One For You: Real Numbers

1) Let $Z$ denote the integers $Z = \{0, \pm 1, \pm 2, \ldots\}$. Prove $Z$ is not bounded below.

2) Let $S = \{-\frac{1}{n} \mid n \text{ is a natural number}\}$. Find $\sup S$ and prove your answer.

3) Assume $S, T$ are non-empty sets of reals and for all $s \in S$ and for all $t \in T$, $s \leq t$. Prove that $\sup S \leq \inf T$. 
One for you too: Number solutions

1) Assume \( \mathbb{Z} \) is bounded below; we'll get a contradiction.
   We'll use two ideas:
   a) \( N \) are not bounded above
   b) negative numbers switch inequalities:
      \[ 2 < B \implies -B < -2. \]

Here goes: so there's a \( B \in \mathbb{Z} \) with \( j \in \mathbb{Z} \)
implies \( B \leq j \). Assume \( n \in \mathbb{N} \) and then \( -n \in \mathbb{Z} \), so
\( n \in \mathbb{N} \implies B < -n \), \( \rightarrow n \leq -B \), so \( -B \)
is an upper bound for \( \mathbb{N} \), contradicting Archimedean.

2) Again, we'll use the idea that negatives switch
   inequalities and we know that \( \text{cuf} \{ \frac{j}{n} | n \in \mathbb{N} \} = 0 \).
To prove \( \text{sup} S = 0 \) we need to show
   i) \( 0 \) is an upper bound of \( S \):
      \[ a \in S \implies a = \frac{j}{n} \text{ and } n \in \mathbb{N} \text{ but } n \in \mathbb{N} \implies n \geq 1 \]
      so \( -\leq -1 < 0 \). \( \frac{j}{n} < 0 \) and \( 0 \) is an upper bound.
   ii) If \( b \) is an upper bound of \( S \), \( 0 \leq b \),
      since \( n \in \mathbb{N} \implies -\frac{j}{n} \leq b \),
      \( n \in \mathbb{N} \implies \frac{j}{n} \geq 2 - b \). So \(-b \text{ is a lower bound for }
      \{ \frac{j}{n} | n \in \mathbb{N} \} \). But \( \text{inf} \{ \frac{j}{n} | n \in \mathbb{N} \} \text{ is the }
      greatest lower bound, so \(-b \leq \text{inf} \{ \frac{j}{n} | n \in \mathbb{N} \} \implies
      -b \leq 0 \implies 0 \leq b \).
here's a kind of picture

\[
( S ) \quad ( T )
\]

in terms of \( u^n, s^n, \quad ( s ) \quad ( t ) \)

\[ s^n \quad u^n \quad t^n \]

unfortunately, we can't pass directly \( s, t \) to their \( u^n \)'s and \( s^n \)'s, so here's an interim medley idea

\[
( S ) \quad ( T )
\]

\[ \Rightarrow s^n \quad s^n \rightarrow s^n \quad s^n \rightarrow s^n \quad s^n \rightarrow \quad \text{lower bounds of } T \\]
\[ \Rightarrow s^n \quad s^n \rightarrow s^n \quad s^n \rightarrow \text{lower bound for } T \]

Let's do it. Pick any \( a \in S, t \in T \), we need to show

\[ a \in s^n \rightarrow u^n \text{ is an upper bound for } S. \]
\[ u^n \text{ is an upper bound for } S \rightarrow a \in s^n \quad u^n \text{ is lower bound for } T. \]

Now \( s^n \) is a lower bound for \( T \).
\[ \text{lower bound for } T 
\Rightarrow \]
\[ a \in s^n \quad u^n \]
\[ \Rightarrow s^n \leq u^n \quad \text{there.} \]

For practice: go the other way: show

\[ a \in S \text{ is a lower bound for } T \quad \text{so} \]
\[ t \leq \text{infty} \quad \text{etc.} \]