Quiz 2

1) Assume $0 < x_1 < 1$, $0 < x_2 < 1$, $0 < x_3 < 1$ and for $n \geq 3$, define

$$x_{n+1} = \frac{1}{5} (x_n + x_{n-1} + 3x_{n-2})$$

Prove that for all natural numbers $n$, $0 < x_n < 1$

Since the recursion involves three terms we'll use strong induction.

Let $P(n)$ be $0 < x_n < 1$.

1. Check $P(1), P(2), P(3)$

   This was given: $0 < x_j < 1$ for $j = 1, 2, 3$.

2. Check for $k \geq 2$, $P(1) \land P(2) \land \ldots \land P(k) \implies P(k+1)$.

   $0 < x_k < 1 \quad P(k)$

   $0 < x_{k-1} < 1 \quad P(k-1)$

   $0 < x_{k-2} < 1 \quad P(k-2)$

   So $0 < 2x_{k-2} < 3$ algebra

   Add these three in equations

   $0 + 0 + 0 < x_k + x_{k-1} + 3x_{k-2} < 1 + 1 + 3$

   And divide by 5

   $0 < \frac{1}{5} (x_k + x_{k-1} + 3x_{k-2}) < 1$

   Or

   $0 < x_{k+1} < 1 \quad \text{since } k+1 \geq 3$

By strong induction, $P(n)$ for all $n \in \mathbb{N}$. 