Exam II

1)(30 points) Define \( a_n \) as below. Use \( \epsilon \), \( K \) to prove: \( a_n \) converges.

\[
a_n = \frac{n^2 - n + 1}{3n^2 - 4n - 8}
\]

2)(30) Prove or give a counterexample: Assume there exists an \( \epsilon > 0 \) and there exists a \( K \) such that \( n \geq K \) implies \( |a_n - a| < \epsilon \). Then \( a_n \to a \).

3)(40 points) Let \( S \) be a non-empty set of real numbers bounded above. Let \( T = \{upper \ bound \ of \ S\} \). Prove \( \sup(S) = \inf(T) \).
Sample Exam Two Solutions

1) We expect this to be like \( \frac{n^2}{3n^2} \).

So the \( \frac{1}{3} \) is the limit. Let's look

\[
\frac{n^2 - n + 8}{3n^2 - 4n - 8} = \frac{(3n^2 - 3n + 3) - (3n^2 - 4n - 8)}{3(3n^2 - 4n - 8)} = \frac{n + 5}{3(3n^2 - 4n - 8)}
\]

Now to get \( n \) of absolute values:

\( (n + 5) = n + 5 \) so we're OK there.

Getting \( \frac{3n^2 - 4n - 8}{2n} \) is trickier, here's one way:

we'll write it as \( 3n^2 - 4n \geq 8 \) the factors

\( n(3n - 4) \geq 8 \). If each factor is bigger than \( 8 \)

the product will be. So \( n \geq 8 \), \( 3n - 4 \geq 8 \) or \( 3n \geq 12 \)

or \( n \geq 4 \). So \( n \geq 8 \) it is - throw that into OK

So now divide \( \frac{n + 5}{3(3n^2 - 4n - 8)} \) in

To make it easier use \( \frac{1}{2} \leq 1 \) and \( n + 5 \leq n \).

So I want \( \frac{n}{3n^2 - 4n - 8} \leq \frac{1}{n} \) \( \Rightarrow \)

\( n \leq 3n^2 - 4n - 8 \) \( \Rightarrow \)

\( 0 \leq 2n^2 - 4n - 8 \) or \( 0 \leq n^2 - 2n - 4 \)
Ok, same task: make \( n^2 - 2n \geq 4 \)
or \( n(n-2) \geq 4 \). But see

\[ n \geq 4 \text{ or } n-2 \geq 4 \text{ so } n \geq 6 \]

But we already have \( n \geq 8 \).

Alright, \( 1/n < 3 \) if \( n > 1/3 \).

So - now for the proof: given \( \varepsilon > 0 \). Choose \( k = \sup \{ 1/\varepsilon, 8 \} \). Then for \( n \geq k \)

\[
\left| \frac{n^2 - n + 1}{3n^2 - 4n - 8} - \frac{1}{3} \right| = \left| \frac{n+5}{3(3n^2 - 4n - 8)} \right|
\]

\[ \leq \frac{n}{3n^2 - 4n - 8} \leq \frac{1}{n} \quad \text{since } n \geq 8 \]

\[ \leq \frac{1}{k} \quad \text{since } n \geq k \geq 8 \]

\[ \leq \frac{1/n}{\varepsilon} = 3 \quad \text{since } n \geq k > \frac{1}{\varepsilon} \]

v)

What's different from the usual definition of limit? Instead of "for all \( \varepsilon > 0 \) we have "\( \exists k \geq 0 \)"

Now \( \varepsilon \) controls accuracy — how close

On gets to \( a \). If I have "for all \( \varepsilon \)"

I can make \( a \) very close to \( a \) by

choosing \( \varepsilon \) very small.
Using "\( \exists x \)" - well, 0 doesn't have to be close to a at all.

So what we want is a sequence \( a_n \), and an \( a \) such that \( a_n \) does not converge to \( a \).

But for which there exists \( \varepsilon \) etc, etc,

OK - let's take \( a_n = \frac{1}{n} \), \( a = 1 \), \( a_n \) does not \( \to 1 \).

"exists" \( \varepsilon \) - \( \varepsilon \) can be chosen, \( 0' \) use \( \varepsilon = \frac{1}{2} \).

Then I have to show there is an \( a_k \) such that \( |a_k - a| < \varepsilon \).

So I've got "exists \( x \)". and "exists \( \varepsilon \)."

And by 0 have to show \( n \geq k \Rightarrow |a_n - a| < \varepsilon \).

or \( n \geq k \Rightarrow \left| \frac{1}{n} - 1 \right| < \varepsilon \).

But \( \left| \frac{1}{n} - 1 \right| \leq \frac{1}{|n|} + |1| = |1| + 1 = 2 \)

and yes, \( 2 < 3 \).

3) We have a result: If \( t \in T \) and \( t \) is a lower bound of \( T \), then \( t = \inf T \).

So I'll show

a) \( \inf T \)

b) \( \sup S \) is a lower bound of \( T \)

Then it follows \( \sup S = \inf T \)

a) \( \sup S \) is upper bound of \( S \) - def. So \( \sup S \leq \sup S = \inf T \)

b) If \( t \in T \), \( t \) is an upper bound of \( S \), so \( \sup S \leq t \).
Since \( \sup S \) is the least upper bound of \( S \), so \( \sup S \leq \sup T \).