One For You: Limit Laws

1) Give an $\varepsilon$, $K$ proof: If $a_n \to 0$, then $a_n^2 \to 0$.

2) Give an $\varepsilon$, $K$ proof: If $a_n \to a$ then $|a_n - a| \to 0$.

3) Assume for all $n$, $a_n \leq b$. Assume $a_n \to a$. Prove $a \leq b$.

4) Assume $a_n$ is bounded, and $a_n + b_n$ is unbounded. Prove $b_n$ is unbounded.

5) State whether the following are true or false, and prove if true, give a counter-example if false:
   a) Assume $a_n$ converges and $b_n \to 0$. Then $a_n/b_n$ is unbounded.
   b) If $a_n/b_n \to 0$ and $a_n$ is unbounded then $b_n$ is unbounded.
One for you. Limit Sews

1) Hack away at it:

\[ |c_n^2 - 0| < \varepsilon \] simplifies to \[ c_n^2 < \varepsilon \]

or \[ \sqrt{c_n^2} < \sqrt{\varepsilon} \]. Since \( c_n > 0 \), \[ \sqrt{c_n^2} = c_n \]
(otherwise, \( \sqrt{c_n^2} = |c_n| = c_n \))

So \( \sqrt{\varepsilon} \) is a. Since \( \varepsilon > 0 \), there is \( \alpha \) such that \( |c_n - 0| < \sqrt{\varepsilon} \). Since \( c_n \geq 0 \), thus \( c_n \leq \sqrt{\varepsilon} \). We can square since \( c_n \geq 0 \) to get \( c_n^2 \leq \varepsilon \). And then, going since \( c_n^2 \geq 0 \), \( c_n^2 - 0 \leq \varepsilon \).

2) Given \( \varepsilon > 0 \), there is \( K \) such that for \( n \geq K \), \( |a_n - a| < \varepsilon \). Then

\[ |a_n - a| - 0 = |a_n - a| < \varepsilon \] \( \varepsilon \) since \( |a_n - a| \geq 0 \).

3) Assume not. Here's the picture

\[ \frac{a - \varepsilon}{b} \]
all the \( a_n \) are here \( \geq K \)

all the \( a_n \) here \( \geq K \) - but 'here' is \( > \varepsilon \)!!

So \( n \geq K \) \( \Rightarrow \) \( a_n \geq \varepsilon \). But \( \forall n \leq a \leq b \)

contradiction.
So to do a proof we need to implement the picture - we need to assume $a > b$.
Then we need to pick our $\varepsilon$ to get

$$\left| \frac{1}{x} \right| < \frac{a - \varepsilon}{b}$$

that is, we need $b < a - \varepsilon$ or $\varepsilon < a - b$.

Then, we need to get our $c_n > b$.
This would come from $c_n > a - \varepsilon > b$.

Now for the formal proof.

Assume not. Then $a > b$. Let $\varepsilon = a - b > 0$.
Then there exists a $k$ such that

$$n \geq k \implies |a_n - a| < \varepsilon$$

so

$$a - \varepsilon < a_n < a + \varepsilon$$

$$a - \varepsilon < c_n,$$

since $\varepsilon = a - b$,

$$a - (a - b) < c_n$$

$$b < c_n.$$

This contradicts the hypothesis that $c_n \leq b$ for all $n$. 
4) Assume not.

Since $a_n$ is bounded so from $b_n$, $|a_n| \leq A$ for a fixed $A$.

Since we're assuming $b_n$ are bounded, there exist $a, B$ with $|b_n| \leq B$ for all $n$.

Then $|a_n + b_n| \leq (a_n + |b_n|) \leq A + B$.

So $a_n + b_n$ is bounded. Contradict our hypothesis that it's unsounded.

5a) False. Let $a_n = b_n = \sqrt{n}$.

Then $a_n$ converges as $a_n \to 0$.

But $a_n/b_n = 1$ which is bounded. By 4 it's

5b) True. Assume $b_n$ is bounded

for all $n$, $|b_n| \leq B$ for a fixed $B$.

Since $a_n/b_n \to 0$, $a_n/b_n$ is bounded.

Then $|a_n|/b_n \leq C$ for some fixed $C$.

Then $|a_n| = |a_n/b_n| |b_n| \leq C B$.

So $a_n$ is bounded. Thus contradicted the assumption $a_n$ is unsounded.