One For You Standard Induction

A) We’ll show that the sum of odd numbers is a perfect square. Try it:

\[ 1 = 1^2 \]
\[ 1 + 3 = 4 = 2^2 \]
\[ 1 + 3 + 5 = 9 = 3^2 \]

To prove this, we’ll need to relate the left and right sides. Here’s how:

\[ 1 = (2 \cdot 1 - 1) = 1^2 \]
\[ 1 + 3 = 1 + (2 \cdot 2 - 1) = 2^2 \]
\[ 1 + 3 + 5 = 1 + 3 + (2 \cdot 3 - 1) = 3^2 \]
\[ 1 + 3 + 5 + 7 = 1 + 3 + 5 + (2 \cdot 4 - 1) = 4^2 \]

Use these ideas to state a result, then use induction to prove it.

B) Factorials grow very quickly compared to squares:

\[ 1! = 1 \text{ vs } 1^2 = 1 \]
\[ 2! = 1! \cdot 2 = 2 \text{ vs } 2^2 = 4 \]
\[ 3! = 2! \cdot 3 = 6 \text{ vs } 3^2 = 9 \]
\[ 4! = 3! \cdot 4 = 24 \text{ vs } 4^2 = 16 \]
\[ 5! = 4! \cdot 5 = 120 \text{ vs } 5^2 = 25 \]

Well, eventually they do: show that if \( n \geq 4 \), \( n! > n^2 \)

C)

\[ \frac{1}{2} = \frac{1}{2} \]
\[ \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3} \]
\[ \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{4} \]
\[ \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{4}{5} = \frac{1}{5} \]

State and prove the general result.
One for you: basic induction solutions

1) This one is very tricky, because before we can even start, we need to formulate what we want to prove, and set up the algebra to prove it.

a) The problem helps: we have statements like

\[ 1 + 3 + 5 + 7 + \cdots = 1 + 3 + 5 + (2 \cdot 4 - 1) = 4^2 \]

\[ \uparrow \quad \uparrow \]

So, in words, the sum of the first \( n \) natural odd numbers is \( n^2 \). But we need algebra! And the sum of the first \( n \) is not algebraic.

Call "the sum of the first \( n \) odd numbers" \( S_n \).

How do I get from \( S_n \) to \( S_{n+1} \)? Let's try it:

\[ S_3 = 1 + 3 + 5 = 1 + 3 + 2(2) - 1 \]

\[ S_4 = 1 + 3 + 5 + 7 = 1 + 3 + 5 + (2 \cdot 4 - 1) \]

So \[ S_4 = S_3 + (2 \cdot 4 - 1) \]

\[ S_{n+1} = S_n + 2(n+1) - 1 \] or

\[ S_{n+1} = S_n + (2n+1) \rightarrow \text{the recursive step.} \]

Now it's easy. Let \( P(n) \) be the proposition

\[ P(n) \Rightarrow S_n = n^2 \]

i) Check \( P(1) \): \( S_1 = 1^2 \) or \( 1 = 1 \checkmark \)

ii) Check \( P(k) \) implies \( P(k+1) \).

\[ S_k = k^2 \quad P(k) \]

\[ S_k + 2(k+1) - 1 = k^2 + 2k + 1 = (k+1)^2 \quad \text{algebra} \]

\[ S_{k+1} = (k+1)^2 \quad \text{reversal} \checkmark \]

So, by induction, \( P(n) \) is true for all \( n \).
2) Here the set-up is different: we need to start with 4, not 1. So the induction axiom we'll use is from the setup of \( P(n) \)
and \( P(k) \rightarrow P(k+n) \) for \( k \geq 4 \).
Then for all \( n \geq 4 \), \( P(n) \).
And for us, \( n = 4 \).

The recursions are a little unexpected:

\((n+1)! = (n+1)n! \quad ; \quad (n+1)^2 = n^2 + 2n + 1 \).

Let \( P(n) \Rightarrow n! \geq n^2 \)

1) Check \( P(4) \):

\( 4! \geq 4^2 \) \( \checkmark \)

2) Check \( P(n) \rightarrow P(k+n) \)

\( k! \geq k^2 \quad P(k) \)

\( (k+n)k! \geq k^2(k+n) \quad \text{Algebra} \)

\( k^2(k+n) \geq (k+n)^2 \quad \text{Algebra - shown below} \)

\( (k+n)! \geq (k+n)^2 \quad \text{Transitivity of} \geq \)

Algebra below: here we've gone just to check

Is \( k^2(k+n) \geq (k+n)^2 \) ? Well, divide \( k+n \),

It equivalent to asking if \( k^2 \geq k+n \). That's equivalent to asking if \( k^2 - k \geq 0 \), which is equivalent to asking if \( k(k-1) \geq 1 \). But

This is true: \( k \geq 4 \) so \( k-1 \geq 3 \) so

\( k(k-1) \geq 12 \geq 1 \) \( \checkmark \)

So by induction, \( k! \geq k^2 \) for all \( k \geq 4 \).
Here we have the problems similar to 1) -
we don't have a good way to define the recursion.
So let's start by calling the products by a number:

\[ S_1 = \frac{1}{2}; \quad S_2 = \frac{1 \cdot 2}{2 \cdot 3}; \quad S_3 = \frac{1 \cdot 2 \cdot 3}{3 \cdot 4} \]

Now try to relate \( S_3 \) to \( S_2 \) algebraically:

\[ S_3 = \frac{1 \cdot 2 \cdot 3}{3 \cdot 4} = \left(\frac{1 \cdot 2}{3 \cdot 4}\right) = S_2 \cdot \left(\frac{1}{4}\right) \]

\[ S_2 = S_2 \cdot \left(\frac{1}{4}\right) \quad \text{so if } \frac{n}{n} \text{ were 2, we'd have } \frac{n}{n} = S_2 \left(\frac{n}{n}\right), \text{ um...} \]

If 3 is \( n+1 \), 4 is \( n+2 \): \( S_{n+1} = S_n \left(\frac{n+1}{n+2}\right) \)

Now we can start: let \( P(n)\) be \( S_n = \frac{1}{n+1} \)

(i) Check \( P(1) \): \( S_1 = \frac{1}{1+1} \) or \( S_1 = \frac{1}{2} \) \( \checkmark \)

(ii) Check \( P(k) \rightarrow P(k+1) \)

\[ S_k = \frac{1}{k+1} \] \( \rightarrow \) \( P(k) \)

\[ \begin{align*}
\frac{k+1}{k+2} \cdot S_k &= \frac{1}{k+1} \cdot \frac{k+1}{k+2} \\
&= \frac{1}{k+2} \text{ closure}
\end{align*} \]

\[ S_{n+1} = \frac{1}{n+2} \text{ observe and recursion} \]

so for all \( n \), \( S_n = \frac{1}{n+1} \) \( \checkmark \)