GROUP WORK We'll have GW on Wednesday April 5. It will cover limits and inequalities, and monotone sequences.

QUIZ The usual Friday Quiz; covering the same material as the group work.

BOOK PROBLEMS Section 3.3 p77 1, 2, 3, 7, 8

PRACTICE

1) Assume \( x_1 \) satisfies \( 0 < x_1 < 1 \) and define \( x_{n+1} = 0.5 x_n (1 - x_n) \). Prove:
   
   i) For all \( n \), \( 0 < x_n < 1 \)
   
   ii) \( x_n \) converges.
   
   iii) Find the limit of \( x_n \)

2) Let \( x_1 < \frac{1}{2} \) and for \( n \geq 1 \) define
   
   \[ x_{n+1} = 2 x_n (1 - x_n) \]

   Prove that \( x_n \) converges and find the limit.

3) Assume \( a_1 > 0 \) Define
   
   \[ a_{n+1} = \frac{1}{2} \left( a_n + \frac{2}{a_n} \right) \]

   Is \( a_n \) monotone?
Book Section 3.3

1) \( x_1 = 8; x_{n+1} = \frac{1}{2} x_n + 2 \).

Intuition: \( x_2 = \frac{1}{2} 8 + 2 = 4 + 2 = 6 \) so I suspect
\( x_n \) is monotonically decreasing. To prove convergence,
I need to show it is bounded below.

First step: reduce monotonicity to boundedness
\( x_{n+1} \leq x_n \) off \( \frac{1}{2} x_n + 2 \leq x_n \)
off \( 2 \leq \frac{1}{2} x_n \) off \( 4 \leq x_n \).

Second step: prove bounded below by induction.
\( P(1) \Rightarrow x_1 \geq 4 \).
Check \( P(1) \): \( x_1 \geq 4 \Rightarrow 8 \geq 4 \) true.
Check \( P(n) \Rightarrow P(n+1) \)
Assume \( x_n \geq 4 \) (induction hypothesis \( P(n) \))
then \( \frac{1}{2} x_n + 2 \geq \frac{1}{2} \cdot 2 + 2 = 2 + 2 \) close source
\( x_{n+1} \geq 4 \) def \( x_n \).

So: \( x_n \geq 4 \) for all \( n \), so that \( x_n \) is monotonically decreasing, since the \( x_n \) are bounded below.

Then \( x_n \) converges to some \( x \).
But \( x_n \) also converges to \( x \). So
\( \lim_{n \to \infty} x_{n+1} = x \) and \( \lim_{n \to \infty} \frac{1}{2} x_n + 2 = \frac{1}{2} x + 2 \).
\( x = \frac{1}{2} x + 2 \) or \( \frac{1}{2} x = 2 \) or \( x = 2 \).
Step # 2

3) \( x_1 \geq 2 \) and \( x_{n+1} = 1 + \sqrt{x_n - 1} \).

usual check for monotonicity: Let \( x_1 = 5 \),
\( x_2 = 1 + \sqrt{4} = 1 + 2 = 3 \). So I suspect
\( x_n \) is monotone decreasing.

First step \( x_{n+1} \leq x_n \) iff
\( 1 + \sqrt{x_n - 1} \leq x_n \) iff \( \sqrt{x_n - 1} \leq x_n - 1 \).

We'll show below that \( x_n \geq 1 \) so \( x_n - 1 \geq 0 \), so
we can square both sides to get \( x_n - 1 \leq x_n \) iff
\( x_n - 1 \leq (x_n - 1)^2 \) iff \( 1 \leq x_n - 1 \) iff \( 2 \leq x_n \).

Second step: use induction to prove \( x_n \geq 2 \)

(remark: this also proves \( x_n \geq 1 \))

\( P(n) \) : \( x_n \geq 2 \).

Check \( P(1) \) : \( x_1 \geq 2 \) this was given. Nothing to prove.
Check \( P(n) \implies P(n+1) \):
\( x_n \geq 2 \) \text{ conduces hypothesis }
\( 1 + \sqrt{x_n - 1} \geq 1 + \sqrt{2 - 1} = 1 + 1 = 2 \) algebra
\( x_{n+1} \geq 2 \) recursion for \( x_{n+1} \).

So \( x_n \geq 2 \) is so the \( x_n \) are monotone decreasing
and bounded below. Therefor convergent to some \( x \).

Note = \( x_n \geq 2 \) so, class result, \( x \geq 2 \).

So \( 1 + \sqrt{x_n - 1} \to 1 + \sqrt{x - 1} \). But \( x_{n+1} \to x \) so
\( x = 1 + \sqrt{x - 1} \) or \( x - 1 = \sqrt{x - 1} \) or \((x - 1)^2 = x - 1 \)
Since \( x \geq 2 \) I can divide by \( x - 1 \neq 0 \),
or \( x - 1 = 1 \) and \( x = 2 \).
7) \( x_1 > a > 0 \) and \( x_{n+1} = x_n + \frac{1}{x_n} \).

First try: \( x_1 = 1 \) then \( x_2 = 2 \) so suspect \( x_n \) is increasing and bounded above.

\[ x_{n+1} = x_n + \frac{1}{x_n} \Rightarrow x_{n+1} - x_n = \frac{1}{x_n} > 0 \Rightarrow x_{n+1} > x_n \]

So the first thing we'd need to do is show for all \( n \),

\( x_n > 0 \). I'll do this by induction.

\( P(1) : x_1 > 0 \)

\( P(n) \Rightarrow x_n > 0 \) since \( x_1 > 0 \).

\( P(n) \Rightarrow P(n+1) : x_{n+1} = x_n + \frac{1}{x_n} > 0 \) since \( x_n > 0 \) by induction hypothesis.

So \( x_{n+1} > 0 \).

So \( x_n \) are monotone increasing.

If the \( x_n \) are bounded above, they converge to some \( x \). Now the \( x_n \) are monotone increasing so

\( x_n \geq x \geq a \geq 0 \) so \( \lim_{n \to \infty} x_n \geq x > 0 \) so \( x > 0 \).

Therefore by sum and quotient limits laws,

\( x_n + \frac{1}{x_n} \to x + \frac{1}{x} \).

But \( x_n \to x \) so \( x = x + \frac{1}{x} \) or \( 0 = \frac{1}{x} \).

This can't happen for \( x > 0 \).

Therefore then \( x_n \) cannot converge,

so we also know they cannot be bounded.

Sneaky, huh?
8). Since \( a_n \leq b_n \) for all \( n \), and \( b_n \) is decreasing, for all \( n \), \( b_n \leq b_1 \). Then \( a_n \leq b_\frac{a_1}{b_1} \), so \( a_n \) is increasing and bounded above; so \( a_n \) converges.

Since \( a_n \) is increasing, \( a_1 \leq a_n \leq b_n \), so the \( b_n \) are decreasing and bounded below. So the \( b_n \) converge. Then \( b_n-a_n \) converges by limit laws but \( 0 \leq b_n-a_n \) so by class result,

\[
0 \leq \lim (b_n-a_n) = \lim b_n - \lim a_n
\]

so \( \lim a_n \leq \lim b_n \).

Practice Problems

1) \( 0 < x_1 < 1 \) and \( x_{n+1} = \frac{1}{2} x_n (1-x_n) \).

2) \( 0 \leq x_n < 1 \) by induction.

\( P(n) : \ 0 < x_n < 1 \).

Check \( P(1) : \ 0 < x_1 < 1 \) given.

Assume \( P(k) : \ 0 < x_k < 1 \)

\[
-0 > -x_k > -1
\]

\[
1 - 0 > 1 - x_k > 1 - 1
\]

\[
1 > 1 - x_k > 0
\]

so \( 0 < x_k < 1 \) and \( 0 < 1-x_k < 1 \)

so \( 0 < x_{k+1} (1-x_{k+1}) < 1 \)

so \( 0 < \frac{1}{2} x_k (1-x_k) < 1 \)

so \( 0 < x_{n+1} < 1 \)
ii) \( x_n \) converges if it is monotone.

Which direction? If \( x_1 = \frac{1}{3}, x_2 = \frac{1}{2}(\frac{1}{3} + \frac{2}{3}) = \frac{1}{2} \), so \( x_2 \neq 0 \).

So I suspect \( x_n \) is decreasing.

\[
x_n \leq x_{n+1} \implies \frac{1}{2} x_n + (1-x_n) \leq x_{n+1}
\]

\[
\implies \frac{1}{2} (1-x_n) \leq x_{n+1} \quad \text{since} \quad x_n \neq 0
\]

\[
\implies 1-x_n \leq 2
\]

\[
\implies -x_n \leq 1
\]

\[
\implies x_n \geq -1 \quad \text{and} \quad x_n > 0.
\]

So \( x_n \) is decreasing bounded below 0 by \( y \geq 0 \), so

\( x_n \to x \). As usual,

\[ x = \frac{1}{2} x (1-x) \quad \text{so} \quad 2x = x - x^2 \quad \text{so} \]

\[ x = -x^2 \quad \text{so} \quad x^2 + x = 0 \quad \text{so} \quad x(x+1) = 0. \]

Since \( x_n > 0 \), \( \lim x_n \geq 0 \), so

\[ x > 0 \quad \text{so} \quad x(x+1) = 0 \quad \text{so} \quad x = 0 \quad \text{or} \quad x = -1. \]

2) Shown.

3) Show \( x_1 = 1, x_2 = \frac{1}{2} (1 + \frac{2}{3}) = \frac{1}{2} (\frac{5}{3}) = 2/3 = 1.4 \).

So \( x_2 > x_1 \), and \( x_3 = \frac{1}{2} (1 + \frac{2}{3}) = \frac{1}{2} (1 + \frac{2}{3}) \)

\[ = \frac{7}{6} = 1 + \frac{1}{6} < 1 + \frac{1}{2} \quad \text{so} \quad x_3 < x_2. \]

So \( x_n \) is not monotone.