesis in the proof to rule out the existence of nontrivial gradient Ricci soliton limits.

We believe that the arguments we use to prove Theorem 1 could easily be extended to construct κ-noncollapsed examples on $\mathbb{R}^k \times \mathcal{S}^p \times \mathcal{S}^p$ for any $k \geq 1$ and $p \geq 2$ with the same properties that (a) their singularities occur at spatial infinity and (b) distinct subsequential blowup limits are possible.

1.3. **Manifolds.** We now proceed to discuss our general results. Let $(\mathcal{B}^n, g_{\mathcal{B}})$ be a complete noncompact Riemannian manifold. For $\alpha \in \{1, \dots, A < \infty\}$, let $(\mathcal{F}_\alpha^{n_\alpha}, g_{\mathcal{F}_\alpha})$ be a collection of space forms,[2] and let μ_α be constants such that $\mu_\alpha g_{\mathcal{F}_\alpha} = 2\,\mathrm{Rc}[g_{\mathcal{F}_\alpha}]$. Given functions $u_\alpha : \mathcal{B}^n \to \mathbb{R}_+$, there is a warped product metric g on the manifold $\mathcal{M}^{\mathcal{N}} = \mathcal{B}^n \times \mathcal{F}_1^{n_1} \times \cdots \times \mathcal{F}_A^{n_A}$, where $\mathcal{N} = n + \sum_{\alpha=1}^A n_\alpha$, given by

$$g = g_{\mathcal{B}} + \sum_{\alpha=1}^{A} u_\alpha g_{\mathcal{F}_\alpha}. \tag{2}$$

For brevity, we omit the dimensions of the manifold $\mathcal{M}^{\mathcal{N}}$ and its factors $\mathcal{F}_\alpha^{n_\alpha}$ in what follows.

Under Ricci flow, the structure (2) of the multiply-warped product metric is preserved, and the base metric $g_{\mathcal{B}}$ and warping functions u_α evolve by the coupled diffusion-reaction system

$$\partial_t\, g_{\mathcal{B}} + 2\,\mathrm{Rc}[g_{\mathcal{B}}] = -2 \sum_{\alpha=1}^{A} n_\alpha u_\alpha^{-1/2} \nabla^2 (u_\alpha^{1/2}), \tag{3a}$$

$$(\partial_t - \Delta)\, u_\alpha = -\mu_\alpha - u_\alpha^{-1} |\nabla u_\alpha|^2, \qquad (\alpha \in \{1, \dots, A\}). \tag{3b}$$

3. **Remark.** *Throughout this paper, undecorated geometric quantities are computed with respect to the metric g on $\mathcal{M}$ and its Levi–Civita connection. In particular, the Laplacian in (3) denotes that of the metric g,* i.e., *$\Delta \equiv \Delta_{\mathcal{M}}$, rather than the Laplacian $\Delta_{\mathcal{B}}$ of the metric $g_{\mathcal{B}}$ on the base. Given any smooth function $\varphi(x)$ depending only on $x \in \mathcal{B}$, the two differential operators are related by*

$$\Delta_{\mathcal{M}} \varphi = \Delta_{\mathcal{B}} \varphi + \frac{1}{2} \sum_{\alpha=1}^{A} n_\alpha u_\alpha^{-1} \langle \nabla u_\alpha, \nabla \varphi \rangle, \tag{4}$$

as follows easily from Claim 32 of Appendix A. In keeping with this convention, the undecorated symbol ∇^2 above denotes the covariant Hessian on $\mathcal{M}$.

If some $u_\alpha(x, 0)$ is a constant a_α, then $u_\alpha(x,t) = a_\alpha - \mu_\alpha t$ is an explicit solution of (3b) for as long as the flow remains smooth. Since we are interested in studying perturbations (though not necessarily small everywhere) of spatially homogeneous solutions, we set $a_\alpha = \inf_{x \in \mathcal{B}} u_\alpha(x, 0)$ and define $v_\alpha(\cdot, 0) : \mathcal{B} \to \mathbb{R}_+$ by

$$v_\alpha(x, 0) = u_\alpha(x, 0) - a_\alpha, \tag{5}$$

for $\alpha \in \{1, \dots, A\}$. We observe that for as long as a smooth solution of system (3) exists, the metric has the form

$$g(x,t) = g_{\mathcal{B}}(x,t) + \sum_{\alpha=1}^{A} \big\{ (a_\alpha - \mu_\alpha t) + v_\alpha(x,t) \big\} g_{\mathcal{F}_\alpha}, \tag{6}$$

[2]Our theorems in this paper directly imply the same results in the more general case that the factors $\mathcal{F}_\alpha^{n_\alpha}$ are Einstein manifolds, but the space form hypothesis facilitates the explicit curvature calculations that we perform in Appendix A.

where $u_\alpha(x,t) = (a_\alpha - \mu_\alpha t) + v_\alpha(x,t)$.

4. **Remark.** *The construction outlined above ensures that* $\inf_{x\in\mathcal{B}} v_\alpha(x,0) = 0$. *Because our solutions are not compact, it is not automatic that* $\inf_{x\in\mathcal{B}} v_\alpha(x,t) = 0$ *for those* $t > 0$ *for which a solution exists. However, this is true and follows from results we prove below.*

In Appendix A, we compute the curvatures of $(\mathcal{M}, g)$. Here, for $\alpha \in \{1,\dots,A\}$ and all $t \geq 0$ that a Ricci flow solution exists, we define the functions

$$\gamma_\alpha(x,t) = |\nabla v_\alpha(x,t)|^2, \tag{7a}$$

$$\chi_\alpha(x,t) = |\nabla^2 v_\alpha(x,t)|^2_{g_\mathcal{B}}, \tag{7b}$$

$$\rho(x,t) = \left|\operatorname{Rm}[g_\mathcal{B}](x,t)\right|^2_{g_\mathcal{B}}, \tag{7c}$$

where the second and third norms are computed with respect to $g_\mathcal{B}$.[3] To motivate these quantities, we note that it follows from Remark 34 in Appendix A that there is a universal constant C depending only on the dimensions such that

$$\left|\operatorname{Rm}[g] - \sum_{\alpha=1}^{A} u_\alpha \operatorname{Rm}[g_{\mathcal{F}_\alpha}]\right|_g \leq C\left\{\rho^{1/2} + \sum_{\alpha=1}^{A}\left(u_\alpha^{-2}\gamma_\alpha + u_\alpha^{-1}\chi_\alpha^{1/2}\right)\right\}, \tag{8}$$

where g is the metric on the total space $\mathcal{M}$. So in any open set in which the quantities v_α are small relative to u_α (which need not be true globally but is always true sufficiently close to spatial infinity), control of ρ, γ_α/u_α^2, and χ_α/u_α^2 indicates that the curvature is pointwise close to that of an un-warped product.

1.4. Main results.

In this paper, we assume that γ_α, χ_α, and ρ are bounded on our initial data in terms of a constant C_{init} and functions G_α and H_α in a manner that we call our Main Assumptions and make precise in Section 2.1. (Specifically, we use G_α to bound γ_α and H_α to bound χ_α.)

Our first general result provides an asymptotic description of all solutions of Ricci flow originating from initial data g_{init} that satisfy those assumptions. Specifically, it shows in a precise sense that the asymptotics of the original data are preserved:

5. **Theorem.** *Let* $\big(\mathcal{M}, g(t)\big)$ *be a solution of the Ricci flow system* (3) *that originates from initial data* g_{init} *satisfying our Main Assumptions and exists for* $t \in [0, T_{\text{small}}]$.

There exists $C_* = C_*(n, n_\alpha, C_{\text{init}})$ *such that for* $t \in [0, \min\{T_{\text{small}}, C_*^{-1}\})$, *the metric can be written as*

$$\begin{aligned} g(x,t) &= g_\mathcal{B}(x,t) \\ &\quad + \sum_{\alpha=1}^{A}\left\{(a_\alpha - \mu_\alpha t) + \left(1 + \mathcal{O}\left(\frac{G_\alpha\left(v_\alpha(x,0)\right)}{v_\alpha^2(x,0)}\right)\right) v_\alpha(x,0)\right\} g_{\mathcal{F}_\alpha}, \end{aligned}$$

where $1/C_* \leq |g_\mathcal{B}(x,t)| \leq C_*$.

[3]Strictly speaking, these are the norms of the projections of $\nabla^2 v_\alpha$ and Rm, respectively, onto the subspace of $T^*_\mathcal{M} \otimes \cdots \otimes T^*_\mathcal{M}$ on which $g_\mathcal{B}^{-1}$ is positive definite. We follow this notational simplification freely below. The terms that do not appear in these norms can be seen explicitly in Claim 31 and equation (66). (Because ∇v_α is the same on the base as it is on the total space, it does not matter which metric is used for the first norm.)

It follows from the Main assumptions that the terms $G_\alpha\big(v_\alpha(x,0)\big)/v_\alpha^2(x,0)$ are bounded. By those assumptions, those terms bound $|\nabla \log v_\alpha(x,0)|^2$, which in turn implies that the functions $v_\alpha(\cdot,0)$ can decay at most exponentially (specifically, see Remark 11 below). In fact, if the functions G_α are chosen so that the quantities $G_\alpha\big(v_\alpha(x,0)\big)/v_\alpha^2(x,0)$ are comparable to $|\nabla \log v_\alpha(x,0)|^2$, then $G_\alpha(x,0)/v_\alpha^2(x,0) \searrow 0$ as $v_\alpha(x,0) \searrow 0$ if and only if $v_\alpha(\cdot,0)$ decays more slowly than exponentially.

We prove Theorem 5 in the course of proving the following stronger but more technical result:

6. **Theorem.** *Let* $(\mathcal{M}, g_{\text{init}})$ *satisfy the Main Assumptions stated in Section 2.1. Then there exists a constant* $C_* = C_*(n, n_\alpha, C_{\text{init}})$ *such that the following are true:*

A solution

$$g(x,t) = g_{\mathcal{B}}(x,t) + \textstyle\sum_{\alpha=1}^{A} \big\{a_\alpha - \mu_\alpha t + v_\alpha(x,t)\big\} g_{\mathcal{F}_\alpha}$$

of the Ricci flow initial value problem with $g(x,0) = g_{\text{init}}(x)$ *exists with curvatures bounded in space at all times* $t \in [0, T_*)$, *where* $T_* := \min\{T_{\text{sing}}, C_*^{-1}\}$, *and* T_{sing} *is the (finite) singularity time,* i.e., *the maximal existence time of a smooth solution with bounded curvature.*

The v_α *are uniformly equivalent for* $t \in [0, T_*)$. *Specifically, one has*

$$\frac{1}{C_*} v_\alpha(x,t) \le v_\alpha(x,0) \le C_* v_\alpha(x,t).$$

Moreover, for each $x \in \mathcal{B}$ *and* $t \in [0, T_*)$, *one has*

$$\rho(x,t) \le C_{\text{init}}\,(1 + C_* t), \tag{9a}$$

and for $\alpha \in \{1, \dots, A\}$,

$$\gamma_\alpha(x,t) \le C_{\text{init}} \left(1 + C_* t\, \frac{G_\alpha\big(v_\alpha(x,t)\big)}{v_\alpha^2(x,t)}\right) G_\alpha(v_\alpha(x,t)), \tag{9b}$$

$$\chi_\alpha(x,t) \le C_{\text{init}}\,(1 + C_* t)\, H_\alpha\big(v_\alpha(x,t)\big), \tag{9c}$$

where G_α *and* H_α *are functions specified in the Main Assumptions.*

We prove Theorem 6 in Section 3.3 below after precisely stating our assumptions in Section 2.1 and establishing preliminary estimates in Sections 2.2–3.2.

If $C_*^{-1} < T_{\text{sing}}$, then the theorem above cannot describe the solution up to the singular time. However, by appropriately modifying the initial data in such a way that T_{sing} may be reduced if necessary, we can always arrange that our results do apply up to T_{sing}, as we now explain.

A key strength of the theorem is that the constant C_* is independent of the quantities a_α. One sees from (8) that the curvature can be very large if some a_α is very small. But even in that case, the bounds (9) persist. This leads directly to our next result. We let ς be such that $a_\varsigma/\mu_\varsigma = \min\{a_\alpha/\mu_\alpha \colon \mu_\alpha > 0\}$. By (6), the metric on $\mathcal{F}_\varsigma$ has the form $\big\{(a_\varsigma - \mu_\varsigma t) + v_\varsigma(x,t)\big\} g_{\mathcal{F}_\varsigma}$. By Remark 4, $\inf v_\varsigma(\cdot,0) = 0$, and by Theorem 6, this infimum is preserved. Thus the solution cannot exist past the formal singularity time $T_{\text{form}} := a_\varsigma/\mu_\varsigma$. Hence we have the following corollary:

7. **Corollary.** *There exist initial data* $(\mathcal{M}, g'_{\text{init}})$ *satisfying the Main Assumptions stated in Section 2.1 with the same constant* C_{init}, *the same initial values* v_α, *the same real-valued functions* G_α *and* H_α, *but with possibly changed constants* a_α, *such that the conclusions of Theorem 6 hold for the Ricci flow evolution of* $(\mathcal{M}, g'_{\text{init}})$ *at*

all times $[0, T_{\text{sing}})$. *Moreover,* $T_{\text{sing}} = T_{\text{form}}$*; there are no finite singular points in space; and the singularity is Type-I and occurs at spatial infinity.*

A proof of this corollary is found in Section 3.3, following the proof of Theorem 6.

A schematic outline of our proof of Theorem 6 is as follows. The proof relies on two pairs of supporting results, with Propositions 14 and 20 composing the first pair and Propositions 21 and 22 composing the second. In the process, we obtain Theorem 5 as a consequence of the arguments we employ to prove Proposition 21.

Standard short-time existence results give us a smooth Ricci flow solution on some time interval $[0, T_{\min}]$, with some curvature bound. Propositions 14 and 20 take as their input a curvature bound on $[0, T_{\min}]$; they output linear growth estimates for $\rho, \gamma_\alpha, \chi_\alpha$ on an interval $[0, T_1] \subseteq [0, T_{\min}]$, albeit with a possibly large constant that depends on the input curvature bound. As noted below the statement of Theorem 6, we ultimately do not want estimates that directly depend on the curvature. The fact that we get linear growth estimates for $\rho, \gamma_\alpha, \chi_\alpha$, however, lets us then apply Propositions 21 and 22, which take as their input uniform bounds on a suitable subinterval $[0, T_2] \subseteq [0, T_1]$ and yield the conclusions of the theorem on some time interval $[0, T_3] \subseteq [0, T_2]$. Finally, we use an "open-closed" argument to show that the supremum of $t > 0$ such that the theorem holds cannot be too small, *i.e.,* that it extends to $\min\{T_{\text{sing}}, C_*^{-1}\}$.

1.5. **Application: weak stability of generalized cylinders.** Our second application of our main results concerns stability of cylinders $\mathbb{R}^k \times \mathcal{S}^p$ under Ricci flow. This is a subtle question. Even though a round cylinder $\mathbb{R}^k \times \mathcal{S}^p$ is expected to be a stable singularity model in a suitable sense, it is not immediately clear how to define its stability.

For mean curvature flow, there is a rich collection of recent results addressing the stability of generalized cylinders. For example, it is shown in [CM12] that the only entropy-stable[4] shrinkers are spheres, hyperplanes, and generalized cylinders. See also [SS20], [SZ20], and [Zhu20].

Currently, we know of no analogue of such powerful results for Ricci flow. Accordingly, we adopt the following:

8. **Definition.** *We say a solution* $g(\cdot, t)$ *of Ricci flow is* weakly stable *if for every* $\epsilon > 0$*, there exists* $\delta > 0$ *such that for any other Ricci flow solution* $\tilde{g}(\cdot, t)$ *satisfying* $\|g(\cdot, 0) - \tilde{g}(\cdot, 0)\|_{g(0)} < \delta$*, one has* $\|g(\cdot, t) - \tilde{g}(\cdot, t)\|_{g(t)} < \epsilon$ *for all* $t \geq 0$ *that both solutions exist.*

For "outer" perturbations (*i.e.,* those for which $g(0) \geq g_{\text{cyl}}(0)$ and such that the cylindrical ends are preserved in a C^0 sense) that also satisfy the admissable conditions of Definition 23, we prove the following result, which is stated more precisely as Theorem 29 in the text below.

9. **Theorem.** *Ricci flow of a direct product metric* g_{cyl} *on* $\mathbb{R}^k \times \mathcal{S}^p$ *is weakly stable with respect to admissible perturbations of* g_{cyl}.

Moreover, if $g(\cdot, 0)$ *is an admissible perturbation of* $g_{\text{cyl}}(\cdot, 0)$*, then both flows* $g(\cdot, t)$ *and* $g_{\text{cyl}}(\cdot, t)$ *develop a singularity at the same finite time and remain close to each other in the* C^0 *norm up to that singular time.*

See §4.2 for further context regarding this result.

[4] See definitions (0.5) and (0.6) in [CM12].

10. **Remark.** *We note that the proof and conclusion of Theorem 9 also apply for any direct product metric on* $\mathbb{R}^k \times \mathbb{S}^p \times \mathbb{S}^q$*, for any nonnegative integers* p, q*, as long as at least one of them is nonzero.*

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2. Assumptions and preliminary estimates

2.1. **Assumptions.** We begin by establishing some notation.

Given a smooth function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, we define

$$\|\varphi\|_{2,\mathrm{mon}} = \sup_{s\in\mathbb{R}_+} \left(1 + \frac{s|\varphi'(s)|}{\varphi(s)} + \frac{s^2|\varphi''(s)|}{\varphi(s)}\right).$$

We caution the reader that this is not a norm. The double bars are a reminder that $\|\cdot\|_{2,\mathrm{mon}}$ is a supremum rather than a pointwise bound. The subscript is a reminder that $\|\varphi\|_{2,\mathrm{mon}}$ depends on two derivatives of φ, and that the quantity in parenthesis is constant if φ is a monomial.

Given a smooth function $\psi : \mathcal{B} \times [0, T] \to \mathbb{R}_+$, we define

$$|\psi|_{2,\exp} = \frac{\left|(\partial_t - \Delta)\psi\right|}{\psi} + \frac{|\nabla\psi|^2}{\psi^2}.$$

The single bars in $|\cdot|_{2,\exp}$ are a reminder that it is a pointwise bound, *i.e.,* a function of $x \in \mathcal{B}$ rather than a supremum. The subscript is a reminder that $|\cdot|_{2,\exp}$ depends on two derivatives, and that $|\nabla\psi|^2/\psi^2$ is constant in space if $\psi(x, t) = e^{d_{g(t)}(x',x)}$ for some $x' \in \mathcal{B}$, where $d_{g(t)}(x', x)$ represents distance with respect to the metric $g(t)$.

In Section 2.2, we state some useful properties satisfied by $\|\cdot\|_{2,\mathrm{mon}}$ and $|\cdot|_{2,\exp}$.

We next define

$$\mathcal{G} = \left\{ G : \mathbb{R}_+ \to \mathbb{R}_+ : \|G\|_{\mathcal{G}} := \|G\|_{2,\mathrm{mon}} + \sup_{s\in\mathbb{R}_+} \frac{G(s)}{s^2} < \infty \right\}. \tag{10}$$

We note that $s^2 \in \mathcal{G}$, so $\mathcal{G} \neq \emptyset$. We again caution the reader that we are once more using nonstandard notation: the symbol $\|\cdot\|_{\mathcal{G}}$ defined here is not a norm, and $\mathcal{G}$ is not a vector space.

Any choices of $G_\alpha \in \mathcal{G}$ generate associated functions $H_\alpha \in \mathcal{G}$ defined by

$$H_\alpha[s_1, \ldots, s_A](s_\alpha) = \left(\sum_{\beta=1}^{A} \frac{G_\beta(s_\beta)}{s_\beta^2}\right) G_\alpha(s_\alpha). \tag{11}$$

The notation reflects the fact that H_α is intended to control the geometry on the fiber $\mathcal{F}_\alpha$, but inputs information from the functions $G_1, \ldots, G_A$ used to control the geometry of all fibers $\mathcal{F}_1, \ldots, \mathcal{F}_A$. For brevity, we write $H_\alpha[s_1, \ldots, s_A](s_\alpha) \equiv H_\alpha(s_\alpha)$ below. The mnemonic theme is that we find it convenient to use $G_\alpha, H_\alpha \in \mathcal{G}$ to control gradient and Hessian terms in (12b) and (12c), respectively. We assume below that our choices of G_α satisfy the inequalities $\|G_\alpha\|_{\mathcal{G}} \leq \bar{C}_\alpha$ for some constants $\bar{C}_\alpha$, $\alpha \in \{1, \ldots, A\}$.

Throughout this paper, we assume that our initial data consist of a metric

$$g_{\text{init}}(x) = g_{\mathcal{B}}(x,0) + \sum_{\alpha=1}^{A} \big\{a_\alpha + v_\alpha(x,0)\big\} g_{\mathcal{F}_\alpha}$$

on the manifold $\mathcal{M} = \mathcal{B} \times \mathcal{F}_1 \times \cdots \times \mathcal{F}_A$ satisfying the following:

Main Assumptions. *There exist a constant C_{init} and functions $G_\alpha \in \mathcal{G}$ such that for $\alpha \in \{1, \ldots, A\}$,*

$$\|G_\alpha\|_{\mathcal{G}} \leq C_{\text{init}}, \tag{12a}$$

$$\gamma_\alpha(x,0) \leq C_{\text{init}}\, G_\alpha\big(v_\alpha(x,0)\big) \qquad \textit{for all } x \in \mathcal{B}, \tag{12b}$$

$$\chi_\alpha(x,0) \leq C_{\text{init}}\, H_\alpha\big(v_\alpha(x,0)\big) \qquad \textit{for all } x \in \mathcal{B}, \tag{12c}$$

$$\rho(x,0) \leq C_{\text{init}} \qquad \textit{for all } x \in \mathcal{B}. \tag{12d}$$

We further assume that $|\nabla \operatorname{Rm}[g(\cdot,0)]|_{g(\cdot,0)}$ is bounded and that at least one $\mu_\alpha > 0$, i.e., *that at least one fiber is a space form of positive Ricci curvature.*

We note that our choices of $G_\alpha \in \mathcal{G}$ may depend on the initial data, and that it follows from our main results that the choice $\mu_\alpha > 0$ forces a singularity at a time $T_{\text{sing}} < \infty$.

11. **Remark.** *We observe that part (12b) of the Main Assumptions requires that $|\nabla \log v_{\alpha,\text{init}}|^2$ is bounded, which implies that $\inf u_{\alpha,\text{init}}$ cannot be attained. This means that Theorem 6 and Corollary 7 are primarily useful in analyzing singularities that occur at spatial infinity.*

Given initial data in which $\inf u_\alpha$ is attained in a compact set, one could adjust $a_{\alpha,\text{init}}$ downward in order to apply those results. However, their output would not be sharp in that case, because it then cannot describe the developing singularity all the way up to the singular time.

2.2. **Basic inequalities.** It is not difficult to verify the following useful properties of $\|\cdot\|_{2,\text{mon}}$ and $|\cdot|_{2,\exp}$:

$$\begin{aligned}
\|\varphi_1 + \varphi_2\|_{2,\text{mon}} &\leq \|\varphi_1\|_{2,\text{mon}} + \|\varphi_2\|_{2,\text{mon}}, \\
\|\varphi_1 \varphi_2\|_{2,\text{mon}} &\leq 2\|\varphi_1\|_{2,\text{mon}} \|\varphi_2\|_{2,\text{mon}}, \\
\|\varphi_1 \circ \varphi_2\|_{2,\text{mon}} &\leq \|\varphi_1\|_{2,\text{mon}} \|\varphi_2\|_{2,\text{mon}}^2, \\
|\varphi \circ \psi|_{2,\exp} &\leq \|\varphi\|_{2,\text{mon}}^2 |\psi|_{2,\exp}, \\
|\psi_1 \psi_2|_{2,\exp} &\leq |\psi_1|_{2,\exp} + |\psi_2|_{2,\exp}, \\
|\psi_1 + \psi_2|_{2,\exp} &\leq 2\left(|\psi_1|_{2,\exp} + |\psi_2|_{2,\exp}\right).
\end{aligned} \tag{13}$$

We explicitly verify the fourth inequality, whose proof is slightly less straightforward than the proofs of the others.

Proof. Let $\psi : \mathcal{M} \times [0,T] \to \mathbb{R}_+$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. Then $(\varphi \circ \psi)_t = \varphi'(\psi)\psi_t$, $\nabla_i(\varphi \circ \psi) = \varphi'(\psi)\nabla_i \psi$, and $\Delta(\varphi \circ \psi) = \varphi'(\psi)\Delta\psi + \varphi''(\psi)|\nabla\psi|^2$. Thus one has

$$\begin{aligned}
\frac{\big(\partial_t - \Delta\big)(\varphi \circ \psi)}{\varphi \circ \psi} + \frac{|\nabla(\varphi \circ \psi)|^2}{(\varphi \circ \psi)^2} &= \frac{\varphi'(\psi)\big(\partial_t - \Delta\big)\psi - \varphi''(\psi)|\nabla\psi|^2}{\varphi(\psi)} + \frac{(\varphi'(\psi))^2 |\nabla\psi|^2}{(\varphi(\psi))^2} \\
&= \frac{\psi\varphi'(\psi)}{\varphi(\psi)} \frac{\big(\partial_t - \Delta\big)\psi}{\psi} - \frac{\psi^2 \varphi''(\psi)}{\varphi(\psi)} \frac{|\nabla\psi|^2}{\psi^2} \\
&\quad + \left\{\frac{\psi\varphi'(\psi)}{\varphi(\psi)}\right\}^2 \frac{|\nabla\psi|^2}{\psi^2},
\end{aligned}$$

from which it is easy to see that $|\varphi\circ\psi|_{2,\exp}\le\|\varphi\|^2_{2,\mathrm{mon}}|\psi|_{2,\exp}$. □

2.3. **Differential inequalities.** We now estimate the evolution equations of the quantities we work with throughout this paper: γ_α, χ_α, and ρ.

12. **Lemma.** *If γ_α, χ_α, and ρ are as in (7), then there exists a uniform constant C_N that depends only on the dimension vector $\vec{N}=(n,n_\alpha)$ such that we have the estimates*

$$\big(\partial_t-\Delta\big)\gamma_\alpha\le-\frac12\,\frac{|\nabla\gamma_\alpha|^2}{\gamma_\alpha}+6\left(\frac{\gamma_\alpha}{u_\alpha^2}\right)\gamma_\alpha, \tag{14}$$

$$\big(\partial_t-\Delta\big)\chi_\alpha\le-\frac12\,\frac{|\nabla\chi_\alpha|^2}{\chi_\alpha}+C_NL\chi_\alpha+C_NL\sum_{\beta=1}^{A}\frac{\gamma_\beta}{u_\beta^2}\,\gamma_\alpha, \tag{15}$$

and

$$\big(\partial_t-\Delta\big)\rho\le-\frac12\,\frac{|\nabla\rho|^2}{\rho}+C_NL^3, \tag{16}$$

where

$$L:=\rho^{1/2}+\sum_{\beta=1}^{A}\frac{\gamma_\beta}{u_\beta^2}+\sum_{\beta=1}^{A}\frac{\chi_\beta^{1/2}}{u_\beta}.$$

The useful structure here is that we have negative gradient terms appearing on the right-hand sides of all three inequalities. In (14) and (15), we also have what may be regarded as linear terms with coefficients that can be bounded in terms of the quantities under consideration; in (15) and (16), we have inhomogeneous terms that may be similarly bounded.

Proof. In the proof, we use the same symbol C_N to denote constants that might differ from line to line but that all depend only on the dimension vector $\vec{N}=(n,n_\alpha)$.

An easy computation (see Appendix C) implies that

$$\big(\partial_t-\Delta\big)\gamma_\alpha=-2|\nabla^2v_\alpha|^2+2\,\frac{|\nabla v_\alpha|^4}{u_\alpha^2}-4\,\frac{\nabla^2v_\alpha(\nabla v_\alpha,\nabla v_\alpha)}{u_\alpha}.$$

Using Cauchy–Schwarz and Kato's inequality ($|\nabla|\nabla v_\alpha||\le|\nabla^2v_\alpha|$), we get (14).

To obtain (15), note that in Appendix C we compute that

$$\begin{aligned}\big(\partial_t-\Delta\big)\chi_\alpha\le&-2|\nabla^3v_\alpha|^2_{g_{\mathcal{B}}}+4\,\mathrm{Rm}_{\mathcal{B}}(\nabla^2v_\alpha,\nabla^2v_\alpha)+2u_\alpha^{-2}\gamma_\alpha\chi_\alpha\\&-2u_\alpha^{-3}\langle\nabla v_\alpha,\nabla\gamma_\alpha\rangle\gamma_\alpha+4u_\alpha^{-2}\langle\nabla^2v_\alpha,\nabla v_\alpha\otimes\nabla\gamma_\alpha\rangle\\&-2u_\alpha^{-1}\langle\nabla^2v_\alpha,\nabla^2\gamma_\alpha\rangle_{g_{\mathcal{B}}}+Nu_\alpha^{-2}\gamma_\alpha\big\{-\chi_\alpha+\frac14u_\alpha^{-1}\langle\nabla\ v_\alpha,\nabla\gamma_\alpha\rangle\big\}\\&+C_N\Big(\sum_{\alpha=1}^{A}|\nabla\log u_\alpha|^2\Big)|\nabla^2v_\alpha||\nabla^2v_\alpha|_{g_{\mathcal{B}}}.\end{aligned}$$

An easy computation using results about the Levi-Civita connection Γ derived in Appendix A yields

$$-|\nabla^3v_\alpha|^2_{g_{\mathcal{B}}}\le-\frac{|\nabla\,|\nabla^2v_\alpha|^2_{g_{\mathcal{B}}}\,|^2_{g_{\mathcal{B}}}}{2\chi_\alpha}=-\frac{|\nabla\chi_\alpha|^2_{g_{\mathcal{B}}}}{2\chi_\alpha}=-\frac{|\nabla\chi_\alpha|^2}{2\chi_\alpha}$$

and

$$\begin{aligned}\big|u_\alpha^{-1}\langle\nabla^2 v_\alpha,\nabla^2\gamma_\alpha\rangle_{g_{\mathcal{B}}}\big| &\le C_N\left(\frac{\chi_\alpha^{3/2}}{u_\alpha}+\frac{\gamma_\alpha^{1/2}\,\chi_\alpha^{1/2}}{u_\alpha}\,|\nabla^3 v_\alpha|_{g_{\mathcal{B}}}\right)\\ &\le C_N\left(\frac{\chi_\alpha^{3/2}}{u_\alpha}+\frac{\gamma_\alpha\,\chi_\alpha}{u_\alpha^2}\right)+|\nabla^3 v_\alpha|_{g_{\mathcal{B}}}^2.\end{aligned}$$

Again using results about the Hessian from Appendix A, we see that

$$|\nabla^2 v_\alpha|\le C_N\left(\chi_\alpha^{1/2}+\frac{\gamma_\beta^{1/2}}{u_\beta}\,\gamma_\alpha^{1/2}\right),$$

implying that

$$|\nabla^2 v_\alpha||\nabla^2 v_\alpha|_{g_{\mathcal{B}}}\le C_N\left(\chi_\alpha+\sum_{\beta=1}^{A}\frac{\gamma_\beta}{u_\beta^2}\,\gamma_\alpha\right),$$

where we use the Cauchy–Schwarz inequality. Putting these estimates together yields

$$\begin{aligned}(\partial_t-\Delta)\chi_\alpha &\le -\frac{|\nabla\chi_\alpha|^2}{2\chi_\alpha}+C_N\chi_\alpha\left(\rho^{1/2}+\frac{\gamma_\alpha}{u_\alpha^2}+\frac{\chi_\alpha^{1/2}}{u_\alpha}+\sum_{\beta=1}^{A}\frac{\gamma_\beta}{u_\beta^2}\right)\\ &\quad+C_N\left(\frac{\chi_\alpha^{1/2}\gamma_\alpha^2}{u_\alpha^3}+\Big(\sum_{\beta=1}^{A}\frac{\gamma_\beta}{u_\beta^2}\Big)^2\gamma_\alpha\right)\\ &\le -\frac{|\nabla\chi_\alpha|^2}{2\chi_\alpha}+C_N\,L\,\chi_\alpha+C_N\,\gamma_\alpha\sum_{\beta=1}^{A}\frac{\gamma_\beta}{u_\beta^2}\left(\sum_{\beta=1}^{A}\frac{\chi_\beta^{1/2}}{u_\beta}+\sum_{\beta=1}^{A}\frac{\gamma_\beta}{u_\beta^2}\right)\\ &\le -\frac{|\nabla\chi_\alpha|^2}{2\chi_\alpha}+C_N\,L\,\chi_\alpha+C_N\,L\sum_{\beta=1}^{A}\frac{\gamma_\beta}{u_\beta^2}\,\gamma_\alpha,\end{aligned}$$

as claimed.

Finally, as in Appendix C, we denote by $\mathcal{H}$ the (integrable) horizontal distribution of $\mathcal{M}$ and by $\mathrm{Rm}_{\mathcal{H}\otimes\mathcal{H}}$ the restriction

$$\mathrm{Rm}_{\mathcal{H}\otimes\mathcal{H}}:=\mathrm{Rm}\big|_{\mathcal{H}\otimes T\mathcal{M}\otimes T\mathcal{M}\otimes\mathcal{H}}.$$

Our computation in Appendix C shows that ρ evolves by

$$\begin{aligned}(\partial_t-\Delta)\rho &\le -2|\nabla\,\mathrm{Rm}\,|_{g_{\mathcal{B}}}^2+C_n\rho^{3/2}\\ &\quad+2\sum_{\alpha=1}^{A}n_\alpha\Big\{u_\alpha^{-2}\,\mathrm{Rm}_{\mathcal{B}}(\nabla^2 v_\alpha,\nabla^2 v_\alpha)\\ &\qquad\qquad-2u_\alpha^{-3}\,\mathrm{Rm}_{\mathcal{B}}(\nabla^2 v_\alpha,\nabla v_\alpha\otimes\nabla v_\alpha)\Big\}\\ &\quad+C_N\Big(\sum_{\alpha=1}^{A}|\nabla\log u_\alpha|^2\Big)|\,\mathrm{Rm}\,|_{g_{\mathcal{B}}}|\,\mathrm{Rm}_{\mathcal{H}\otimes\mathcal{H}}\,|_g.\end{aligned}$$

Claim 35 in Appendix C shows that $\nabla \operatorname{Rm}$ vanishes if exactly one index is vertical. Thus by Kato's inequality for tensors, we have

$$-|\nabla \operatorname{Rm}|^2_{g_{\mathcal{B}}} \leq -|\nabla| \operatorname{Rm}|_{g_{\mathcal{B}}}|^2_{g_{\mathcal{B}}} = -\frac{1}{4}\frac{|\nabla\rho|^2_{g_{\mathcal{B}}}}{\rho} = -\frac{1}{4}\frac{|\nabla\rho|^2}{\rho}.$$

Moreover, using our computations of curvature components in Appendix A, we immediately get

$$|\operatorname{Rm}_{\mathcal{H}\otimes\mathcal{H}}|_g \leq C_N\left(\sum_{\beta=1}^{A}\frac{\gamma_\beta}{u_\beta^2} + \sum_{\beta=1}^{A}\frac{\chi_\beta^{1/2}}{u_\beta} + \rho^{1/2}\right).$$

All of these together imply that

$$\begin{aligned}(\partial_t - \Delta)\rho &\leq -\frac{1}{2}\frac{|\nabla\rho|^2}{\rho} + C_N\rho^{3/2} + C_N\sum_{\beta=1}^{A}\frac{\rho^{1/2}\chi_\beta}{u_\beta^2} + C_N\sum_{\beta=1}^{A}\frac{\rho^{1/2}\chi_\beta^{1/2}\gamma_\beta}{u_\beta^3}\\ &\quad + C_N\sum_{\beta=1}^{A}\frac{\gamma_\beta}{u_\beta^2}\rho^{1/2}\left(\sum_{\beta=1}^{A}\frac{\gamma_\beta}{u_\beta^2} + \sum_{\beta=1}^{A}\frac{\chi_\beta^{1/2}}{u_\beta} + \rho^{1/2}\right)\\ &\leq -\frac{1}{2}\frac{|\nabla\rho|^2}{\rho} + C_N L^3,\end{aligned}$$

yielding (16). This completes the proof. □

3. Analysis

In this section, we prove estimates for solutions of parabolic equations on noncompact manifolds evolving by Ricci flow. Among the results we obtain below are Propositions 14, 20, 21, and 22 which, as discussed in the introduction, play a major role in the proof of Theorem 6.

3.1. A noncompact maximum principle.

The goal of our first result, Lemma 13, is to obtain estimates for a function U in terms of a "comparison function" V and a "control function" W. For example, we often take U to be a function that we want to estimate on a short time interval $[t_0, t_1]$, V to be the same function at the initial time t_0, and W to be a large constant that depends on bounds for the curvatures on $[t_0, t_1]$. Our proof of the lemma proceeds by applying a noncompact maximum principle to the quantity U/V, thereby allowing us to bound it suitably from above. We use Lemma 13 extensively in the proofs below.

13. **Lemma.** *Let $(\mathcal{M}, g(t))$ be a smooth solution of Ricci flow for $t \in [0,T]$, and let $U, V, W : \mathcal{M}\times[0,T] \to \mathbb{R}_+$ be smooth functions. Suppose that there exist constants $0 < c \ll C$ such that*

$$\begin{aligned}(\partial_t - \Delta)U &\leq C(UW + VW) - c\frac{|\nabla U|^2}{U},\\ \frac{|(\partial_t-\Delta)V|}{V} + \frac{|\nabla V|^2}{V^2} &\leq CW,\\ |(\partial_t - \Delta)W| + |\nabla W|^2 &\leq CW,\\ W &\leq C,\end{aligned}$$

where the Laplacian and norms above are computed with respect to the solution $g(t)$ of Ricci flow.

Then there exist $\lambda = \lambda(c, C)$ *and* $T' = T'(c, C, T) \in (0, T]$ *such that for all* $t \in [0, T']$,

$$\big(\partial_t - \Delta\big)\left\{\frac{U}{V} - \lambda t\left(1 + \frac{U}{V}\right)W\right\} \le 0.$$

Moreover, if there exist a point $x' \in \mathcal{B}$ *and a constant* C' *such that one has* $U(x,t) \le C' e^{C' d^2_{g(t)}(x',x)} V(x,t)$ *on* $[0, T']$*, then for* $t \in [0, T']$,

$$\frac{U(x,t)}{V(x,t)} \le \sup_{y \in \mathcal{M}} \frac{U(y,0)}{V(y,0)} + 2\lambda t W(x,t)\left(1 + \sup_{y \in \mathcal{M}} \frac{U(y,0)}{V(y,0)}\right).$$

Proof. We define $X = U/V$ and compute that

$$\big(\partial_t - \Delta\big)X = \frac{\big(\partial_t - \Delta\big)U}{V} + 2X\frac{\langle \nabla U, \nabla V\rangle}{UV} - 2X\frac{|\nabla V|^2}{V^2} - X\frac{\big(\partial_t - \Delta\big)V}{V}.$$

We split the second term on the RHS above as follows:

$$2X\frac{\langle \nabla U, \nabla V\rangle}{UV} = (2-c)X\frac{\langle \nabla U, \nabla V\rangle}{UV} + cX\frac{\langle \nabla U, \nabla V\rangle}{UV}. \tag{17}$$

In what follows, we denote by $C' = C'(c, C)$ a constant that may change from line to line. We use the weighted Cauchy–Schwarz inequality to estimate the first term on the RHS of (17) by

$$(2-c)X\frac{|\langle \nabla U, \nabla V\rangle|}{UV} \le \frac{c}{2}\frac{|\nabla U|^2}{UV} + C'\frac{|\nabla V|^2}{V^2}X,$$

and rewrite the second term as

$$cX\frac{\langle \nabla U, \nabla V\rangle}{UV} = \frac{c}{2}\Big(\frac{|\nabla U|^2}{UV} + X\frac{|\nabla V|^2}{V^2} - \frac{|\nabla X|^2}{X}\Big),$$

obtaining

$$\big(\partial_t - \Delta\big)X \le \frac{1}{V}\left\{\big(\partial_t - \Delta\big)U + c\frac{|\nabla U|^2}{U}\right\} + X\left\{\frac{\big|\big(\partial_t - \Delta\big)V\big|}{V} + C'\frac{|\nabla V|^2}{V^2}\right\} - \frac{c}{2}\frac{|\nabla X|^2}{X}.$$

Thus our assumptions on U and V imply that

$$\begin{aligned} \big(\partial_t - \Delta\big)X &\le \frac{C(U+V)W}{V} + C'CWX - \frac{c}{2}\frac{|\nabla X|^2}{X} \\ &\le C'CW(1+X) - \frac{c}{2}\frac{|\nabla X|^2}{X}. \end{aligned} \tag{18}$$

Now for $\lambda = \lambda(c, C) > 0$ to be chosen, we define $Y = X - (\lambda t)(1+X)W$ and compute that

$$\begin{aligned} \big(\partial_t - \Delta\big)Y = &(1 - \lambda t W)\big(\partial_t - \Delta\big)X - \lambda t(1+X)\big(\partial_t - \Delta\big)W \\ &- \lambda(1+X)W + 2\lambda t\langle \nabla X, \nabla W\rangle. \end{aligned}$$

If $t \le T_1 := 1/(\lambda C)$, then $1 - \lambda t W \ge 0$, so we may apply estimate (18) to the first term on the RHS above. We then use our assumption on $\big|\big(\partial_t - \Delta\big)W\big|$ to estimate the second term and apply Cauchy–Schwarz to the last term, obtaining

$$\begin{aligned}\big(\partial_t-\Delta\big)Y &\le (1-\lambda tW)\left\{C'CW(1+X)-\frac{c}{2}\frac{|\nabla X|^2}{X}\right\}+C\lambda tW(1+X)\\ &\quad -\lambda W(1+X)+\lambda t\left(\frac{|\nabla X|^2}{X}+X|\nabla W|^2\right).\end{aligned}$$

By using our assumption that $|\nabla W|^2\le CW$, we simplify this to

$$\begin{aligned}\big(\partial_t-\Delta\big)Y &\le W(1+X)\big\{-\lambda+CC'(1-\lambda tW)+2C\lambda t\big\}\\ &\quad+\frac{|\nabla X|^2}{X}\big\{-\frac{c}{2}(1-\lambda tW)+\lambda t\big\}.\end{aligned}$$

Then choosing $\lambda=2CC'$ and using our upper bound for W, we obtain

$$\big(\partial_t-\Delta\big)Y\le W(1+X)CC'(-1+4Ct)+\frac{|\nabla X|^2}{X}\big\{-\frac{c}{2}+CC'(Cc+2)t\big\}.$$

The RHS is nonpositive provided that $t\le T_2:=\frac{1}{4C}$ and $t\le T_3:=\frac{c}{2CC'(Cc+2)}$. Thus we choose $T'=\min\{T_1,T_2,T_3\}$.

Finally, we justify applying the weak maximum principle on the noncompact manifold $\mathcal{M}$ in the form detailed in Theorem 12.22 of [CCG08]. Specifically, since W is bounded, the assumption that $U(x,t)/V(x,t)\le C'e^{C'd^2_{g(t)}(x',x)}$ implies easily that Theorem 12.22 applies to $Y(x,t)-\sup_{y\in\mathcal{M}}Y(y,0)$, allowing us to conclude

$$X(x,t)\le\lambda tW(x,t)(1+X(x,t))+\sup_y X(y,0),$$

implying that

$$X(x,t)\le\frac{\lambda t}{1-\lambda tW(x,t)}W(x,t)+\frac{1}{1-\lambda tW(x,t)}\sup_{y\in\mathcal{M}}X(y,0).$$

We can decrease T' if necessary to make $\lambda tW(x,t)$ small enough for all $t\in[0,T']$ so that the following holds:

$$\begin{aligned}X(x,t)&\le\lambda t\Big(1+2\lambda tW(x,t)\Big)W(x,t)+\Big(1+2\lambda tW(x,t)\Big)\sup_{y\in\mathcal{M}}X(y,0)\\ &=\sup_{y\in\mathcal{M}}X(y,0)+W(x,t)\Big(\lambda t+2\lambda^2t^2W(x,t)+2\lambda t\sup_{y\in\mathcal{M}}X(y,0)\Big)\\ &\le\sup_{y\in\mathcal{M}}X(y,0)+2\lambda tW(x,t)\Big(1+\sup_{y\in\mathcal{M}}X(y,0)\Big).\end{aligned}$$

This completes the proof. □

3.2. **Main estimates.** We now establish two pairs of Propositions that provide the key results we need to prove Theorem 6.

In Propositions 14 and 20, we obtain bounds for γ, χ, and ρ on a solution that is smooth on a compact time interval $[t_0,t_1]$.[5] A strength of these results is that they allow us to extend bounds that hold at t_0 to the entire interval $[t_0,t_1]$, which cannot be taken for granted because $\mathcal{M}$ is noncompact; a weakness is that the constant we obtain for these bounds depends on an upper bound for the full curvature tensor on $[t_0,t_1]$.

[5] In *Step 1* of our proof of Theorem 6, we initially apply Propositions 14 and 20 with $t_0=0$. In *Step 2* however, we need to apply them at some $t_0>0$.

In Propositions 21 and 22, we show that if the functions v_α and their derivatives satisfy uniform bounds on an interval $[0,T]$, then those bounds can be improved, independent of the curvature, at least on an interval $[0,T_*]$, with $0<T_*\le T$.

14. **Proposition.** *Let* $(\mathcal{M},g_{\mathrm{init}})$ *satisfy the Main Assumptions in Section 2.1.*

Suppose a solution $g(t)$ *of Ricci flow exists for* $[t_0,t_1]$*, satisfying the bounds* $\rho(x,t_0)\le C_0C_{\mathrm{init}}$ *and*

(19)
$$v_\alpha(x,t_0)>0,\quad \gamma_\alpha(x,t_0)\le C_0C_{\mathrm{init}}\,G_\alpha\big(v_\alpha(x,t_0)\big),\quad \chi_\alpha(x,t_0)\le C_0C_{\mathrm{init}}\,H_\alpha\big(v_\alpha(x,t_0)\big),$$

for some $C_0\ge 1$,[6] *along with a uniform bound* $\sup_{(x,t)\in\mathcal{B}\times[t_0,t_1]}|\operatorname{Rm}(x,t)|\le C_1$*.*

Then there exists $C'=C'(C_{\mathrm{init}},C_0,C_1)$ *and* $T'=T'(C_{\mathrm{init}},C_0,C_1)\in(t_0,t_1]$ *such that for all* $t\in[t_0,T']$*, one has*

$$v_\alpha(x,t)\ge\frac{v_\alpha(x,t_0)}{1+C'(t-t_0)}, \tag{20a}$$

$$\gamma_\alpha(x,t)\le C_0C_{\mathrm{init}}(1+C'\,(t-t_0))\,G_\alpha\big(v_\alpha(x,t_0)\big), \tag{20b}$$

$$\chi_\alpha(x,t)\le C_0C_{\mathrm{init}}(1+C'(t-t_0))\,H_\alpha\big(v_\alpha(x,t_0)\big), \tag{20c}$$

$$\rho(x,t)\le C_0C_{\mathrm{init}}\,(1+C'(t-t_0)). \tag{20d}$$

Because its proof is lengthy, we prove Proposition 14 in a series of steps that are contained in Lemmas 15–19. In the course of the proof, we use the same symbols C' and T' for possibly different constants that depend only on C_{init}, C_0, and C_1 — with C' allowed to grow but remain finite, and T' allowed to shrink but remain positive.

Our first observation is needed because to prove our main results (Theorem 6 and Corollary 7), we need to apply Lemma 13 in cases where V may be independent of time, but ΔV and $|\nabla V|^2$ are computed with respect to $g(t)$.

15. **Lemma.** *Suppose that the assumptions of Proposition 14 hold.*

Then there are a constant $C'(C_{\mathrm{init}},C_0,C_1)$ *and time* $T'(C_{\mathrm{init}},C_0,C_1)\in(t_0,t_1]$ *such that on* $[t_0,T']$*, we have*

$$|\nabla\operatorname{Rm}[g(t)]|_{g(t)}\le C',$$
$$\frac{|\nabla v_\alpha(x,t_0)|^2_{g(t)}}{v_\alpha(x,t_0)^2}\le C',$$
$$\frac{|\Delta_{g(t)}v_\alpha(x,t_0)|_{g(t)}}{v_\alpha(x,t_0)}\le C'.$$

Note that the final two collections of inequalities can be summarized as

$$|v_\alpha(x,t_0)|_{2,\exp}\le C'. \tag{21}$$

Proof. If $t_0>0$, our assumed bound on $|\operatorname{Rm}|$ at time $t=t_0$ and regularity theory for Ricci flow imply the stated bound for $|\nabla\operatorname{Rm}|$. If $t_0=0$, we note that the Main Assumptions outlined in Section 2.1 include an upper bound for $|\nabla\operatorname{Rm}|$ at time $t=0$. Then Theorem 14.16 of [CCG08]) lets us bound $|\nabla\operatorname{Rm}|$ on $[t_0,T']$.

The subsequent inequalities follow because they hold at time t_0, and because $\partial_t g$ and $\partial_t\Gamma$ are controlled by our bounds on $|\operatorname{Rm}|$ and $|\nabla\operatorname{Rm}|$. □

[6] We take $C_0=1$ if $t_0=0$ but allow $C_0>1$ if $t_0>0$.

16. **Lemma.** *Suppose that the assumptions of Proposition 14 hold.*

Then there exist a constant $C'(C_{\text{init}}, C_0, C_1)$ *and time* $T'(C_{\text{init}}, C_0, C_1) \in (t_0, t_1]$ *so that for all* $t \in [t_0, T']$*, estimate* (20a) *holds.*

Proof. First we claim there exists a T' so that $v_\alpha \geq 0$ on $\mathcal{M} \times [t_0, T']$.

The Ricci flow equation restricted to the metric on $\mathcal{F}_\alpha$ is

$$\partial_t(u_\alpha g_\alpha) = -2\operatorname{Rc}\big|_\alpha,$$

where $\operatorname{Rc}\big|_\alpha$ denotes the Ricci curvature of planes tangent to $\mathcal{F}_\alpha$. Using the fact that g_α is independent of time, we can rewrite this as

$$\partial_t(\log u_\alpha) u_\alpha g_\alpha = -2\operatorname{Rc}\big|_\alpha.$$

Since $|\operatorname{Rm}[g(t)]| \leq C_1$ on $[t_0, t_1]$, we get a comparable bound for $|\operatorname{Rc}|$, implying that

$$\big|\partial_t \log u_\alpha\big| \leq C',$$

where $C' = C'(C_1)$. Hence for all $t \in [t_0, t_1]$, we have

$$u_\alpha(x,t) \geq e^{-C'(t-t_0)}\,(a_\alpha - \mu_\alpha t_0). \tag{22}$$

To prove the claim that v remains nonnegative for a short time, we first show that given any $\delta > 0$, we have $v_\alpha \geq -\delta t$ on a time interval $[t_0, T']$, where T' could possibly decrease in the proof but is independent of δ.

Equation (22) implies that $u_\alpha \geq (a_\alpha - \mu_\alpha t_0)\,(1 - C'(t - t_0))$, so

$$v_\alpha(x,t) = u_\alpha - (a_\alpha - \mu_\alpha t) \geq -C'\,(t - t_0). \tag{23}$$

We fix T' so that $T' - t_0 \leq (a - \mu_\alpha t_1)/(2C_1)$ and let $\tau \in [t_0, T']$ be arbitrary. Because $C_1(\tau - t_0) \leq C_1(T' - t_0) \leq \frac{1}{2}(a_\alpha - \mu t_1)$, we may let

$$\epsilon \in \big(C_1(\tau - t_0),\, a_\alpha - \mu_\alpha t_1\big) \tag{24}$$

be arbitrary. Then $v_\alpha + \epsilon > 0$ on $[t_0, \tau]$, so each function $v_{\alpha,\epsilon} := (v_\alpha + \epsilon)^{-1}$ is well defined on that time interval. Using that $v_\alpha(x,t)$ evolves by

$$\big(\partial_t - \Delta\big) v_\alpha = -\frac{\gamma_\alpha}{u_\alpha}, \tag{25}$$

a straightforward computation yields

$$\big(\partial_t - \Delta\big) v_{\alpha,\epsilon} = |\nabla \log v_{\alpha,\epsilon}|^2 v_{\alpha,\epsilon}\left(\frac{v_\alpha + \epsilon}{u_\alpha} - 2\right).$$

Our choice of ϵ implies that

$$\frac{v_\alpha + \epsilon}{u_\alpha} = \frac{v_\alpha + \epsilon}{v_\alpha + a_\alpha - \mu_\alpha t} \leq \frac{v_\alpha + \epsilon}{v_\alpha + a_\alpha - \mu_\alpha t_1} \leq 1$$

for all $t \in [t_0, T']$ and thus that

$$\big(\partial_t - \Delta\big) v_{\alpha,\epsilon} \leq -|\nabla \log v_{\alpha,\epsilon}|^2 v_{\alpha,\epsilon} = -\frac{|\nabla v_{\alpha,\epsilon}|^2}{v_{\alpha,\epsilon}}.$$

Let $U(x,t) = v_{\alpha,\epsilon}(x,t)$, $V(x,t) = v_{\alpha,\epsilon}(x,t_0)$, and $W = C'$. Observing that V is independent of time and using Lemma 15, one sees that

$$\frac{\big|\big(\partial_t - \Delta\big)V\big|}{V} + \frac{|\nabla V|^2}{V^2} \leq \frac{|\Delta_{g(t)} v_\alpha(x,t_0)|}{v_\alpha(x,t_0) + \epsilon} + \frac{|\nabla v_\alpha(x,t_0)|^2_{g(t)}}{(v_\alpha(x,t_0) + \epsilon)^2} \tag{26}$$
$$\leq C'.$$

Note in particular that (26) is independent of ϵ. For a sufficiently short time, $v_{\alpha,\epsilon} \leq (\epsilon - C'(\tau - t_0))^{-1} < \infty$. So U is bounded in space, and the bound (26) implies $|\nabla \log V|$ is bounded, so V decays at most exponentially. Thus, Lemma 13 can be applied to U, V, and W as defined above to conclude that

$$v_{\alpha,\epsilon}(x,t) \leq (1 + C'(t - t_0))\, v_{\alpha,\epsilon}(x,t_0), \qquad t \in [t_0, \tau],$$

where $C' = C'(C_{\text{init}}, C_0, C_1)$ is independent of ϵ. Letting $\epsilon \searrow C_1(\tau - t_0)$, which is the lower bound imposed by (24), we find that

$$v_\alpha + C_1(\tau - t_0) \geq \frac{C_1(\tau - t_0)}{1 + C'(\tau - t_0)} \geq \frac{C_1(\tau - t_0)}{1 + C'(t_1 - t_0)}.$$

Because $\tau \in [t_0, T']$ is arbitrary, this implies that

$$v_\alpha \geq -\left(1 - \frac{1}{1 + C'(t_1 - t_0)}\right) C_1\,(t - t_0)$$

for all $t \in [t_0, T']$, which improves (23) by a fixed factor. Repeating this bootstrap argument k times (which can be done without changing T'), where

$$\left(1 - \frac{1}{1 + C'(t_1 - t_0)}\right)^k C_1 \leq \delta,$$

proves that $v_\alpha \geq -\delta(t - t_0)$ on $[t_0, T']$. Because $\delta > 0$ is arbitrary and T' is independent of δ, it follows that $v_\alpha \geq 0$ on $[t_0, T']$, as claimed.

We next prove a better quantitative lower bound for v_α, as long as $v_\alpha(x,t) \geq 0$ holds, that is, for $t \in [t_0, T']$, where T' is some possibly smaller time $T'(C_{\text{init}}, C_0, C_1)$.

The method is very close to that used in the proof of the claim that $v_\alpha \geq 0$, so we avoid unnecessary repetition. Let $\epsilon \in (0, a_\alpha - \mu_\alpha t_1)$ be arbitrary, and again let $v_{\alpha,\epsilon} := (v_\alpha + \epsilon)^{-1}$. Note that in contrast to the previous argument, where (24) is needed, we have proven that $v_\alpha \geq 0$ above, hence we know that $v_{\alpha,\epsilon}$ is well-defined and bounded by ϵ^{-1} for all $\epsilon > 0$. Then as in the arguments above, we find that

$$\big(\partial_t - \Delta\big) v_{\alpha,\epsilon} \leq -\frac{|\nabla v_{\alpha,\epsilon}|^2}{v_{\alpha,\epsilon}}.$$

Now let $U(x,t) = v_{\alpha,\epsilon}(x,t)$, $V(x,t) = v_{\alpha,\epsilon}(x,t_0)$, and $W = C'$. Again using equation (26) and the fact that U is bounded, we can apply Lemma 13 to obtain

$$v_{\alpha,\epsilon}(x,t) \leq (1 + C'(t - t_0))\, v_{\alpha,\epsilon}(x,t_0),$$

where $C' = C'(C_{\text{init}}, C_0, C_1)$ is independent of ϵ. We let $\epsilon \searrow 0$ to conclude that

$$\frac{v_\alpha(x,t_0)}{1 + C'(t - t_0)} \leq v_\alpha(x,t)$$

on $\mathcal{M} \times [t_0, T']$. □

17. **Lemma.** *Under the assumptions of Proposition 14, there exist $C'(C_{\text{init}}, C_0, C_1)$ and $T' = T'(C_{\text{init}}, C_0, C_1) \in (t_0, t_1]$ such that for all $t \in [t_0, T']$, estimate (20b) holds.*

Proof. Since we have (22), we can find T' sufficiently small so that

$$u_\alpha(x,t) \geq \frac{1}{2} \inf_{\mathcal{M}} u_\alpha(\cdot, t_0) > 0 \qquad \text{for} \quad t \in [t_0, T']. \tag{27}$$

Recalling that[7] $\operatorname{Rm}[g_{\mathcal{F}_\alpha}] = c_\alpha\, g_{\mathcal{F}_\alpha} \owedge g_{\mathcal{F}_\alpha}$ and using formula (67), which we derive in Lemma 33 in Appendix A, one sees easily that the bound $|\operatorname{Rm}[g(t)]| \le C_1$ on $[t_0, t_1]$ implies that

$$\left|c_\alpha u_\alpha^{-1} - \frac{1}{2}|\nabla(\log u_\alpha^{1/2})|^2\right| \le C_1.$$

Combining this with (27) implies the existence of $C' = C'(\inf_{\mathcal{M}} u_\alpha(\cdot, t_0), C_1)$ such that

$$\frac{\gamma_\alpha}{u_\alpha^2} = 4|\nabla(\log u_\alpha^{1/2})|^2 \le C' \tag{28}$$

for all $(x,t) \in \mathcal{M} \times [t_0, T']$. Using (14) and (28) yields

$$\big(\partial_t - \Delta\big)\gamma_\alpha \le C'\,\gamma_\alpha - \frac{1}{2}\,\frac{|\nabla\gamma_\alpha|^2}{\gamma_\alpha}. \tag{29}$$

We apply Lemma 13 to (29) with $U(x,t) = \gamma_\alpha(x,t)$, $V(x,t) = G_\alpha\big(v_\alpha(x,t_0)\big)$, and $W(x,t) = C'$. To see that all assumptions of Lemma 13 are satisfied, we need to check that $|V|_{2,\exp}$ is bounded by C'. Indeed, by (13) and (21) we have

$$|V|_{2,\exp} \le \|G_\alpha\|_{2,\mathrm{mon}}\,|v_\alpha(x,t_0)|_{2,\exp} \le C'. \tag{30}$$

We also need to check that $\frac{U(x,t)}{V(x,t)} \le C'\,e^{C' d^2_{g(t)}(x,x_0)}$ for $t \in [t_0, T']$, where x_0 is some fixed point in $\mathcal{B}$. Indeed, since $\nabla u_\alpha = \nabla v_\alpha$, (28) implies that for every $t \in [t_0, T']$, the function $u_\alpha(\cdot, t)$ grows at most exponentially in space and thus, for every $t \in [t_0, T']$, $U(x,t) = \gamma_\alpha(x,t)$ grows at most exponentially in space as well. On the other hand, by (30), $|\nabla \log V|$ is bounded, so $V(x,t)$ has at most exponential decay. Hence,

$$\frac{U}{V}(x,t) \le C' e^{C' d_t(x,x_0)} \qquad \text{on } \ \mathcal{M} \times [t_0, t_1],$$

as desired.

We can finally apply Lemma 13 as indicated above to conclude that for $t \in [t_0, T']$, we have

$$\gamma_\alpha(x,t) \le (1 + C'\,(t - t_0))\,C_0 C_{\mathrm{init}} G_\alpha(v_\alpha(x,t_0)), \tag{31}$$

as desired. □

18. **Lemma.** *Under the assumptions of Proposition 14, there exist a constant $C' = C'(C_{\mathrm{init}}, C_0, C_1)$ and a time $T' = T'(C_{\mathrm{init}}, C_0, C_1) \in (t_0, t_1]$ such that for all times $t \in [t_0, T']$, estimate (20c) holds.*

Proof. Assume $T' \in (t_0, t_1]$ is chosen so that both (20a) and (20b) hold on $[t_0, T']$.

Recall that in Lemma 12, we compute that χ_α evolves by

$$\big(\partial_t - \Delta\big)\chi_\alpha \le C_N L \chi_\alpha + C_N L \sum_{\beta=1}^{A} \left(\frac{\gamma_\beta}{u_\beta^2}\right) \gamma_\alpha - \frac{1}{2}\,\frac{|\nabla \chi_\alpha|^2}{\chi_\alpha},$$

where $L := \sum_{\beta=1}^{A} \frac{\gamma_\beta}{u_\beta^2} + \sum_{\beta=1}^{A} \frac{\chi_\beta^{1/2}}{u_\beta} + \rho^{1/2}$, $\rho := |\operatorname{Rm}_{\mathcal{B}}|^2$, and C_N depends only on the dimension vector $\vec{N} = (n, n_\alpha)$. The curvature bound $|\operatorname{Rm}[g(t)]| \le C_1$ implies

[7] See (65) for our normalization of the Kulkarni–Nomizu product $\owedge$.

that the terms in (67d) are bounded. Then using estimate (28) and increasing C' if necessary, we find that $|L| \le C'$ on $\mathcal{M} \times [t_0, T']$. Hence we have

$$(\partial_t - \Delta)\chi_\alpha \le C'\chi_\alpha + C' \left(\sum_{\beta=1}^{A} \frac{\gamma_\beta}{u_\beta^2} \right) \gamma_\alpha - \frac{1}{2}\frac{|\nabla \chi_\alpha|^2}{\chi_\alpha}.$$

By (20a) and (20b), we see that for $t \in [t_0, T']$,

$$\left(\sum_{\beta=1}^{A} \frac{\gamma_\beta}{u_\beta^2} \right) \gamma_\alpha \le C' \sum_{\beta=1}^{A} \frac{G_\beta(v_\beta(x,t_0))}{v_\beta^2(x,t_0)} G_\alpha(v_\alpha(x,t_0)).$$

Let $U(x,t) = \chi_\alpha(x,t)$, $V(x,t) = \sum_{\beta=1}^{A} \frac{G_\beta(v_\beta(x,t_0))}{v_\beta^2(x,t_0)} G_\alpha(v_\alpha(x,t_0)) = H_\alpha(v_\alpha(x,t_0))$, and $W = C'$. We verify that $V(x,t)$ satisfies the hypotheses of Lemma 13. By using estimate (13), we obtain

$$\begin{aligned} |V|_{2,\exp} &\le \Big| \sum_{\beta=1}^{A} \frac{G_\beta(v_\beta(x,t_0))}{v_\beta^2(x,t_0)} \Big|_{2,\exp} + |G_\alpha(v_\alpha(x,t_0))|_{2,\exp} \\ &\le 2 \sum_{\beta=1}^{A} \Big| \frac{G_\beta(v_\beta(x,t_0))}{v_\beta^2(x,t_0)} \Big|_{2,\exp} + |G_\alpha(v_\alpha(x,t_0))|_{2,\exp}. \end{aligned}$$

Note that by (13) and (21), we also have

$$|G_\alpha(v_\alpha(x,t_0))|_{2,\exp} \le \|G_\alpha\|_{2,\mathrm{mon}}^2 \, |v_\alpha(x,t_0)|_{2,\exp} \le \bar{C}_\alpha^2 C'.$$

Moreover, we may regard $\frac{G_\beta(v_\beta(x,t_0))}{v_\beta^2(x,t_0)}$ as a composition of functions $\varphi_\beta(s) := \frac{G_\beta(s)}{s^2}$ and $v_\beta(x,t_0)$. Then using (13) again, we obtain

$$\Big| \frac{G_\beta(v_\beta(x,t_0))}{v_\beta^2(x,t_0)} \Big|_{2,\exp} \le \|\varphi_\beta\|_{2,\mathrm{mon}}^2 \, |v_\beta(x,t_0)|_{2,\exp}.$$

By (21), we have $|v_\beta(x,t_0)|_{2,\exp} \le C'$. It is easy to see that

$$\frac{s\varphi_\beta'}{\varphi_\beta} \le \frac{s|G_\beta'(s)|}{G_\beta(s)} + 2 \le \bar{C}_\beta + 2$$

and

$$\frac{s^2|\varphi_\beta''|}{\varphi_\beta(s)} \le \frac{s^2|G_\beta''(s)|}{G_\beta(s)} + 4\,\frac{s|G_\beta'(s)|}{G_\beta(s)} + 6 \le 5\,\bar{C}_\beta + 6.$$

These imply that

$$\Big| \frac{G_\beta(v_\beta(x,t_0))}{v_\beta^2(x,t_0)} \Big|_{2,\exp} \le C',$$

and hence that

$$|V|_{2,\exp} \le C'. \tag{32}$$

Recall that in this proof, we choose $U = \chi_\alpha = |\nabla\nabla u_\alpha|_{g_{\mathcal{B}}}$. Our assumption that the curvature is bounded by C_1 implies in particular (by Remark 34) that $|u^{-1}\nabla\nabla u_\alpha - 1/2 u_\alpha^{-2}\nabla u_\alpha \otimes \nabla u_\alpha|$ is bounded by C'. Then (28) implies firstly that $|u_\alpha^{-1}\nabla\nabla u_\alpha| < C'$, and secondly that u_α grows at most exponentially in space, so that χ_α grows at most exponentially in space. On the other hand, by (32), we have that $V(x,t)$ decays at most exponentially in space. These two estimates yield the bound $\frac{U}{V} \le C' e^{C'\, d_{g(t)}(x,x_0)}$.

We can now apply Lemma 13 to our choice of $U(x,t)$, $V(x,t)$, and $W(x,t)$ to conclude

$$\chi_\alpha(x,t) \le C_0 C_{\text{init}}\,(1 + C'\,(t-t_0))H_\alpha(v_\alpha(x,t_0)),$$

where we use the initial condition that $\chi_\alpha(x,t_0) \le C_0 C_{\text{init}}\, H_\alpha\big(v_\alpha(x,t_0)\big)$. □

19. **Lemma.** *Under the assumptions of Proposition 14, there exist a constant $C' = C'(C_{\text{init}}, C_0, C_1)$ and a time $T' = T'(C_{\text{init}}, C_0, C_1) \in (t_0, t_1]$ such that for all times $t \in [t_0, T']$, estimate (20d) holds.*

Proof. Recall that in Lemma 12, we compute that $\rho(x,t) = |\operatorname{Rm}[g_{\mathcal{B}}](x,t)|$ evolves by

$$\big(\partial_t - \Delta\big)\rho \le C_N L^3 - \frac{|\nabla\rho|^2}{\rho},$$

where $L = \sum_{\beta=1}^{A} \frac{\gamma_\alpha}{u_\alpha^2} + \sum_{\beta=1}^{A} \frac{\chi_\alpha^{1/2}}{u_\alpha} + \rho^{1/2}$, and C_N depends only on the dimension vector $\vec{N} = (n, n_\alpha)$. As in the proof of Lemma 18, we conclude that $|L| \le C'$ on $\mathcal{M} \times [t_0, t_1]$ and hence that

$$\big(\partial_t - \Delta\big)\rho \le C_* C'\, L^2 - \frac{|\nabla\rho|^2}{\rho}.$$

By Lemma 17 and Lemma 18, there exist constants C' and $T' \in (t_0, t_1]$ so that for all $t \in [t_0, T']$, we have

$$\begin{aligned} L &\le C' \left(\sum_{\alpha=1}^{A} \frac{G_\alpha(v_\alpha(x,t_0))}{v_\alpha^2(x,t_0)} + \sum_{\alpha=1}^{A} \frac{H_\alpha^{1/2}(v_\alpha(x,t_0))}{v_\alpha(x,t_0)} + \rho^{1/2} \right) \\ &\le C'(\bar{C}_\alpha + \rho^{1/2}), \end{aligned}$$

where $\bar{C}$ is a bound on $\sup_{s_\alpha\in\mathbb{R}_+}\left(\sum_{\alpha=1}^{A} \frac{G_\alpha(s_\alpha)}{s_\alpha^2} + \sum_{\alpha=1}^{A} \frac{H_\alpha(s_\alpha)}{s_\alpha}\right)$, *i.e.*, a uniform constant. Hence,

$$\big(\partial_t - \Delta\big)\rho \le C_* C'\,(\rho + 1) - \frac{|\nabla\rho|^2}{\rho}.$$

We apply Lemma 13 with $U(x,t) = \rho(x,t)$, $V(x,t) = 1$, and $W(x,t) = C'$ to conclude that for all $t \in [t_0, T']$, we have

$$\rho(x,t) \le C_0 C_{\text{init}}(1 + C'(t-t_0)).$$

□

Combining Lemmas 15–19 completes the proof of Proposition 14.

Recall that the estimates (20b) and (20c) for $\gamma_\alpha(x,t)$ and $\chi_\alpha(x,t)$, respectively, which we prove in Proposition 14 have $v_\alpha(x,t_0)$ on the RHS. Our next result improves those by substituting $v_\alpha(x,t)$ for $v_\alpha(x,t_0)$.

20. **Proposition.** *Suppose the assumptions of Proposition 14 hold.*
Then there exists $C' = C'(C_{\text{init}}, C_0, C_1, t_1 - t_0)$ so that we have

$$\gamma_\alpha(x,t) \le (1 + C'(t-t_0))\, C_0 C_{\text{init}} G_\alpha(v_\alpha(x,t)),$$

$$\chi_\alpha(x,t) \le (1 + C'(t-t_0))\, C_0 C_{\text{init}} H_\alpha(v_\alpha(x,t)),$$

for all $t \in [t_0, T']$, where T' is the same as in Proposition 14.

Proof. By the chain rule, we have

$$\partial_t\big(G_\alpha(v_\alpha)\big) = G'_\alpha(v_\alpha)\,\partial_t v_\alpha,$$

which implies that

$$\begin{aligned} |\partial_t G_\alpha(v_\alpha(x,t))| &= \left|\frac{v_\alpha(x,t)G'_\alpha(v_\alpha(x,t))}{G_\alpha(v_\alpha(x,t))}\right| \frac{G_\alpha(v_\alpha(x,t)}{v_\alpha(x,t)}\,|\partial_t v_\alpha(x,t)| \\ &\le \bar{C}_\alpha G_\alpha(v_\alpha(x,t))\left(\frac{|\nabla^2 v_\alpha(x,t)|}{v_\alpha(x,t)} + \frac{|\nabla v_\alpha(x,t)|^2}{v_\alpha^2(x,t)}\right). \end{aligned} \tag{33}$$

By (20a) and (20b), which hold for $t \in [t_0, T']$, we have

$$\begin{aligned} |\nabla v_\alpha|^2(x,t) &\le (1 + C'\,(t-t_0))C_0 C_{\mathrm{init}}\bar{C}_\alpha v_\alpha^2(x,t_0) \\ &\le (1 + C'\,(t-t_0))^3 C_0 C_{\mathrm{init}}\bar{C}_\alpha v_\alpha^2(x,t) \end{aligned}$$

for all $t \in [t_0, T']$. This yields

$$\frac{|\nabla v_\alpha|^2}{v_\alpha^2} \le (1 + C'\,(t-t_0))^3 C_0 C_{\mathrm{init}}\bar{C}_\alpha \qquad \text{for} \quad t \in [t_0, T']. \tag{34}$$

To bound $\frac{|\nabla^2 v_\alpha(x,t)|}{v_\alpha(x,t)}$, we note that by (20a) and (20c), we have

$$\begin{aligned} \chi_\alpha(x,t) &\le (1 + C'\,(t-t_0))C_0 C_{\mathrm{init}} H_\alpha(v_\alpha(x,t_0)) \\ &\le (1 + C'\,(t-t_0))C_0 C_{\mathrm{init}}\bar{C}_\alpha v_\alpha^2(x,t_0) \\ &\le (1 + C'\,(t-t_0))^3 C_0 C_{\mathrm{init}}\bar{C}_\alpha v_\alpha(x,t)^2, \end{aligned}$$

implying that

$$\frac{|\nabla^2 v_\alpha|^2}{v_\alpha^2} \le \Big(1 + C'\,(t-t_0)\Big)^3 C_0 C_{\mathrm{init}}\bar{C}_\alpha, \qquad \text{for} \;\; t \in [t_0, T'],$$

where $\bar{C}_\alpha$ is a uniform constant. Combining this estimate with (34) and (33) yields

$$|\partial_t \log G_\alpha(v_\alpha(x,t))| \le (1 + C'\,(t-t_0))^3\, C_0 C_{\mathrm{init}}\bar{C}_\alpha,$$

and hence

$$G_\alpha(v_\alpha(x,t)) \le (1 + C'(t-t_0))\, G_\alpha(v_\alpha(x,t_0)) \qquad \text{for all} \;\; t \in [t_0, T']. \tag{35}$$

We combine (20b) and (35) to conclude that for all $t \in [t_0, T']$, we have

$$\gamma_\alpha(x,t) \le (1 + C'\,(t-t_0))C_0 C_{\mathrm{init}}\, G_\alpha(v_\alpha(x,t)).$$

Finally, using (20a), (20c), and (35) yields

$$\chi_\alpha(x,t) \le (1 + C'\,(t-t_0))C_0 C_{\mathrm{init}} H_\alpha(v_\alpha(x,t))$$

for all $t \in [t_0, T']$, as claimed. □

We now prove our second pair of Propositions, which provide control of the curvatures by a constant that depends only on the initial data. Specifically, in contrast to Propositions 14 and 20, the constant we obtain below is independent of the bound $\sup_{(x,t)\in\mathcal{B}\times[t_0,t_1]} |\operatorname{Rm}(x,t)| \le C_1$.

21. **Proposition.** *Let* $(\mathcal{M}, g_{\text{init}})$ *satisfy the Main Assumptions in Section 2.1. Suppose a solution* $g(t)$ *of Ricci flow exists for* $[0,T]$ *satisfying* $v_\alpha(x,0) > 0$ *and*

$$\gamma_\alpha(x,t) \le 2\,C_{\text{init}}\,G_\alpha\big(v_\alpha(x,t)\big), \tag{36a}$$

$$\chi_\alpha(x,t) \le 2\,C_{\text{init}}\,H_\alpha\big(v_\alpha(x,t)\big), \tag{36b}$$

$$\rho(x,t) \le 2\,C_{\text{init}} \tag{36c}$$

for $t \in [0,T]$.

Then there exists C_* *depending only on* C_{init} *and* $\vec{N} = (n, n_\alpha)$ *such that for* $t \in [0, \min\{T, C_*^{-1}\}]$, *one has the bounds*

$$\begin{aligned} \gamma_\alpha(x,t) &\le C_{\text{init}} G_\alpha(v_\alpha(x,t)) \Big(1 + C_* t E_\alpha\big(v_\alpha(x,t)\big)\Big), \\ \chi_\alpha(x,t) &\le C_{\text{init}}\, H_\alpha\big(v_\alpha(x,t)\big) \Big(1 + C_*\, t\Big), \\ \rho(x,t) &\le C_{\text{init}}\,(1 + C_*\, t), \end{aligned}$$

where

$$E_\alpha\big(v_\alpha(x,t)\big) := \frac{G_\alpha\big(v_\alpha(x,t)\big)}{v_\alpha(x,t)^2}$$

is bounded by our Main Assumptions.

Proof. The proof is very similar to the proof of Proposition 14. We let $C_*' = C_*'(C_{\text{init}})$ be a uniform constant that may increase from line to line, whereas C_* is the final constant that appears in the statement above.

To obtain the desired bound for γ_α, we recall estimate (14),

$$\big(\partial_t - \Delta\big)\gamma_\alpha \le 6 \left(\frac{\gamma_\alpha}{v_\alpha^2}\right) \gamma_\alpha - \frac{1}{2}\frac{|\nabla \gamma_\alpha|^2}{\gamma_\alpha}.$$

By (36), we have

$$\gamma_\alpha(x,t) \le 2\,C_{\text{init}} G_\alpha(v_\alpha(x,t)),$$

and hence

$$\big(\partial_t - \Delta\big)\gamma_\alpha \le C_*'\, E_\alpha(v_\alpha)\,\gamma_\alpha - \frac{1}{2}\frac{|\nabla \gamma_\alpha|^2}{\gamma_\alpha}.$$

Our goal is to apply Lemma 13 to $U(x,t) = \gamma_\alpha(x,t)$, $V(x,t) = G_\alpha(v_\alpha(x,t))$, and $W(x,t) = E_\alpha(v_\alpha(x,t))$. In order to do this, we need to verify that the hypotheses of Lemma 13 are satisfied. By (13), we have

$$|G_\alpha(v_\alpha)|_{2,\exp} \le \|G_\alpha\|_{2,\text{mon}}^2\, |v_\alpha|_{2,\exp} \le C_*'\, |v_\alpha|_{2,\exp}$$

and

$$|v_\alpha|_{2,\exp} = 2\frac{\gamma_\alpha}{v_\alpha^2} \le C_*'\, E_\alpha(v_\alpha). \tag{38}$$

Thus,

$$|G_\alpha(v_\alpha)|_{2,\exp} \le C_*'\, E_\alpha(v_\alpha).$$

On the other hand, $E_\alpha(v_\alpha) \le \|G_\alpha\|_{\mathcal{G}} \le \bar{C}_\alpha$ and

$$|E_\alpha(v_\alpha)|_{2,\exp} \le \|E_\alpha\|_{2,\text{mon}}^2 |v_\alpha|_{2,\exp}.$$

It is easy to see that $E_\alpha(s) = \frac{G_\alpha(s)}{s^2} \le C$ satisfies $\|E_\alpha\|_{2,\text{mon}} \le C_*'$ and that $E_\alpha(v_\alpha) \le C_*'$ is bounded. Hence we have

$$|E_\alpha(v_\alpha)|_{2,\exp} \le C_*',$$

implying by definition of $|\cdot|_{2,\exp}$ that

$$\big|\big(\partial_t-\Delta\big)E_\alpha(v_\alpha)\big|+\frac{|\nabla E_\alpha(v_\alpha)|^2}{E_\alpha(v_\alpha)}\le C'_*\,E_\alpha(v_\alpha).$$

Then using the fact that $|E_\alpha(v_\alpha)|$ is bounded and increasing C'_* if necessary, we obtain

$$\big|\big(\partial_t-\Delta\big)E_\alpha(v_\alpha)\big|+|\nabla E_\alpha(v_\alpha)|^2\le C'_*\,E_\alpha(v_\alpha).$$

We can now apply Lemma 13 as indicated above to conclude that there exists $C'_*=C'_*(C_{\rm init},N)$ and $T_*=T_*(C_{\rm init},N)$ so that for all $t\in[0,\min\{T_*,T\}]$, we have

$$\begin{aligned}\frac{\gamma_\alpha(x,t)}{G_\alpha(v_\alpha(x,t))}&\le C_{\rm init}+C'_*t\,(1+C_{\rm init})\,E_\alpha\big(v_\alpha(x,t)\big)\\&\le C_{\rm init}\Big(1+C_*\,t\,E_\alpha\big(v_\alpha(x,t)\big)\Big).\end{aligned}\tag{39}$$

This yields

$$\gamma_\alpha(x,t)\le C_{\rm init}\,G_\alpha(v_\alpha(x,t))\Big(1+C_*\,t\,E_\alpha\big(v_\alpha(x,t)\big)\Big)$$

for $t\in[0,\min\{T,T_*\}]$, as claimed.

To obtain the bound for χ_α, we recall estimate (15), namely

$$\big(\partial_t-\Delta\big)\chi_\alpha\le C_N\,L\chi_\alpha+C_N\,L\sum_{\beta=1}^{A}\frac{\gamma_\beta}{u_\beta^2}\,\gamma_\alpha-\frac12\,\frac{|\nabla\chi_\alpha|^2}{\chi_\alpha},$$

where $L=\sum_{\beta=1}^{A}\frac{\gamma_\beta}{u_\beta^2}+\sum_{\beta=1}^{A}\frac{\chi_\beta^{1/2}}{u_\beta}+\rho^{1/2}$, and the constant C_N depends only on $\vec N=(n,n_\alpha)$. By the assumptions in (36), we have $L\le C'_*$ and

$$\left(\sum_{\beta=1}^{A}\frac{\gamma_\beta}{u_\beta^2}\right)\gamma_\alpha\le C'_*\,H_\alpha(v_\alpha),$$

where C'_* is a uniform constant depending only on $C_{\rm init}$. Thus,

$$\big(\partial_t-\Delta\big)\chi_\alpha\le C_N\,\chi_\alpha+C_N\,H_\alpha(v_\alpha)-\frac12\,\frac{|\nabla\chi_\alpha|^2}{\chi_\alpha}.$$

We want to apply Lemma 13 to $U(x,t)=\chi_\alpha(x,t)$, $V(x,t)=H_\alpha(v_\alpha(x,t))$, and $W(x,t)=C'_*$. By (13), the fact that $\|G_\alpha\|_{2,\rm mon}+\|E_\alpha\|_{2,\rm mon}\le C'_*$, (36), and (38), we have

$$\begin{aligned}|H_\alpha(v_\alpha)|_{2,\exp}&\le\Big|\sum E_\beta(v_\beta)|_{2,\exp}+|G_\alpha(v_\alpha)|_{2,\exp}\\&\le2\sum|E_\beta(v_\beta)|_{2,\exp}+|G_\alpha(v_\alpha)|_{2,\exp}\\&\le C'_*\,|v_\alpha|_{2,\exp}=2C'_*\,\frac{|\nabla v_\alpha|^2}{v_\alpha^2}\\&\le C'_*\frac{G_\alpha(v_\alpha)}{v_\alpha^2}\le C'_*.\end{aligned}$$

By Lemma 13 applied as above, there exist $C'_*=C'_*(C_{\rm init},C_N)$ and $T_*=T_*(C_{\rm init},C_N,)$ such that for all $t\in[0,\min\{T,T_*\}]$, we have

$$\frac{\chi_\alpha(x,t)}{H_\alpha(v_\alpha(x,t))}\le C_{\rm init}+C'_*\,t\,(1+C_{\rm init}),$$

which implies that

$$\chi_\alpha(x,t) \le C_{\mathrm{init}}\, H_\alpha(v_\alpha(x,t))\,(1 + C_* t).$$

To obtain the desired bound for $\rho(x,t)$, we recall estimate (16),

$$\big(\partial_t - \Delta\big)\rho \le C_N L^3 - \frac{|\nabla\rho|^2}{\rho},$$

where L is as above, and C_N depends only on $\vec{N} = (n, n_\alpha)$. Hence,

$$\big(\partial_t - \Delta\big)\rho \le C'_* L^2 - \frac{|\nabla\rho|^2}{\rho}.$$

As in Lemma 19, using (36), we obtain

$$\big(\partial_t - \Delta\big)\rho \le C'_*\,(\rho + 1) - c_N\,\frac{|\nabla\rho|^2}{\rho}.$$

We take $U(x,t) = \rho(x,t)$, $V(x,t) = 1$ and $W(x,t) = C'_*$ and apply Lemma 13. It gives us the existence of $C'_* = C'_*(C_{\mathrm{init}}, T, C_N)$ and $T_* = T_*(C_{\mathrm{init}}, T, C_N)$ such that for all $t \in [0, \min\{T, T_*\}]$, we have

$$\rho(x,t) \le C_{\mathrm{init}} + C'_*\, t\,(1 + C_{\mathrm{init}}),$$

implying that

$$\rho(x,t) \le C_{\mathrm{init}}\,(1 + C_*\, t),$$

for all $t \in [0, \min\{T, T_*\}]$, where C_* is a uniform constant.

Finally, we increase C_* if necessary so that $C_* T_* \ge 1$. This concludes the proof of Proposition 21. ☐

Next, we improve Proposition 14 by showing in a precise sense that the quantities v_α are uniformly equivalent for $t \in [0, \min\{T, C_*^{-1}\}]$, where $[0,T]$ is the time interval on which the hypotheses of Proposition 21 hold. As we note in the Introduction, Theorem 5 follows easily from the arguments that we use to prove Proposition 22.

22. **Proposition.** *Let $(\mathcal{M}, g_{\mathrm{init}})$ satisfy the Main Assumptions in Section 2.1, and let T and C_* be as in the statement of Proposition 21. Then the quantities v_α are uniformly equivalent: for $t \in [0, \min\{T, C_*^{-1}\}]$, we have*

$$\frac{1}{C_*} v_\alpha(x,t) \le v_\alpha(x,0) \le C_* v_\alpha(x,t).$$

Furthermore, if in addition to the Main Assumptions, it is also true that

$$\frac{G_\alpha(s)}{s^2} = o(1;\ s \searrow 0), \tag{40}$$

then for all $t \in [0, \min\{T, C_*^{-1}\}]$,

$$v_\alpha(x,t) = \big(1 + o(1;\ v_\alpha(x,0) \searrow 0)\big)\, v_\alpha(x,0).$$

Proof. It follows easily from (3b) that there exists $C_N = C_N(n, n_\alpha)$ such that

$$|\partial_t v_\alpha| = \left|\Delta v_\alpha - \frac{|\nabla v_\alpha|^2}{a_\alpha - \mu_\alpha t + v_\alpha}\right| \le C_N \left(\frac{|\nabla^2 v_\alpha|}{v_\alpha} + \frac{v_\alpha}{a_\alpha - \mu_\alpha t + v_\alpha}\,\frac{|\nabla v_\alpha|^2}{v_\alpha^2}\right) v_\alpha. \tag{41}$$

By Claim 32, there exists C'_* such that

$$|\nabla^2 v_\alpha| \le \chi_\alpha^{1/2} + C'_* \sum_{\beta=1}^{A} \frac{\gamma_\beta^{1/2}}{u_\beta}\, \gamma_\alpha^{1/2}.$$

Then we may apply Proposition 21 to obtain

$$\frac{|\nabla^2 v_\alpha|}{v_\alpha} \le \frac{\chi_\alpha^{1/2}}{v_\alpha} + C'_* \left(\sum_{\beta=1}^{A} \frac{\gamma_\beta}{v_\beta^2}\right)^{1/2} \frac{\gamma_\alpha^{1/2}}{v_\alpha} \le C'_* \sqrt{\frac{H_\alpha\big(v_\alpha(x,t)\big)}{v_\alpha(x,t)^2}}.$$

Applying Proposition 21 again to bound the gradient term, we find that the absolute value of the quantity in parentheses on the RHS of (41) is bounded by

$$C'_* \left(\sqrt{\frac{H_\alpha\big(v_\alpha(x,t)\big)}{v_\alpha(x,t)^2}} + \frac{G_\alpha\big(v_\alpha(x,t)\big)}{v_\alpha(x,t)^2}\right). \tag{42}$$

It follows that

$$|\partial_t v_\alpha| \le C_N C_* \|G_\alpha\|_{\mathcal{G}}\, v_\alpha.$$

This proves the first claim.

To prove the second claim, we observe that it follows from assumption (40) at $t = 0$ and the first claim that for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $v_\alpha(x,0) < \delta$, then $G_\alpha\big(v_\alpha(x,t)\big) < \varepsilon\, v_\alpha^2(x,t)$ uniformly for $t \in [0,T)$. At any x with $v_\alpha(x,0) < \delta$, one can then bound the quantity in parentheses on the RHS of (41) in absolute value by

$$\sqrt{\frac{H_\alpha\big(v_\alpha(x,t)\big)}{v_\alpha(x,t)^2}} + \frac{G_\alpha\big(v_\alpha(x,t)\big)}{v_\alpha(x,t)^2} \le \varepsilon.$$

Hence at such x, one has $\partial_t v_\alpha(x,t) \le \varepsilon\, C_N\, v_\alpha(x,t)$. The second claim follows. ☐

3.3. **Proofs of main results.** In this section, we prove Theorem 6 and Corollary 7.

Proof of Theorem 6. We define

$$T_{\sup} := \sup\{T \in [0, T_{\rm sing})\colon \text{ the conclusions of Theorem 6 hold on } [0,T]\}.$$

We prove the theorem in two steps.

Step 1. We claim that $T_{\sup} > 0$. To see this, we recall that by [Shi89] and [CZ06], there exists $T_{\min}(n, n_\alpha, C_{\rm init}) > 0$ such that Ricci flow with initial data $(\mathcal{M}, g_{\rm init})$ has a unique smooth solution on $[0, T_{\min}]$. We first apply Propositions 14 and 20 with $t_0 = 0$, $t_1 = T_{\min}$, $C_0 = 1$, and $C_1 = C_1(T_{\min}) < \infty$. They yield a constant C' and a time $T_1 \in (0, T_{\min}]$ such that the estimates

$$\begin{aligned} v_\alpha(x,t) &\ge \frac{v_\alpha(x,0)}{1+C't},\\ \rho(x,t) &\le C_{\rm init}\,(1+C't),\\ \gamma_\alpha(x,t) &\le C_{\rm init}(1+C't)\, G_\alpha\big(v_\alpha(x,t)\big),\\ \chi_\alpha(x,t) &\le C_{\rm init}(1+C't)\, H_\alpha\big(v_\alpha(x,t)\big), \end{aligned}$$

hold on $[0, T_1]$. We may then choose $T_2 \in (0, T_1]$ small enough that $C'T_2 \le 1$. This ensures that the hypotheses of Propositions 21 and 22 are satisfied on $[0, T_2]$.

Next we apply Propositions 21 and 22 on $[0, T_2]$. They yield a time $T_3 \in (0, T_2]$ depending only on $\{C_{\rm init}, n, n_\alpha\}$ such that the estimates claimed in Theorem 6 hold on $[0, T_3]$. It follows that $T_{\sup} \geq T_3 > 0$, thus proving the claim.

Step 2. We next claim that $T_{\sup} \geq \min\{T_{\rm sing}, C_*^{-1}\}$. We prove the claim by contradiction, so we may suppose that $T_{\sup} < \min\{T_{\rm sing}, C_*^{-1}\}$. Then because $T_{\sup} < T_{\rm sing}$ and the inequalities in Theorem 6 are of the form $\leq$ rather than $<$, they hold on $[0, T_{\sup}]$ by continuity. So we may apply Propositions 14 and 20 with $t_0 = T_{\sup}$, $t_1 = T_1 \in (T_{\sup}, T_{\rm sing})$ arbitrary, $C_0 = 1 + C_* T_{\sup}$, and $C_1 = C_1(T_1) < \infty$. They yield a constant C'' and a time $T_4 \in (T_{\sup}, T_1]$ such that the estimates

$$
\begin{aligned}
v_\alpha(x,t) &\geq \frac{v_\alpha(x, T_{\sup})}{1 + C''(t - T_{\sup})} \geq \frac{v_\alpha(x,0)}{C_*\big(1 + C''(t - T_{\sup})\big)},\\
\rho(x,t) &\leq (1 + C_* T_{\sup})\, C_{\rm init}\,\big(1 + C''(t - T_{\sup})\big),\\
\gamma_\alpha(x,t) &\leq (1 + C_* T_{\sup})\, C_{\rm init}\big(1 + C''(t - T_{\sup})\big)\, G_\alpha\big(v_\alpha(x,t)\big),\\
\chi_\alpha(x,t) &\leq (1 + C_* T_{\sup})\, C_{\rm init}\big(1 + C''(t - T_{\sup})\big)\, H_\alpha\big(v_\alpha(x,t)\big),
\end{aligned}
$$

hold on $[0, T_4]$. Because $1 + C_* T_{\sup} < 2$ by assumption, these estimates let us apply Propositions 21 and 22 and thus obtain the conclusions of Theorem 6 on $[0, T_5]$ for some $T_5 > T_{\sup}$. By definition of $T_{\sup}$, this is a contradiction, which proves the claim. □

Proof of Corollary 7. We recall that by assumption, at least one fiber is a positively curved space form, and that we have chosen ς so that $\frac{a_\varsigma}{\mu_\varsigma} = \min\{\frac{a_\alpha}{\mu_\alpha} : \mu_\alpha > 0\}$. Because the constant C_* in Theorem 6 depends only on our Main Assumptions, which are independent of a_ς, we may without creating circular dependencies shrink a_ς (where, abusing notation, we continue to denote this quantity by a_ς) to create new initial data $g'_{\rm init}$ for which $a_\varsigma = \mu_\varsigma / C_*$. Note that we do not change $v_\alpha(\cdot, 0)$.

Now we apply Theorem 6 to Ricci flow originating from $g'_{\rm init}$. The Theorem controls the evolving metric on $[0, \min\{T_{\rm sing}, C_*^{-1}\})$, for the same constant C_*. Since the v_α are uniformly equivalent in time on $[0, \min\{T_{\rm sing}, C_*^{-1}\})$, the condition that $\inf_{x \in \mathcal{B}} v_\alpha(x,t) = 0$, which holds at $t = 0$ by construction (see Remark 4), also holds on that entire interval. But this means that $T_{\rm sing}$ can be no larger than the formal vanishing time $T_{\rm form} = a_\varsigma / \mu_\varsigma = C_*^{-1}$. This in particular implies that the conclusion of Theorem 6 holds for all $t \in [0, T_{\rm sing})$. However, we have $T_{\rm sing} = T_{\rm form}$, because for $t \in [0, C_*^{-1})$, Theorem 6 implies positivity of v_ς, hence that $u_\varsigma > a_\varsigma - \mu_\varsigma t > 0$, and gives control on the remaining curvatures.

Next we prove that solutions originating from initial data satisfying our Main Assumptions develop Type-I singularities at spatial infinity. We recall inequality (8):

$$
\Big| \mathrm{Rm}[g] - \sum_{\alpha=1}^{A} u_\alpha \,\mathrm{Rm}[g_{\mathcal{F}_\alpha}] \Big|_g \leq C \left\{ \rho^{1/2} + \sum_{\alpha=1}^{A} (u_\alpha^{-2} \gamma_\alpha + u_\alpha^{-1} \chi_\alpha^{1/2}) \right\}.
$$

Theorem 6 implies that the RHS is bounded by a constant C' depending only on $C_* = C_*\big(C_{\rm init}, \vec{N} = (N, n_\alpha)\big)$ and $T_{\rm sing}$. Thus we find that as $t \nearrow T_{\rm sing}$,

$$
|\,\mathrm{Rm}\,| - \sum_{\alpha=1}^{A} \frac{c_\alpha}{u_\alpha} \leq C', \tag{43}
$$

where the constants c_α depend only on $\vec{N}$ and the Ricci constants μ_α. Moreover, Proposition 22 implies that on any compact set, the functions

$$u_\alpha(x,t) = (a_\alpha - \mu_\alpha t) + v_\alpha(x,t)$$

are bounded from below, and thus the curvature is bounded from above. But by Remark 4, $\inf v_\alpha(x,0) = 0$, and by Theorem 6, this remains true for all $t > 0$ that the solution exists. Thus at any such time, the warping function of the fiber $\mathcal{F}_\varsigma$ satisfies

$$\sup_{x\in\mathcal{B}} u_\varsigma^{-1}(x,t) = (a_\varsigma - \mu_\varsigma t)^{-1},$$

which shows that the singularity is Type-I and forms at spatial infinity, as claimed. □

4. Applications

4.1. **Essential blowup sequences on noncompact manifolds.** Assume that a Ricci flow solution $(\mathcal{M}, g(t))$ develops a singularity at some time $T < \infty$. This means that $\limsup_{t\nearrow T} \mathcal{R}(t) = \infty$, where $\mathcal{R}(t) := \sup_{\mathcal{M}} |\operatorname{Rm}[g(t)]|$. A Ricci flow solution $(\mathcal{M}, g(t))$ that becomes singular at $T < \infty$ is called *Type-I* if there exists a constant $C > 0$ such that for all $t \in [0,T)$, one has $\mathcal{R}(t) \leq \frac{C}{T-t}$.

If $\big(\mathcal{M}, g(t)\big)$ is a Type-I Ricci flow solution, then a point $p \in \mathcal{M}$ is called a *Type-I singular point* if there exists an essential blowup sequence $(p_i, t_i) \in \mathcal{M} \times [0,T)$ so that $\lim_{i\to\infty} p_i = p$ and $\lim_{i\to\infty} t_i = T$. By definition, to be an *essential blowup sequence* means that there exists a constant $c > 0$ such that

$$|\operatorname{Rm}[g(t_i)]|_{g(t_i)}(p_i, t_i) \geq \frac{c}{T - t_i}.$$

Because $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}(t) \leq C_n \mathcal{R}^2(t)$, the curvature of a developing singularity always grows at least at a Type-I rate, and so such sequences always exist. If $\mathcal{M}$ is noncompact, however, it might be the case that an essential blowup sequence does not limit to any Type-I singular point in $\mathcal{M}$.

In [EMT11], it is proven that if $(\mathcal{M}, g(t))$ is a Type-I Ricci flow on $[0,T)$, and if $p \in \mathcal{M}$ is a Type-I singular point, then for every sequence $\lambda_j \to \infty$, the corresponding rescaled Ricci flows $(\mathcal{M}, g_j(t), p)$, defined on $[-\lambda_j T, 0)$ by $g_j(t) := \lambda_j\, g(T + \lambda_j^{-1} t)$ subconverge to a normalized nontrivial gradient shrinking Ricci soliton in canonical form. This is a solution $(\mathcal{N}, g, f)$ that exists on a time interval $(-\infty, T]$ and satisfies

$$\operatorname{Rc} + \nabla^2 f = \frac{1}{2(T-t)} g \qquad \text{and} \qquad \frac{\partial}{\partial t} f = |\nabla f|^2.$$

The result in [EMT11] applies in the case that $(\mathcal{M}, g(t))$ is a Type-I Ricci flow on a compact manifold $\mathcal{M}$, or if $\mathcal{M}$ is noncompact and $p \in \mathcal{M}$ is a Type-I singular point. On the other hand, if $\mathcal{M}$ is noncompact and $(\mathcal{M}, g(t))$ is a Type-I flow with a singularity forming at spatial infinity, then a Type-I singular point may not exist — see the example suggested in Remark 1.3 of [EMT11]. In this case, the results of [EMT11] do not preclude the possibility that the limit along some blow up sequences is a nontrivial gradient shrinking Ricci soliton, while along some other blow up sequences it is not. Our goal in this section is to use the results we have proven here to produce an example exhibiting this phenomenon.

Assume that $(\mathcal{M}, g(t))$ is a complete noncompact Type-I Ricci flow that develops a singularity at spatial infinity at some time $T < \infty$. That is, there exist sequences

$p_j \in \mathcal{M}$, $t_j \in [0,T)$, and a uniform constant $c_0 > 0$ such that $(p_j, t_j) \to (\infty, T)$ as $j \to \infty$, and

$$|\operatorname{Rm}|(p_j, t_j) \geq \frac{c_0}{T - t_j}.$$

This in particular implies that (p_j, t_j) is an essential blow up sequence for the singularity developing at spatial infinity at time T. If λ_j is a sequence such that $\lim_{j\to\infty} \lambda_j = \infty$ and if $g_j(\cdot, t) := \lambda_j\, g(\cdot, T + t\lambda_j^{-1})$, then a *blowup limit* of the flow along the sequence p_j is a pointed subsequential limit of $(\mathcal{M}, g_j(\cdot, T), p_j)$, if it exists.

We now explore what blowup limits are possible for a particular family of noncompact Ricci flow solutions of the type considered in this paper.

23. **Definition.** *Let g_{Eucl} denote the Euclidean metric on $\mathbb{R}^k$. A family of functions $\delta_\alpha(x) : \mathbb{R}^k \to \mathbb{R}_+$ specifies an* admissible perturbation *of $g = g_{\mathrm{Eucl}} + \sum_{\alpha=1}^{A} a_\alpha\, g_{\mathcal{F}_\alpha^{n_\alpha}}$ on $\mathbb{R}^k \times \mathcal{F}^{n_1} \times \cdots \times \mathcal{F}^{n_A}$ if $\lim_{|x|\to\infty} \delta_\alpha(x) = 0$ and there exist functions $G_\alpha \in \mathcal{G}$ satisfying*[8]

$$\frac{G_\alpha(\delta_\alpha(x))}{\delta_\alpha^2(x)} = o(1; \delta_\alpha(x) \searrow 0)$$

such that the metric

$$g_{\mathrm{init}} = g_{\mathrm{Eucl}} + \sum_{\alpha=1}^{A} \big(a_\alpha + \delta_\alpha(x)\big)\, g_{\mathcal{F}^{n_\alpha}} \tag{44}$$

satisfies the Main Assumptions.[9]

We define the set

$$\mathcal{A} := \{\delta_\alpha(x) : \mathbb{R}^k \to \mathbb{R}_+ : \delta_\alpha(x) \text{ is admissible in the sense of Definition 23}\}. \tag{45}$$

24. **Lemma.** *For any manifold $\mathbb{R}^k \times \mathcal{S}^p \times \mathcal{S}^q$, the set $\mathcal{A}$ is nonempty.*

Proof. In polar coordinates on $\mathbb{R}^k$, choose rotationally-symmetric warping functions $\delta_1(r) = \delta_2(r) = \frac{1}{1+r^2}$, along with control functions $G_1(s) = G_2(s) = \frac{s^3}{1+s}$. Then it is straightforward to verify that the Main Assumptions are satisfied for the metric (44), because

$$\gamma_\alpha(r) = |\nabla \delta_\alpha(r)|^2 = \frac{4r^2}{(1+r^2)^4} \leq \frac{4}{(1+r^2)^3} \leq 8\, G_\alpha(\delta_\alpha(r)),$$

and

$$\chi_\alpha(r) = |\nabla\nabla \delta_\alpha(r)|^2 \leq C_{init} \left(\frac{1}{(1+r^2)^4} + \frac{r^4}{(1+r^2)^6} \right)$$
$$\leq \frac{C_{\mathrm{init}}}{(1+r^2)^4} \leq C_{\mathrm{init}} H_\alpha(\delta(r)).$$

Here we use the fact that $H_\alpha(\delta_\alpha(r)) \geq \frac{1}{2(1+r^2)^4}$, which is easy to check. We also have $\rho \equiv 0$ for the Euclidean metric. Thus we conclude that $\delta_1, \delta_2 \in \mathcal{A}$. $\square$

---

[8]We write $o(1; \delta_\alpha(x) \searrow 0)$ to denote a quantity that becomes arbitrarily small as $\delta_\alpha(x) \searrow 0$, *i.e.,* as $|x| \to \infty$. We use similar expressions below, *mutatis mutandis.*

[9]We note that by Remark 11, satisfying the Main Assumptions forces the functions δ_α to be strictly positive everywhere.

For the purpose of the applications that we discuss in this subsection, it suffices to consider a doubly-warped product. Thus we fix $k = 1$, $A = 2$, and spherical fibers $\mathcal{F}^{n_1} = \mathcal{F}^{n_2} = \mathcal{S}^p$ $(p \geq 2)$ in the remainder of this subsection.

To prove Theorem 1, we consider $\mathcal{M} = \mathbb{R} \times \mathcal{S}^p \times \mathcal{S}^p$ $(p \geq 2)$, with initial metric

$$g_{\text{init}} = (\mathrm{d}x)^2 + \big(a_* + v_1(x,0)\big)\, g_{\mathcal{S}^p} + \big(a_* + v_2(x,0)\big)\, g_{\mathcal{S}^p}, \tag{46}$$

where $g_{\mathcal{S}^p}$ is the round metric scaled so that $2\,\mathrm{Rc}_{g_{\mathcal{S}^p}} = g_{\mathcal{S}^p}$, and $v_1(x,0) = \delta_1(x)$ and $v_2(x,0) = \delta_2(x)$, where $\delta_1, \delta_2 \in \mathcal{A}$ are functions such that

$$\lim_{|x|\to\infty} \frac{\delta_1(x)}{\delta_2(x)} = \eta \in \mathbb{R}_+ \setminus \{0,1\}.$$

The fact that $a_* > 0$ ensures that the curvatures of g_{init} are uniformly bounded, hence that Ricci flow with this initial data has short-time existence.

We require $\eta > 0$ to ensure that δ_1 and δ_2 remain comparable, so that we may take appropriate limits. We further require that $\eta \neq 1$ to demonstrate the existence of sequences that cannot limit to nontrivial gradient shrinking Ricci solitons.

A slight modification of the construction in this section shows that there exist noncompact Ricci flow solutions that develop Type-I singularities for which there can be no blowup limits $(\mathcal{M}_\infty, g_\infty)$. Indeed, if we consider $\mathcal{M} = \mathbb{R} \times \mathcal{S}^1 \times \mathcal{S}^p$ with an initial metric that is not κ-noncollapsed and has curvatures uniformly bounded by $C = C(a_*)$,

$$g_{\text{init}} = (\mathrm{d}x)^2 + \delta_1(x)(\mathrm{d}\theta)^2 + (a_* + \delta_2(x))g_{\mathcal{S}^p},$$

where $\delta_1, \delta_2 \in \mathcal{A}$, then Ricci flow starting at g_{init} has short-time existence, and our work in proving Theorem 1 goes through, *mutatis mutandis*, establishing the following:

25. **Remark.** *There exist complete, noncompact, collapsed Ricci flow solutions* $\big(\mathcal{M}, g(t)\big)$ *that develop Type-I singularities at spatial infinity. For each of these solutions, Type-I blowups have no Cheeger–Gromov limits.*

As we note in the introduction, blowup limits may exist as étale groupoids, in the sense considered in [Lott10].

We now show how Theorem 6 and Corollary 7 lead to our main application, as stated in Theorem 1 and Corollary 2.

Proof of Theorem 1. By the proof of Corollary 7, we may choose $a_* > 0$ sufficiently small so that the estimates of Theorem 6 hold for the solution originating from initial data (46) up to the singular time $T_{\text{sing}} = a_*$, at which time it encounters a Type-I singularity. For as long as it exists, the Ricci flow solution has the form

$$g(x,t) = \mathrm{d}x^2 + \big(a_* - t + v_1(x,t)\big)\, g_{\mathcal{S}^p} + \big(a_* - t + v_2(x,t)\big)\, g_{\mathcal{S}^p}. \tag{47}$$

By the second part of Proposition 22, we have

$$v_\alpha(x,t) = \big(1 + o(1;\, \delta_\alpha(x) \searrow 0)\big)\, \delta_\alpha(x) \tag{48}$$

for all $t \in [0, a_*)$.

Initially, we consider <u>any</u> sequence $(x_j, t_j) \to (\infty, a_*)$ for which

$$\lim_{j\to\infty} \frac{\delta_\alpha(x_j)}{a_* - t_j} =: c_\alpha \in (0,\infty), \qquad (\alpha = 1, 2). \tag{49}$$

Then our curvature estimates above easily imply that there exists a uniform constant $c_0 > 0$ such that $|\mathrm{Rm}(x_j,t_j)| \geq \frac{c_0}{a_* - t_j}$. Let $\lambda_j \to \infty$ be any sequence such that $\limsup_{j\to\infty}(T-t_j)\,\lambda_j<\infty$, and let $g_j(t)=\lambda_j\,g(a_*+t\lambda_j^{-1})$. This rescaling ensures that each g_j exists up to $t=0$; indeed, because we have shown that the singularity is Type-I, it immediately follows that

$$|\operatorname{Rm}[g_j(t)]|\le\frac{C}{-t} \tag{50}$$

for a uniform constant C. Note that in our discussion of convergence below, we always mean in the sense of subsequential convergence, even if we do not explicitly pass to subsequences.

We rewrite $g_j(t)=dy^2+u_{1j}\,g_{S^p}+u_{2j}\,g_{S^p}$, where

$$\begin{aligned} u_{\alpha j}(y,t) &= \lambda_j\,u_\alpha(x_j+y\lambda_j^{-1},a_*+t\lambda_j^{-1})\\ &=(-t)+\lambda_j v_\alpha(x_j+y\lambda_j^{-1},a_*+t\lambda_j^{-1}).\end{aligned}$$

Because our manifold is $\mathcal{M}=\mathbb{R}\times\mathcal{S}^p\times\mathcal{S}^p$ with $p\ge 2$, the metrics g_j are κ-noncollapsed. Hence by (50), a pointed sequence of Ricci flow solutions $(\mathcal{M},g_j(t),x_j)$ smoothly converges in the Cheeger–Gromov sense to an ancient Ricci flow solution $(\mathcal{M}_\infty,g_\infty(t),o)$ that exists for $t\in(-\infty,0)$.

26. **Claim.** *There exist smooth limits $u_{\alpha\infty}(y,t)$ of $u_{\alpha j}(y,t)$ as $j\to\infty$.*

Thus the limit of the convergent subsequence $(\mathcal{M},g_j(t),x_j)$ is $\mathcal{M}_\infty=\mathbb{R}\times\mathcal{S}^p\times\mathcal{S}^p$ with the metric

$$g_\infty(y,t)=dy^2+u_{1\infty}(y,t)g_{\mathcal{S}^p}+u_{2\infty}(y,t)\,g_{\mathcal{S}^p}.$$

Proof. Recall that by Theorem 6, we have

$$c_*\,\delta_\alpha(x_j+y\lambda_j^{-1},0)\le v_\alpha(x_j+y\lambda_j^{-1},a_*+t\lambda_j^{-1})\le C_*\delta_\alpha(x_j+y\lambda_j^{-1},0)$$

for uniform constants c_*,C_*. Putting $y=0=o$ as above, this yields

$$c_*\lambda_j\delta_\alpha(x_j,0)\le v_{\alpha j}(0,t)\le C_*\lambda_j\delta_\alpha(x_j,0).$$

Because $\lim_{j\to\infty}\frac{\delta_\alpha(x_j)}{a_*-t_j}=c_\alpha\in(0,\infty)$ and $0<\limsup_{j\to\infty}\lambda_j(T-t_j)<\infty$, we immediately get

$$c_0\le v_{\alpha j}(0,t)\le C_0 \tag{51}$$

for all $t\in(-a_*\lambda_j,0)$, for uniform constants $0<c_0<C_0<\infty$.

On the other hand, by Theorem 6, we also have

$$|\nabla v_\alpha(x,t)|^2\le C_0\,G\big(v_\alpha(x,t)\big)\qquad\text{for all}\qquad t\in[0,a_*),$$

where C_0 is another uniform constant. This in particular implies that

$$|\nabla\log u_\alpha(x,t)|^2\le|\nabla\log v_\alpha(x,t)|^2\le C_0\qquad\text{on}\qquad \mathcal{M}\times[0,a_*). \tag{52}$$

Estimates (51) and (52) imply that $v_{\alpha j}(y,t)$ converges uniformly to $v_{\alpha\infty}(y,t)$ on compact sets of $\mathcal{M}\times[-a_*\lambda_j,0)$ as $j\to\infty$, in a $C^{0,\mu}$ norm, for some $\mu\in(0,1)$. This together with smooth Cheeger–Gromov convergence implies the claim. □

27. **Claim.** *For $\alpha\in\{1,2\}$ and every $t\in(-\infty,0)$, both $u_{\alpha\infty}(y,t)$ are constant in space.*

Proof. We fix any $t\in(-\infty,0)$, and let $t_j=a_*+t\lambda_j^{-1}$. Then we observe that estimate (52) scales as follows:

$$\lambda_j^2|\nabla_{g_j(t_j)}\log u_{\alpha j}|^2\le C_0.$$

Taking $j \to \infty$ and using the smooth convergence of the metrics proves that $\log u_{\alpha\infty}$ is constant in space. □

To finish the proof of the first part of Theorem 1, we need to show that the limit $(\mathcal{M}, g_\infty(\cdot,t), o)$ cannot be a gradient shrinking Ricci soliton if

$$\lim_{|x|\to\infty} \frac{\delta_1(x)}{\delta_2(x)} = \eta \notin \{0,1\}$$

and if the spacetime sequence (x_j, t_j) is such that the constants $\lim_{j\to\infty} \frac{\delta_\alpha(x_j)}{a_*-t_j} = c_\alpha$ defined in (49) satisfy $c_1 = c_2\eta$ with $\eta \in \mathbb{R}_+ \backslash \{0,1\}$. Using Proposition 22 and (47), we have

$$\begin{aligned}
\frac{u_{1j}(0, -(a_*-t_j)\lambda_j)}{u_{2j}(0, -(a_*-t_j)\lambda_j)} &= \frac{u_1(x_j,t_j)}{u_2(x_j,t_j)} \\
&= \frac{a_* - t_j + \big(1 + o(1, \delta_1(x_j) \searrow 0)\big)\,\delta_1(x_j)}{a_* - t_j + \big(1 + o(1, \delta_2(x_j) \searrow 0)\big)\,\delta_2(x_j)} \qquad (53)\\
&= \frac{1 + \big(1 + o(1, \delta_1(x_j) \searrow 0)\big)\,\frac{\delta_1(x_j)}{a_*-t_j}}{1 + \big(1 + o(1, \delta_2(x_j) \searrow 0)\big)\,\frac{\delta_2(x_j)}{a_*-t_j}}.
\end{aligned}$$

Recall that $\lim_{j\to\infty}(a_*-t_j)\lambda_j = -t_0 \in (0,\infty)$. We let $j \to \infty$ in (53) to obtain

$$\frac{u_{1\infty}(0,t_0)}{u_{2\infty}(0,t_0)} = \frac{1 + c_2\eta}{1 + c_2} \neq 1.$$

In particular, by Claim 27, we have

$$\frac{u_{1\infty}(y,t_0)}{u_{2\infty}(y,t_0)} = \frac{1 + c_2\eta}{1 + c_2} \neq 1 \qquad \text{for all} \qquad y \in \mathbb{R}. \tag{54}$$

28. **Claim.** *At no time $t \in (-\infty, 0)$ is $g_\infty(\cdot,t)$ a gradient shrinking Ricci soliton.*

Proof. Recall that a gradient shrinking soliton is a metric g that satisfies

$$-2\,\mathrm{Rc} + \mathcal{L}_X(g) = \lambda g,$$

where X is the gradient vector field of a potential function and $\lambda < 0$. Restricting the equation above to a spherical fiber, we find that any fiberwise components of $\mathcal{L}_X(g)$ must be multiples of g restricted to the fibers, hence must be constant. Therefore, in analyzing the soliton structure, we may assume that $X = f(y)\,\frac{\partial}{\partial y}$.

It is shown in [AK19] that a metric

$$g = (\mathrm{d}y)^2 + \varphi_1(y)^2 g_{\mathcal{S}^{p_1}} + \varphi_2(y)^2 g_{\mathcal{S}^{p_2}}$$

on a doubly-warped product $\mathbb{R} \times \mathcal{S}^{p_1} \times \mathcal{S}^{p_2}$ is a gradient shrinking soliton with vector field $X = f(y)\,\frac{\partial}{\partial y}$ if and only if the functions f, φ_1, φ_2 satisfy the ODE system

$$f_y = p_1 \frac{(\varphi_1)_{yy}}{\varphi_1} + p_2 \frac{(\varphi_2)_{yy}}{\varphi_2} - \lambda, \tag{55a}$$

$$\frac{(\varphi_1)_{yy}}{\varphi_1} = (p_1 - 1)\frac{1 - (\varphi_1)_y^2}{\varphi_1^2} - p_2 \frac{(\varphi_1)_y(\varphi_2)_y}{\varphi_1\varphi_2} + \frac{(\varphi_1)_y}{\varphi_1} f + \lambda, \tag{55b}$$

$$\frac{(\varphi_2)_{yy}}{\varphi_2} = (p_2 - 1)\frac{1 - (\varphi_2)_y^2}{\varphi_2^2} - p_1 \frac{(\varphi_1)_y(\varphi_2)_y}{\varphi_1\varphi_2} + \frac{(\varphi_2)_y}{\varphi_2} f + \lambda. \tag{55c}$$

The only solutions of this system with φ_1 and φ_2 constant in space are

$$f(y) = -\lambda y, \qquad \varphi_1^2 = \frac{p_1 - 1}{-\lambda}, \quad \text{and} \quad \varphi_2^2 = \frac{p_2 - 1}{-\lambda}.$$

So if $g_\infty = dy^2 + u_{1\infty} g_{S^p} + u_{2\infty} g_{S^p}$ were a gradient shrinking Ricci soliton at some $t \in (\infty, 0)$, then the constants $u_{1\infty}$ and $u_{2\infty}$ would have to be equal. But this contradicts (54). □

We now prove the remainder of Theorem 1 and Corollary 2 by obtaining necessary and sufficient conditions for a limit to be a gradient shrinking soliton.

If $\lambda_j \to \infty$ is such that $\lim_{j\to\infty} \lambda_j(a_* - t_j) = -t_0 > 0$, estimate (53) implies that

$$\frac{u_{1\infty}(0, t_0)}{u_{2\infty}(0, t_0)} = 1 \tag{56}$$

if and only if (x_j, t_j) is a sequence converging to (∞, a_*) such that

$$\lim_{j\to\infty} \frac{\delta_\alpha(x_j)}{a_* - t_j} = 0 \qquad \text{for} \qquad \alpha \in \{1, 2\}, \tag{57}$$

in contrast to (49).

If (57) holds, then Claim 27 implies that for every $t \in (-\infty, 0)$, we have $u_{1\infty}(y, t) = u_{2\infty}(y, t) = u(t)$, where u depends only on time. Thus $u_{1\infty} = u_{2\infty} = -\lambda(p-1)$ and $f(y) = -\lambda y$ satisfy the system (55), implying that the metric $g_\infty = dy^2 + u_{1\infty}\, g_{S^p} + u_{2\infty}\, g_{S^p}$ is a gradient shrinking Ricci soliton.

Finally, since we have the bounds $\rho + \sum_{\alpha=1}^{A} u_\alpha^{-1} \chi_\alpha + u_\alpha^{-2} \gamma_\alpha \le C_0$ for all $t \in [0, a_*)$, the curvature estimate (8) implies that

$$\sup_{\mathcal{M}} |\operatorname{Rm}[g(t)]| = \sup_{\mathcal{M}} \frac{\mu_\alpha}{2(p-1)u_\alpha} = \frac{\mu_\alpha}{2(p-1)(a_* - t)}. \tag{58}$$

Note that to obtain the last identity, we use Proposition 22 and the fact that $\lim_{|x|\to\infty} \delta_\alpha(x) = 0$. On the other hand, Proposition 22 also implies that

$$v_\alpha(x_j, t_j) = \Big(1 + o(1; \delta_\alpha(x_j) \searrow 0)\Big)\, \delta_\alpha(x_j).$$

Combining this with (8), we find that

$$|\operatorname{Rm}(x_j, t_j)| = \frac{(1 + o(1))}{a_* - t_j} \frac{\mu_\alpha}{2(p-1)\Big(1 + (1 + o(1; \delta_\alpha(x_j) \searrow 0)\frac{\delta_\alpha(x_j)}{a_* - t_j}\Big)} + O(1). \tag{59}$$

Thus (58) and (59) imply that

$$\lim_{j\to\infty} \frac{|\operatorname{Rm}(x_j, t_j)|}{\sup_{\mathcal{M}} |\operatorname{Rm}(\cdot, t_j)|} = 1$$

if and only if (57) holds. We have seen above that (57) is equivalent to (56), and that by Claim 27, (56) is equivalent to g_∞ being a gradient shrinking Ricci soliton. This concludes the proof of Theorem 1 and verifies Corollary 2. □

4.2. **Weak stability of generalized cylinders under Ricci flow.** Stability of generalized cylinders $\mathbb{R}^k \times \mathcal{S}^p$ under Ricci flow is a subtle question. Even though a round cylinder $\mathbb{R}^k \times \mathcal{S}^p$ is expected to be a stable singularity model in some sense, it is not immediately clear how to define this stability. One reason for this is the following example. Start with a cylindrical metric $g_{\mathrm{cyl}} = (\mathrm{d}x)^2 + g_{\mathcal{S}^p}$ on $\mathbb{R} \times \mathcal{S}^p$ with $p \geq 2$. Let T_0 denote the time at which the spherical fibers vanish. Now consider an ϵ-perturbation of the initial data g_{cyl}: an initial metric $g_\epsilon = (\mathrm{d}x)^2 + (1+\epsilon)\, g_{\mathcal{S}^p}$ with $|\epsilon| < 1$. Ricci flow originating from g_ϵ will also become singular but at a different singularity time. If we rescale the perturbed flow by $\frac{1}{T_0 - t}$, then the rescaled perturbed solution will encounter a singularity before T_0 if $\epsilon < 0$ or will become infinitely large as $t \nearrow T_0$ if $\epsilon > 0$. In other words, no matter how small a perturbation is, if we chose a cylinder of a different radius, it will not naturally converge after rescaling to the solution originating at g_{cyl}.

Now let g_{Eucl} denote the flat Euclidean metric on $\mathbb{R}^k$, and let $\mathcal{S}^p$ be a round sphere scaled so that $2\,\mathrm{Rc}_{g_{\mathcal{S}^p}} = g_{\mathcal{S}^p}$. We take as initial data $g_{\mathrm{cyl}}(0) = g_{\mathrm{Eucl}} + a_* g_{\mathcal{S}^p}$. Then $g_{\mathrm{cyl}}(t) = g_{\mathrm{Eucl}} + (a_* - t)\, g_{\mathcal{S}^p}$ is a generalized cylinder that solves Ricci flow up to time $a_* > 0$. Consider perturbed initial data

$$g(x,0) = g_{\mathrm{Eucl}} + u(x,0) g_{\mathcal{S}^p}, \tag{60}$$

where $u(x,0) = a_* + \delta(x)$, with $\delta(x) \in \mathcal{A}$ as defined in (45).

29. **Theorem.** *Choose δ from the set $\mathcal{A}$, and let $g(x,t)$ be a Ricci flow solution on $\mathbb{R}^k \times S^p$ with initial metric $g(x,0)$ given in (60). Then there exists a constant C_* depending only on a_* so that for all $x \in \mathbb{R}^k$ and all $t \in [0, a_*)$, one has*

$$\begin{aligned} \frac{1}{C_*}\,\delta(x) &= \frac{1}{C_*}\,|g(x,0) - g_{\mathrm{cyl}}(0)|_{g_{\mathrm{cyl}}(0)} \\ &\leq \sup_{t\in[0,a_*)} |g(x,t) - g_{\mathrm{cyl}}(t)|_{g_{\mathrm{cyl}}(0)} \\ &\leq C_* |g(x,0) - g_{\mathrm{cyl}}(0)|_{g_{\mathrm{cyl}}(0)} = C_*\,\delta(x), \end{aligned} \tag{61}$$

and

$$\begin{aligned} \sup_{t\in[0,a_*)} |g(x,t) - g_{\mathrm{cyl}}(t)|_{g_{\mathrm{cyl}}(0)} &= \Big(1 + o(1; \delta(x) \searrow 0)\Big)\, |g(x,0) - g_{\mathrm{cyl}}(0)|_{g_{\mathrm{cyl}}(0)} \\ &= \Big(1 + o(1; \delta(x) \searrow 0)\Big)\, \delta(x), \end{aligned} \tag{62}$$

which implies that the $g_{\mathrm{cyl}}(0)$-distance between a perturbed solution $g(x,t)$ and an evolving generalized cylinder $g_{\mathrm{cyl}}(t)$ approaches zero as $|x| \to \infty$, uniformly in time $t \in [0, a_*)$. Moreover, the flow $g(x,t)$ develops a Type-I singularity at spatial infinity as $t \nearrow a_*$.

30. **Remark.** *For $\delta \in \mathcal{A}$, we say a solution $g(x,t)$ on $\mathbb{R}^k \times \mathcal{S}^p$ stays in a $\delta(x)$-neighborhood of g_{cyl} if*

$$\sup_{t\in[0,a_*)} |g(x,t) - g_{\mathrm{cyl}}(t)|_{g_{\mathrm{cyl}}(0)} \leq \delta(x).$$

Theorem 29 implies that for admissible perturbations, the perturbed solution never leaves the $\delta(x)$-neighborhood of g_{cyl}. On the other hand, (61) implies that no matter how small $\delta(x) > 0$ may be, after performing a Type-I rescaling by $\frac{1}{a_* - t}$ of both flows $g(x,t)$ and $g_{\mathrm{cyl}}(t)$, the rescaled solutions $\tilde{g}$ and $\tilde{g}_{\mathrm{cyl}}$, respectively, have the property that $\tilde{g}(\cdot,\tau)$ does not converge to $\tilde{g}_{\mathrm{cyl}}(\tau)$ as $\tau \to \infty$, where $\tau = -\log(a_* - t)$. This *behavior is consistent with the example discussed in the opening paragraph of this subsection.*

Proof of Theorem 29. By Proposition 22, if $g(x,t) = a_* - t + v(x,t)$, then there exists a constant $C_*(a_*)$ such that $v(x,t) \le C_*\,\delta(x)$ and

$$v(x,t) = \Big(1 + o(1; \delta(x) \searrow 0)\Big)\,\delta(x)$$

for all $t \in [0, a_*)$. This implies (61) and (62). This further implies that the distance between the perturbed solution $g(x,t)$ and an evolving cylinder $g_{\mathrm{cyl}}(t)$ approaches zero as $|x| \to \infty$, uniformly in time $t \in [0, a_*)$.

Arguments exactly like those that prove Theorem 1 establish that the perturbed solution $g(x,t)$ has the same singular time a_* as the generalized cylinder $g_{\mathrm{cyl}}(t)$; the singularity is Type-I; and it occurs at spatial infinity. □

Appendix A. Curvatures of multiply-warped products

We begin by recalling classical formulas for the curvatures[10] of a simple warped product $\mathcal{B} \times \mathcal{F}$. Let $(\mathcal{B}, \check{g})$ and $(\mathcal{F}, \hat{g})$ be complete Riemannian manifolds. In this Appendix, unlike the rest of this paper, we do not assume that $\mathcal{F}$ is a space form. Let $u\colon \mathcal{B} \to \mathbb{R}_+$ be a smooth warping function. To facilitate working in local coordinates, we denote our warped product metric on $\mathcal{B} \times \mathcal{F}$ by $g = \check{g} + u\hat{g}$.

We begin by working in local coordinates, using lowercase Roman indices (*e.g.,* i, j, k, ℓ) on the base $\mathcal{B}$, lowercase Greek indices (*e.g.,* $\sigma, \tau, \nu, \omega$) on the fiber $\mathcal{F}$, and allowing capital Roman letters to range over both sets. We denote the Christoffel symbols of g by

$$\Gamma_{IJ}^{K} = \frac{1}{2} g^{KL}(\partial_I g_{JL} + \partial_J g_{IL} - \partial_L g_{IJ}),$$

and those of $\check{g}$ and $\hat{g}$ by $\check{\Gamma}_{ij}^{k}$ and $\hat{\Gamma}_{\sigma\tau}^{\nu}$, respectively. We follow the same convention for other geometric quantities, including curvatures. We order the Christoffel symbols by the number of vertical (Greek) indices that appear (in order: $0, 1, 2, 3$) and calculate that

$$\Gamma_{ij}^{k} = \check{\Gamma}_{ij}^{k}, \tag{63a}$$

$$\Gamma_{\sigma j}^{k} = \Gamma_{i\tau}^{k} = \Gamma_{ij}^{\nu} = 0, \tag{63b}$$

$$\Gamma_{\sigma\tau}^{k} = -\frac{1}{2}\,\check{g}^{k\ell} u^{-1}\,\partial_\ell u\,(u\hat{g}_{\sigma\tau}), \tag{63c}$$

$$\Gamma_{i\tau}^{\nu} = \frac{1}{2}\,u^{-1}\,\partial_i u\,\delta_\tau^\nu, \tag{63d}$$

$$\Gamma_{\sigma j}^{\nu} = \frac{1}{2}\,u^{-1}\,\partial_j u\,\delta_\sigma^\nu, \tag{63e}$$

$$\Gamma_{\sigma\tau}^{\nu} = \hat{\Gamma}_{\sigma\tau}^{\nu}. \tag{63f}$$

[10] Throughout this paper, we follow the curvature conventions detailed in Sections 5–6 of [CK04]. Briefly, $R(X,Y)Z = \nabla^2 Z(X,Y) - \nabla^2 Z(Y,X)$ for the $(3,1)$-tensor, and we lower the raised index into the fourth position so that, say, $R_{1221} > 0$ on the round 2-sphere.

Given a function $f : \mathcal{B} \to \mathbb{R}$, there is a natural function $\tilde{f} : \mathcal{B} \times_u \mathcal{F} \to \mathbb{R}$ defined by $\tilde{f}(x, y) = f(x)$. We wish to compare the covariant Hessian of $\tilde{f}$ with respect to g with that of f with respect to $\check{g}$.

31. **Claim.** *If f and $\tilde{f}$ are as above, then*

$$\nabla\nabla\tilde{f} = \check{\nabla}\nabla f + \left\langle \nabla(\log u^{1/2}), \nabla f \right\rangle (u g_{\mathcal{F}}).$$

Proof. This is a straightforward application of (63). We can write

$$\begin{aligned}\nabla_I \nabla_J \tilde{f} &= \check{\nabla}_I \nabla_J f + (\nabla - \check{\nabla})_I \nabla_J f \\ &= \nabla_i \nabla_j f + g_{JK} (\nabla - \check{\nabla})_{IL}^K \nabla^L f.\end{aligned}$$

Since ∇f is horizontal, the only quantity from (63) that appears in the last term above is (63e), which proves the claim. □

We now compute the curvatures of g at the origin of a coordinate system that is normal for $\check{g}$ and $\hat{g}$, but not necessarily so for g. That is to say, we may assume that $\check{\Gamma}_{ij}^k = 0$ and $\hat{\Gamma}_{\sigma\tau}^\nu = 0$ at the origin, hence that $\partial_i \check{g}_{jk} = 0$ and $\partial_\sigma \hat{g}_{\tau\nu} = 0$ there, but we must use the full formula

$$R_{IJKL} = g_{LP} \left(\partial_I \Gamma_{JK}^P - \partial_J \Gamma_{IK}^P + \Gamma_{IQ}^P \Gamma_{JK}^Q - \Gamma_{JQ}^P \Gamma_{IK}^Q \right)$$

to calculate the $(4, 0)$-Riemann curvature tensor of g. Again ordering formulas by the number of vertical indices that appear (in order: $0, 1, 2, 3, 4$), we compute that

$$R_{ijk\ell} = \check{R}_{ijk\ell},$$

$$R_{\sigma jk\ell} = R_{i\tau k\ell} = R_{ij\nu\ell} = R_{ijk\omega} = 0,$$

$$\begin{aligned}R_{\sigma\tau k\ell} &= 0, \\ R_{i\tau\nu\ell} &= g_{p\ell}\big(\partial_i \Gamma_{\tau\nu}^p - \Gamma_{\tau\omega}^p \Gamma_{i\nu}^\omega\big) \\ &= (u\hat{g}_{\tau\nu})\Big(-\frac{1}{2} u^{-1} \nabla_i \nabla_\ell u + \frac{1}{4} u^{-2} \nabla_i u \nabla_\ell u \Big),\end{aligned}$$

$$R_{\sigma\tau\nu\ell} = 0,$$

$$\begin{aligned}R_{\sigma\tau\nu\omega} &= g_{\omega\lambda}\big(\hat{R}_{\sigma\tau\nu}^\lambda + \Gamma_{\sigma m}^\lambda \Gamma_{\tau\nu}^m - \Gamma_{\tau m}^\lambda \Gamma_{\sigma\nu}^m\big) \\ &= u\hat{R}_{\sigma\tau\nu\omega} - \frac{1}{4} u^{-2} |\nabla u|^2 \Big((u\hat{g}_{\sigma\omega})(u\hat{g}_{\tau\nu}) - (u\hat{g}_{\tau\omega})(u\hat{g}_{\sigma\nu}) \Big).\end{aligned}$$

For use below, we note that the curvature operator vanishes if a horizontal plane is paired with a plane spanned by two vertical vectors, as follows easily from the observations

$$0 = R_{\sigma\tau k}^\ell \, g_{j\ell} = R_{\sigma\tau kj} = R_{kj\sigma\tau} \qquad \text{and} \qquad 0 = R_{kj\sigma\tau} \, g^{\tau\nu} = R_{kj\sigma}^\nu. \tag{64}$$

There is a more concise way to write these formulas. Recall that the Kulkarni–Nomizu product of symmetric $(2, 0)$-tensors Φ, Ψ is given by

$$(\Phi \owedge \Psi)_{IJKL} := \Phi_{IL} \Psi_{JK} + \Phi_{JK} \Psi_{IL} - \Phi_{IK} \Psi_{JL} - \Phi_{JL} \Psi_{IK}. \tag{65}$$

With this normalization, the $(4,0)$-curvature tensor Rm of a metric g of constant sectional curvature κ is given by $\mathrm{Rm} = \frac{1}{2}\kappa\, g \owedge g$. Noting that

$$u^{-1/2}\check{\nabla}\nabla(u^{1/2}) = \frac{1}{2}u^{-1}\check{\nabla}\nabla u - \frac{1}{4}u^{-2}\nabla u \otimes \nabla u$$

and using the identity $u^{-2}|\nabla u|^2 = 4|\nabla(\log u^{1/2})|^2$, one sees that the curvature formulas above are equivalent to

$$\mathrm{Rm} = \check{\mathrm{Rm}} + u\hat{\mathrm{Rm}} - \frac{1}{2}\left|\nabla(\log u^{1/2})\right|^2(u\hat{g}) \owedge (u\hat{g}) - 2\,u\hat{g} \owedge \left(u^{-1/2}\check{\nabla}\nabla u^{1/2}\right). \tag{66}$$

We now analyze the curvatures of multiply-warped products of the form (2) on a manifold $\mathcal{M} = \mathcal{B} \times \mathcal{F}_1 \times \cdots \times \mathcal{F}_A$. As above, given a function $f : \mathcal{B} \to \mathbb{R}$, there is a natural function $\tilde{f} : \mathcal{M} \to \mathbb{R}$ defined by $\tilde{f}(x, y_1, \ldots, y_A) = f(x)$.

32. **Claim.** *If f and $\tilde{f}$ are as above, then*

$$\nabla\nabla\tilde{f} = \check{\nabla}\nabla f + \sum_{\alpha=1}^{A}\left\langle \nabla(\log u_\alpha^{1/2}), \nabla f\right\rangle (u_\alpha g_{\mathcal{F}_\alpha}).$$

Proof. This follows by induction on the number of fibers in the multiply-warped product, using Claim 31 as the base case. In the induction step, we regard the multiply-warped product with A fibers as a singly-warped product over a base that is a multiply-warped product with $A-1$ fibers. □

Our next result provides the curvature formulas we need for this paper. We show below that it also leads directly to estimate (8). In stating it, we write formula (67) in terms of $u_\alpha^2\,\mathrm{Rm}[g_{\mathcal{F}_\alpha}]$ and $u_\alpha g_{\mathcal{F}_\alpha}$ because, for fixed $g_{\mathcal{F}_\alpha}$, these have constant norms with respect to g if we vary u_α.

33. **Lemma.** *The $(4,0)$-tensor Rm of the metric (2) on the multiply-warped product $\mathcal{M}$ is given by*

$$\mathrm{Rm}[g] = \mathrm{Rm}[g_{\mathcal{B}}] + \sum_{\alpha=1}^{A} u_\alpha^{-1}\left(u_\alpha^2\,\mathrm{Rm}[g_{\mathcal{F}_\alpha}]\right) \tag{67a}$$

$$- \frac{1}{2}\sum_{\alpha=1}^{A}|\nabla(\log u_\alpha^{1/2})|^2\left(u_\alpha g_{\mathcal{F}_\alpha} \owedge u_\alpha g_{\mathcal{F}_\alpha}\right) \tag{67b}$$

$$- \sum_{\alpha=1}^{A}\sum_{\beta=1}^{\alpha-1}\left\langle \nabla(\log u_\alpha^{1/2}), \nabla(\log u_\beta^{1/2})\right\rangle\left(u_\alpha g_{\mathcal{F}_\alpha} \owedge u_\beta g_{\mathcal{F}_\beta}\right) \tag{67c}$$

$$- 2\sum_{\alpha=1}^{A} u_\alpha\, g_{\mathcal{F}_\alpha} \owedge \left(u_\alpha^{-1/2}\nabla_{g_{\mathcal{B}}}\nabla(u_\alpha^{1/2})\right). \tag{67d}$$

Proof. This follows by an induction argument similar to that in Claim 32. The induction hypothesis is that the claim holds for a metric with $A-1$ fibers, which we denote by $g_{(A-1)} := g_{\mathcal{B}} + \sum_{\alpha=1}^{A-1} u_\alpha g_{\mathcal{F}_\alpha}$. We denote the curvature and connection of $g_{(A-1)}$ by $\mathrm{Rm}_{(A-1)}$ and $\nabla_{(A-1)}$.

We may apply formula (66) for the curvature of a singly warped product to write the curvature of $g = g_{(A-1)} + u_A g_{\mathcal{F}_A}$ in terms of $\mathrm{Rm}_{(A-1)}$, obtaining

$$\begin{aligned}\mathrm{Rm}[g] = \mathrm{Rm}_{(A-1)} &+ u_A \,\mathrm{Rm}[g_{\mathcal{F}_A}] \\ &- \frac{1}{2}\big|\nabla(\log u_A^{1/2})\big|^2 (u_A g_{\mathcal{F}_A}) \owedge (u_A g_{\mathcal{F}_A}) \\ &- 2\, u_A g_{\mathcal{F}_A} \owedge \big(u_A^{-1/2}\nabla_{(A-1)}\nabla u_A^{1/2}\big).\end{aligned}$$

Using Claim 32, we rewrite the Hessian term in the last line above as

$$u_A^{-1/2}\nabla_{(A-1)}\nabla u_A^{1/2} = u_A^{-1/2}\nabla_{g_{\mathcal{B}}}\nabla u_A^{1/2} + \sum_{\beta=1}^{A-1}\big\langle \nabla \log u_\beta^{1/2}, \nabla \log u_A^{1/2}\big\rangle (u_\beta g_{\mathcal{F}_\beta}).$$

This completes the induction step.

In summary, this induction argument shows that adding an additional fiber to a multiply-warped product adds an additional term to each (outer) sum in (67). □

34. **Remark.** *It follows easily from Lemma 33 that there exists a universal constant C depending only on the dimensions such that*

$$|\,\mathrm{Rm}\,|_g \le |\,\mathrm{Rm}[g_{\mathcal{B}}]|_{g_{\mathcal{B}}} + C\sum_{\alpha=1}^{A}\Big(u_\alpha^{-1}|u_\alpha^2\,\mathrm{Rm}[g_{\mathcal{F}_\alpha}]|_g + u_\alpha^{-2}|\nabla v_\alpha|_g^2 + u_\alpha^{-1}|\check{\nabla}\nabla v_\alpha|_{g_{\mathcal{B}}}\Big).$$

Furthermore, one sees readily that

$$\left|\mathrm{Rm}[g] - \sum_{\alpha=1}^{A} u_\alpha\,\mathrm{Rm}[g_{\mathcal{F}_\alpha}]\right|_g \le C\left\{\rho^{1/2} + \sum_{\alpha=1}^{A}\Big(u_\alpha^{-2}\gamma_\alpha + u_\alpha^{-1}\chi_\alpha^{1/2}\Big)\right\},$$

where $\rho, \gamma_\alpha, \chi_\alpha$ are defined in (7). This is estimate (8).

To conclude, we calculate the components of the Ricci tensor. We obtain

$$\begin{aligned}R_{ij} &= g^{JK}R_{iJKj} \\ &= \check{R}_{ij} - \sum_{\alpha=1}^{A} n_\alpha\left(\frac{1}{2}u_\alpha^{-1}\nabla_i\nabla_j u_\alpha - \frac{1}{4}u_\alpha^{-2}\nabla_i u_\alpha \nabla_j u_\alpha\right),\end{aligned} \tag{68}$$

and on each fiber $\mathcal{F}_\alpha$,

$$\begin{aligned}R_{\tau\nu} &= R^i_{i\tau\nu} + R^\sigma_{\sigma\tau\nu} \\ &= (\hat{R}_\alpha)_{\tau\nu} - \left(\frac{1}{2}\Delta_{\mathcal{B}}u_\alpha - \frac{1}{2}u_\alpha^{-1}|\nabla u_\alpha|^2 + \frac{1}{2}\sum_{\beta=1}^{A} n_\beta\left\langle \nabla u_\alpha, \nabla \log u_\beta^{1/2}\right\rangle\right)(\hat{g}_\alpha)_{\tau\nu}.\end{aligned}$$

In the last formula, the Laplacian on the RHS is computed with respect to the metric $g_{\mathcal{B}}$ on the base. To match the convention used elsewhere in this paper, we rewrite the expression in terms of the Laplacian $\Delta \equiv \Delta_{\mathcal{M}}$ computed with respect to the metric g on the total space $\mathcal{M}$. Using (4), we obtain

$$R_{\tau\nu} = (\hat{R}_\alpha)_{\tau\nu} - \frac{1}{2}\Big(\Delta u_\alpha - u_\alpha^{-1}|\nabla u_\alpha|^2\Big)(\hat{g}_\alpha)_{\tau\nu}. \tag{69}$$

Formulas (68) and (69) directly imply the system (3) of evolution equations that results if one evolves the metric g on the total space by Ricci flow.

Appendix B. Laplacians of tensor seminorms

For use in Appendix C below, we here compute and estimate the Laplacians of various tensor seminorms. We continue the conventions of Appendix A, using lowercase Roman indices (*e.g.,* i, j, k, ℓ) for horizontal vectors, lowercase Greek indices (*e.g.,* $\sigma, \tau, \nu, \omega$) for vertical vectors, and allowing capital Roman letters to range over both sets of indices. We continue denoting $g_{\mathcal{B}}$ by $\check{g}$ when working in local coordinates.

Before treating Laplacians of seminorms, we establish some preliminary results for first derivatives of tensor fields.

35. **Claim.** *If T is an $(m, 0)$-tensor field such that $T(U_1, U_2, \dots, U_m)$ vanishes if exactly one U_k is vertical, then*

$$\nabla T\Big|_{T\mathcal{M}\otimes(T\mathcal{B})^m} = \nabla_{g_{\mathcal{B}}}\Big(T\Big|_{(T\mathcal{B})^m}\Big).$$

Proof. In the proof, we denote horizontal vector fields by $H_1, H_2, \dots$ and vertical vector fields by V, V'. For simplicity, we illustrate the idea of the proof with $m = 3$. The generalization to arbitrary m is clear. The key fact is that the only components of the connection in (63) that differ from those of a direct (*i.e.,* non-warped) product are those that exchange horizontal and vertical vectors. Specifically, we have

$$\nabla T(V, H_1, H_2, H_3) = -V^\sigma \Gamma_{\sigma\ell}^{\tau}\big(H_1^\ell H_2^j H_3^k T_{\tau jk} + H_1^i H_2^\ell H_3^k T_{i\tau k} + H_1^i H_2^j H_3^\ell T_{ij\tau}\big) = 0.$$

Hence $\nabla T(U, H_1, H_2, H_3)$ can be nonzero only if $U = H_4$ is horizontal.

We note that the assumption that H_1, H_2, H_3 are horizontal is necessary: indeed, similar reasoning shows that terms like $\nabla T(V, H_1, H_2, V')$ are nonzero in general. □

36. **Claim.** *If T is a symmetric $(2, 0)$-tensor field with no nonzero horizontal-vertical components, then all components of ∇T for a warped product are the same as those for a direct product (*i.e., *a metric with u constant) except*

$$\nabla_i T_{\sigma\tau} = -u^{-1}\nabla_i u\, T_{\sigma\tau},$$
$$\nabla_\sigma T_{i\tau} = \nabla_\sigma T_{\tau i} = \frac{1}{2}u^{-1}\nabla^j u\, T_{ji}\, g_{\sigma\tau} - \frac{1}{2}u^{-1}\nabla_i u\, T_{\sigma\tau},$$

which do not vanish in general.

Proof. Direct computation using (63). □

We note for use below that Claim 36 implies easily that all components of $\nabla g_{\mathcal{B}}$ vanish identically except

$$\nabla_\sigma \check{g}_{i\tau} = \nabla_\sigma \check{g}_{\tau i} = \frac{1}{2}u^{-1}\nabla_i u\, g_{\sigma\tau}. \tag{70}$$

For clarity, before deriving an estimate for multiply-warped products, we first perform an exact calculation for a singly-warped product. We continue to assume that T is a symmetric $(2, 0)$-tensor field with no nonzero horizontal-vertical components. Then we have

$$\begin{aligned}\nabla_Q |T|^2_{g_{\mathcal{B}}} &= 2(\nabla_Q \check{g}^{IK})\check{g}^{JL} T_{IJ} T_{KL} + 2\check{g}^{IK}\check{g}^{JL}(\nabla_Q T_{IJ})T_{KL}\\ &= 2\check{g}^{IK}\check{g}^{JL}(\nabla_Q T_{IJ})T_{KL},\end{aligned}\tag{71}$$

because
$$(\nabla_P \check{g}^{IK})\check{g}^{JL}T_{IJ}T_{KL} = \nabla_\sigma \check{g}^{i\tau}\check{g}^{j\ell}T_{ij}T_{\tau\ell} = 0$$
by assumption. Thus we obtain
$$\begin{aligned}\Delta|T|^2_{g_{\mathcal{B}}} &= 2g^{PQ}\nabla_P\big\{\check{g}^{IK}\check{g}^{JL}(\nabla_Q T_{IJ})T_{KL}\big\}\\ &= 2g^{PQ}\check{g}^{IK}\check{g}^{JL}(\nabla_P\nabla_Q T_{IJ})T_{KL} + 2g^{PQ}\check{g}^{IK}\check{g}^{JL}(\nabla_P T_{IJ})(\nabla_Q T_{KL})\\ &\quad + 4g^{PQ}(\nabla_P\check{g}^{IK})\check{g}^{JL}(\nabla_Q T_{IJ})T_{KL}.\end{aligned}$$
Writing this invariantly, we have
$$\Delta|T|^2_{g_{\mathcal{B}}} = 2\langle\Delta T, T\rangle_{g_{\mathcal{B}}} + 2|\nabla T|^2_{g_{\mathcal{B}}} + 4\mathcal{Z}[T],$$
where
$$\begin{aligned}\mathcal{Z}[T] &:= g^{PQ}(\nabla_P\check{g}^{IK})\check{g}^{JL}(\nabla_Q T_{IJ})T_{KL}\\ &= g^{\sigma Q}(\nabla_\sigma\check{g}^{i\tau})\check{g}^{j\ell}(\nabla_Q T_{IJ})T_{\tau\ell} + g^{\sigma Q}(\nabla_\sigma\check{g}^{\nu k})\check{g}^{j\ell}(\nabla_Q T_{\nu j})T_{k\ell}\\ &= g^{\sigma Q}(\nabla_\sigma\check{g}^{\nu k})\check{g}^{j\ell}(\nabla_Q T_{\nu j})T_{k\ell}\\ &= -\frac{1}{2}u^{-1}\check{g}^{j\ell}(\nabla^k u T_{k\ell})(\nabla^\tau T_{\tau j}).\end{aligned}$$
Note that we use (70) in the final step. We expand the divergence factor, obtaining
$$\nabla^\tau T_{\tau j} = \frac{\dim(\mathcal{F})}{2}u^{-1}\nabla^i u T_{ij} - \frac{1}{2}u^{-1}\nabla_j u(\hat{\operatorname{tr}}T),$$
where $\hat{\operatorname{tr}}T := g^{\sigma\tau}T_{\sigma\tau}$ denotes the trace of the vertical components of T. Combining factors, we write $\mathcal{Z}[T]$ invariantly as
$$\mathcal{Z}[T] = \frac{1}{4}u^{-2}(\hat{\operatorname{tr}}T)\langle T, \nabla u\otimes\nabla u\rangle_{g_{\mathcal{B}}} - \frac{\dim(\mathcal{F})}{4}u^{-2}|T(\nabla u)|^2_{g_{\mathcal{B}}},$$
where in the second term, we regard T as an endomorphism of the tangent bundle. This work proves:

37. **Lemma.** *If T is a symmetric $(2,0)$-tensor field with no nonzero horizontal-vertical components on a warped product, then*
$$\begin{aligned}-\Delta|T|^2_{g_{\mathcal{B}}} &= -2\langle\Delta T, T\rangle_{g_{\mathcal{B}}} - 2|\nabla T|^2_{g_{\mathcal{B}}}\\ &\quad + \dim(\mathcal{F})\,u^{-2}|T(\nabla u)|^2_{g_{\mathcal{B}}} - u^{-2}(\hat{\operatorname{tr}}T)\langle T, \nabla u\otimes\nabla u\rangle_{g_{\mathcal{B}}}.\end{aligned}$$

Generalizing this to the multiply-warped products we study in this paper, one readily obtains:

38. **Corollary.** *If T is a symmetric $(2,0)$-tensor field with no nonzero horizontal-vertical components, then there exists a constant C depending only on the dimension vector $\vec{N} = (n, n_\alpha)$ such that*
$$-\Delta|T|^2_{g_{\mathcal{B}}} \le -2\langle\Delta T, T\rangle_{g_{\mathcal{B}}} - 2|\nabla T|^2_{g_{\mathcal{B}}} + C\Big(\sum_{\alpha=1}^{A}|\nabla\log u_\alpha|^2\Big)|T||T|_{g_{\mathcal{B}}}.$$

We now proceed to estimate $-\Delta|\operatorname{Rm}|^2_{g_{\mathcal{B}}}$ on a multiply-warped product. Because the details are so similar to the previous case, we merely sketch the proof. First, Claim 35 shows that $\nabla\operatorname{Rm}$ vanishes if exactly one index is vertical. Thus we see by (70) that
$$\nabla_S|\operatorname{Rm}|^2_{g_{\mathcal{B}}} = 2\check{g}^{IW}\check{g}^{JX}\check{g}^{KY}\check{g}^{LZ}(\nabla_S R_{IJKL})R_{WXYZ},$$

exactly as in (71). Thus we find that

$$\Delta|\operatorname{Rm}|^2_{g_{\mathcal{B}}} = 2\langle\Delta\operatorname{Rm},\operatorname{Rm}\rangle_g + 2|\nabla\operatorname{Rm}|^2_{g_{\mathcal{B}}} + 8\,\mathcal{Z}[\operatorname{Rm}],$$

where

$$\mathcal{Z}[\operatorname{Rm}] := g^{\nu\sigma}(\nabla_\nu \check{g}^{\tau w})\check{g}^{jx}\check{g}^{ky}\check{g}^{\ell z}(\nabla_\sigma R_{\tau jk\ell})R_{wxyz}. \tag{72}$$

39. **Claim.** *The* $(5,0)$*-tensor field* $\nabla\operatorname{Rm}$ *satisfies*

$$\nabla_\sigma R_{\tau jk\ell} = -\Gamma^i_{\sigma\tau}R_{ijk\ell} + \Gamma^\nu_{\sigma k}R_{\tau j\ell\nu} - \Gamma^\nu_{\sigma\ell}R_{\tau jk\nu}.$$

Proof. Using equations (63), (64), and the fact that $R_{\tau jk\ell}=0$, we compute that

$$\begin{aligned}\nabla_\sigma R_{\tau jk\ell} &= -\Gamma^I_{\sigma\tau}R_{Ijk\ell} - \Gamma^I_{\sigma j}R_{\tau Ik\ell} - \Gamma^I_{\sigma k}R_{\tau jI\ell} - \Gamma^I_{\sigma\ell}R_{\tau jkI}\\ &= -\Gamma^i_{\sigma\tau}R_{ijk\ell} - \Gamma^\nu_{\sigma j}R_{\tau\nu k\ell} - \Gamma^\nu_{\sigma k}R_{\tau j\nu\ell} - \Gamma^\nu_{\sigma\ell}R_{\tau jkv}\\ &= -\Gamma^i_{\sigma\tau}R_{ijk\ell} + \Gamma^\nu_{\sigma k}R_{\tau j\ell\nu} - \Gamma^\nu_{\sigma\ell}R_{\tau jk\nu}.\end{aligned}$$

□

We denote by $\mathcal{H}$ the (integrable) horizontal distribution of $\mathcal{M}$ and by $\operatorname{Rm}_{\mathcal{H}\otimes\mathcal{H}}$ the restriction

$$\operatorname{Rm}_{\mathcal{H}\otimes\mathcal{H}} := \operatorname{Rm}\big|_{\mathcal{H}\otimes T\mathcal{M}\otimes T\mathcal{M}\otimes\mathcal{H}},$$

i.e., only those components of Rm having the form $R_{iJK\ell}$. Then equation (70), equation (72), and Claim 39 immediately imply the following:

40. **Corollary.** *There exists a constant* C *depending only on the dimension vector* $\vec{N}=(n,n_\alpha)$ *such that*

$$\begin{aligned}-\Delta|\operatorname{Rm}|^2_{g_{\mathcal{B}}} &\le -2\langle\Delta\operatorname{Rm},\operatorname{Rm}\rangle_{g_{\mathcal{B}}} - 2|\nabla\operatorname{Rm}|^2_{g_{\mathcal{B}}}\\ &\qquad + C\Big(\sum_{\alpha=1}^{A}|\nabla\log u_\alpha|^2\Big)|\operatorname{Rm}|_{g_{\mathcal{B}}}|\operatorname{Rm}_{\mathcal{H}\otimes\mathcal{H}}|_g.\end{aligned}$$

Appendix C. Curvature evolution equations and estimates

We continue the convention of Appendix A, using lowercase Roman indices (*e.g.,* i,j,k,ℓ) for horizontal vectors, lowercase Greek indices (*e.g.,* σ,τ,ν,ω) for vertical vectors, and allowing capital Roman letters to range over both sets of indices. We assume that the metric g is evolving by the Ricci flow system (3).

C.1. **The evolution of** ρ**.** Under Ricci flow, the $(4,0)$-Riemann curvature tensor evolves by (see, *e.g.,* Corollary 6.14 of [CK04])

$$\begin{aligned}\big(\partial_t-\Delta\big)R_{IJKL} &= g^{PQ}\big(R^M_{IJP}R_{MQKL} - 2R^M_{PIK}R_{JQML} + 2R_{PIML}R^M_{JQK}\big)\\ &\quad - (R^P_I R_{PJKL} + R^P_J R_{IPKL} + R^P_K R_{IJPL} + R^P_L R_{IJKP}).\end{aligned}$$

For simplicity, we again begin with an exact calculation for a singly-warped product and generalize this below to an estimate for multiply-warped products. We start by computing the evolution of the curvature tensor acting on horizontal vectors, finding that

$$\big(\partial_t-\Delta\big)R_{ijk\ell} = g^{ab}(R^c_{ija}R_{cbk\ell} - 2R^c_{aik}R_{jbc\ell} + 2R_{aic\ell}R^c_{jbk}) \tag{73a}$$

$$\qquad + g^{\sigma\tau}(R^\gamma_{ij\sigma}R_{\gamma\tau kl} - 2g_{\ell m}R^\gamma_{\sigma ik}R^m_{j\tau\gamma} + 2R_{\sigma i\gamma\ell}R^\gamma_{j\tau k}) \tag{73b}$$

$$\qquad - (R^P_i R_{Pjk\ell} + R^P_j R_{iPk\ell} + R^P_k R_{ijP\ell} + R^P_\ell R_{ijkP}). \tag{73c}$$

We note that (73a) consists of the terms one would see if the base alone were evolving by Ricci flow, while (73c) consists of terms that are cancelled by derivatives of g^{-1} in our calculation of the evolution of $\rho = |\operatorname{Rm}|^2_{g_{\mathcal{B}}}$ below. So we need only to examine the three additional terms in (73b).

By (64), the first term in (73b) vanishes. To evaluate the second and third terms in (73b), we can apply the formulas derived in Appendix A directly, obtaining

$$g_{\ell m} R^{\gamma}_{\sigma i k} R^{m}_{j\tau\gamma} = \hat{g}_{\sigma\tau}\Big(\frac{1}{4}u^{-1}\nabla_i\nabla_k u\nabla_j\nabla_\ell u - \frac{1}{8}u^{-2}\nabla_i\nabla_k u\nabla_j u\nabla_\ell u - \frac{1}{8}u^{-2}\nabla_j\nabla_\ell u\nabla_i u\nabla_k u + \frac{1}{16}u^{-3}\nabla_i u\nabla_j u\nabla_k u\nabla_\ell u\Big)$$

and

$$\begin{aligned} R_{\sigma i\gamma\ell} R^{\gamma}_{j\tau k} &= g_{\gamma\zeta} R^{\zeta}_{\sigma i\ell} R^{\gamma}_{\tau jk} \\ &= \hat{g}_{\sigma\tau}\Big(\frac{1}{4}u^{-1}\nabla_i\nabla_\ell u\nabla_j\nabla_k u - \frac{1}{8}u^{-2}\nabla_i\nabla_\ell u\nabla_j u\nabla_k u \\ &\qquad - \frac{1}{8}u^{-2}\nabla_j\nabla_k u\nabla_i u\nabla_\ell u + \frac{1}{16}u^{-3}\nabla_i u\nabla_j u\nabla_k u\nabla_\ell u\Big). \end{aligned}$$

Combining terms and tracing by $g^{\sigma\tau}$, we conclude that

$$\begin{aligned} \big(\partial_t - \Delta\big) R_{ijk\ell} &= g^{ab}(R^c_{ija}R_{cbk\ell} - 2R^c_{aik}R_{jbc\ell} + 2R_{aic\ell}R^c_{jbk}) \\ &\quad + \frac{\dim(\mathcal{F})}{2}\Big\{u^{-2}\big(\nabla_i\nabla_\ell u\nabla_j\nabla_k u - \nabla_i\nabla_k u\nabla_j\nabla_\ell u\big) \\ &\qquad + \frac{1}{2}u^{-3}\big(\nabla_i\nabla_k u\nabla_j u\nabla_\ell u + \nabla_j\nabla_\ell u\nabla_i u\nabla_k u \\ &\qquad\qquad - \nabla_j\nabla_k u\nabla_i u\nabla_\ell u - \nabla_i\nabla_\ell u\nabla_j u\nabla_k u\big)\Big\} \\ &\quad - (R^P_i R_{Pjk\ell} + R^P_j R_{iPk\ell} + R^P_k R_{ijP\ell} + R^P_\ell R_{ijkP}). \end{aligned} \tag{74}$$

We now estimate the evolution of $\rho(x,t) = \big|\operatorname{Rm}(x,t)\big|^2_{g_{\mathcal{B}}}$ for a multiply-warped product. We note that in the case of a multiply-warped product, the only possible nonzero terms in (73b) occur where the vertical coordinates σ and τ are tangent to the same fiber. Thus we obtain a sum of derivatives of u_α in (74), and using our estimate derived in Corollary 40, we recover the standard estimate for the evolution of the curvature norm (see, *e.g.,* Lemma 7.4 of [CK04]) modified by additional terms coming from the warped-product structure, namely

$$\begin{aligned} \big(\partial_t - \Delta\big)\rho &\le -2|\nabla \operatorname{Rm}|^2_{g_{\mathcal{B}}} + C_n\rho^{3/2} \\ &\quad + 2\sum_{\alpha=1}^{A} n_\alpha\Big\{u_\alpha^{-2}\operatorname{Rm}_{\mathcal{B}}(\nabla^2 v_\alpha, \nabla^2 v_\alpha) \\ &\qquad - 2u_\alpha^{-3}\operatorname{Rm}_{\mathcal{B}}(\nabla^2 v_\alpha, \nabla v_\alpha\otimes\nabla v_\alpha)\Big\} \\ &\quad + C\Big(\sum_{\alpha=1}^{A}|\nabla\log u_\alpha|^2\Big)|\operatorname{Rm}|_{g_{\mathcal{B}}}|\operatorname{Rm}_{\mathcal{H}\otimes\mathcal{H}}|_g, \end{aligned} \tag{75}$$

where $n_\alpha = \dim(\mathcal{F}_\alpha)$, $\operatorname{Rm}_{\mathcal{B}}$ denotes the curvature tensor of $g_{\mathcal{B}}$, and C is a constant depending only on the dimension vector $\vec{N} = (n, n_\alpha)$.

C.2. **The evolution of γ_α.** We next consider the evolution of the curvature tensor acting on vertical vectors in an arbitrary fiber $\mathcal{F}_\alpha$. It follows from (67a) and (67b) that for a multiply-warped product with space-form fibers, it suffices to calculate the evolution of $\gamma_\alpha = |\nabla u_\alpha|^2 = |\nabla v_\alpha|^2$.

As elsewhere in this Appendix, we omit the fiber index for convenience in the computations below. We note that $(\partial_t - \Delta)u$ is given by (3b). It also follows from (3b) that

$$\partial_t \gamma = 2\big\{\langle \nabla \Delta v, \nabla v\rangle - u^{-1}\langle \nabla\gamma, \nabla v\rangle + u^{-2}\gamma^2\big\} + 2\,\mathrm{Rc}(\nabla v, \nabla v),$$

where Rc denotes the Ricci tensor of g acting on horizontal vectors, as in (68). Recalling that $\Delta\gamma = 2\langle \Delta\nabla v, \nabla v\rangle + 2|\nabla\nabla v|^2$, we commute covariant derivatives and conclude that

$$(\partial_t - \Delta)\gamma = -2|\nabla\nabla v|^2 - 2u^{-1}\langle\nabla\gamma, \nabla v\rangle + 2u^{-2}\gamma^2. \tag{76}$$

Observing that $\langle\nabla\gamma, \nabla v\rangle = 2\nabla^2 v(\nabla v, \nabla v)$, we obtain the formula used in Lemma 12.

C.3. **The evolution of χ_α.** We move on to controlling $\chi_\alpha = |\nabla\nabla v_\alpha|^2_{g_{\mathcal{B}}}$. By Remark 34, this is the last quantity needed to control the full curvature tensor. For simplicity, we again fix a fiber and omit subscripts.

We denote the heat operator with the Lichnerowicz Laplacian of the metric g by $(\partial_t - \Delta)_{\mathcal{L}}$. Using the standard formula (see, *e.g.*, Lemma 2.33 of [CLN06])

$$(\partial_t - \Delta)_{\mathcal{L}} \nabla_I \nabla_J v = \nabla_I\nabla_J(\partial_t - \Delta)v,$$

we compute this heat operator acting on the covariant Hessian of v as follows:

$$\begin{aligned}\Big((\partial_t - \Delta)_{\mathcal{L}}(\nabla^2 v)\Big)_{IJ} &= u^{-2}(\nabla_I\nabla_J v)\,\gamma - 2u^{-3}(\nabla_I v\nabla_J v)\,\gamma\\ &\quad + u^{-2}(\nabla_I v\nabla_J\gamma + \nabla_I\gamma\nabla_J v) - u^{-1}\nabla_I\nabla_J\gamma.\end{aligned}$$

Now using the identity $-\Delta = -\Delta_{\mathcal{L}} + 2\,\mathrm{Rm}\,* - 2\,\mathrm{Rc}\,*$, where Rm and Rc are those of the metric g, we convert this formula to one using the standard heat operator:

$$\begin{aligned}\Big((\partial_t - \Delta)\nabla^2 v\Big)_{ij} &= u^{-2}(\nabla_i\nabla_j v)\gamma - 2u^{-3}(\nabla_i v\nabla_j v)\gamma\\ &\quad + u^{-2}\big(\nabla_i v\nabla_j\gamma + \nabla_i\gamma\nabla_j v\big) - u^{-1}\nabla_i\nabla_j\gamma\\ &\quad + 2R_{ik\ell j}\nabla^k\nabla^\ell v - R_i^k\nabla_k\nabla_j v - R_j^k\nabla_i\nabla_k v\\ &\quad + Nu^{-2}\gamma\big(-\frac{1}{2}\nabla_i\nabla_j v + \frac{1}{4}\nabla_i v\nabla_j v\big),\end{aligned}$$

where $N := \sum_{\beta=1}^A \dim\mathcal{F}_\beta$ is the total dimension of the fibers. We obtain the last line above by simplifying $2R_{i\sigma\tau j}\nabla^\sigma\nabla^\tau v$ using the identities

$$R_{i\sigma\tau j} = u^{-1}g_{\sigma\tau}\big(-\frac{1}{2}\nabla_i\nabla_j v + \frac{1}{4}u^{-1}\nabla_i v\nabla_j v\big) \quad \text{and} \quad \nabla^\sigma\nabla^\tau v = \frac{1}{2}u^{-1}\gamma g^{\sigma\tau}.$$

Finally, we apply Corollary 38 to conclude that

$$\begin{aligned}(\partial_t - \Delta)\chi &\le -2|\nabla^3 v|^2_{g_{\mathcal{B}}} + 4\,\mathrm{Rm}_{\mathcal{B}}(\nabla^2 v, \nabla^2 v) + 2u^{-2}\gamma\chi\\ &\quad - 2u^{-3}\langle\nabla v, \nabla\gamma\rangle\gamma + 4u^{-2}\langle\nabla^2 v, \nabla v\otimes\nabla\gamma\rangle\\ &\quad - 2u^{-1}\langle\nabla^2 v, \nabla^2\gamma\rangle_{g_{\mathcal{B}}} + Nu^{-2}\gamma\big\{-\chi + \frac{1}{4}u^{-1}\langle\nabla\ v, \nabla\gamma\rangle\big\}\\ &\quad + C\Big(\sum_{\alpha=1}^A|\nabla\log u_\alpha|^2\Big)|\nabla^2 v||\nabla^2 v|_{g_{\mathcal{B}}},\end{aligned} \tag{77}$$

where $\mathrm{Rm}_{\mathcal{B}}$ again denotes the curvature tensor of $g_{\mathcal{B}}$, and C is a constant depending only on the dimension vector $\vec{N} = (n, n_\alpha)$.

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