I’ve spent over forty years (60% of my life) teaching math. There’s a basic assumption about studying, that seems to be let over from high school: give me lots of problems, and I’ll just work them. That gets me in the groove, then I can do the exam.
That falls apart early in second semester calculus. On methods of integration, infinite series, parameterizations: there are so many techniques, you need to know which technique to use, where.
I spent time as a consultant on a biomedical engineering study for the US Army (it isn’t classified. Classified stuff doesn’t even get mentioned). Now, the Army requires all members of a project to meet, face to face. I flew up to a secure facility in Iowa (why couldn’t it have been like Tokyo) and met with the doctors, engineers, and technicians. The first issue we all addressed was how to collect data on human subjects without electrocuting them. That was one for the electrical engineers. Then there was the question of how to store the data. Everyone turned to me: ‘how many data points do we need to collect per second’.
You know, I couldn’t answer ‘give me ten problems just like that and I’ll answer this one’. I was the expert: it was expected that I had integrated everything I knew about data and about physiology of the heart, and could apply it to new situations. I know no-one is paying you $125 an hour to be an expert on analysis. But you do face the same situation: there aren’t lots of problems. So it’s important to extract information from the problems that do exist. This means going back, looking at problems, theorems, ideas, and organizing those.
The final exam is asking you to do just that: Each person organizes information in their own way. What follows is just one of many methods

A) Definitions: these need to be exact. For example, in the group work on subsequences, many people noted, ”$c_{2k} = b_{2k}$’, which is fine; ’since $b_n \to c$, then $b_{2k} \to c$’ which is fine; ’therefore given $\epsilon > 0$ there exists a $K_b$ such that $2k \geq K_b$ implies $|b_{2k} - c| < \epsilon$’ which is not fine. The definition of limit would give

given $\epsilon > 0$ there exists a $K_b$ such that $k \geq K_b$ implies $|b_{2k} - c| < \epsilon$

So, from memory, give precise definitions of:

a) $a_n$ is bounded
b) $a_n$ converges
c) $b_j$ is a subsequence of $a_n$
d) $a_n$ is a Cauchy sequence
B) Sometimes it’s helpful to have non-definitions in mind. For example, on quiz 7, some people said, ’assume $a_n$ does not converge to zero. Then there’s an $a \neq 0$ with $a_n \to a$’. What’s wrong with this? So:

What does it mean to say $a_n$ diverges? What techniques do we have to show a sequence diverges? What are examples of using those techniques do we have?

C) It’s helpful to have an exact knowledge of results we’ve proved. What results are associated with bounded? Here’s one example: If $a_n \to 0$ and $b_n$ is bounded, then $a_nb_n \to 0$. This could have been used on several problems; for example in the group work involving $(-1)^na_n$. So: what results do you have on bounded sequences?

D) And, subsequences. What results are associated with subsequences? Here’s one example: If $b_j$ is a subsequence of $a_n$ and $c_k$ is a subsequence of $b_j$ is bounded, then $c_k$ is a subsequence of $a_n$. Look at the next page, on limit laws. For results that are false for sequences, are they true for subsequences?

E) What of Cauchy? What results do we know about Cauchy sequences? Are there results in the homework on Cauchy sequences? When is a sequence not Cauchy?

F) It’s useful to have a collection of non-examples in mind. On quiz 7, some people wrote ’since $a_n$ is a sequence, it has a convergent sequence’, even though sample quiz 7 had two problems showing this is false. So what are some examples of non-convergent sequences? Non-Cauchy sequences?

G) Techniques are important as well. What techniques do we have? The problem on quiz 7 gave information on $a_nb_n$ and asked to get information of $a_n$. Most people tried to use an epsilon over 2 proof; none thought that dividing by $b_n$ would isolate $a_n$ and therefore get information about $a_n$. Look at the next page, on limit laws. What sorts of techniques are useful there?

More generally, look at the epsilon over 2 proofs we’ve given. What kinds of problems are amenable to that technique?
True or False? Prove or give a counterexample:

1) If $x_n + y_n$ converges, then at least one of $x_n$, $y_n$ converges.
2) If $x_n$, $y_n$ are bounded then $x_n + y_n$ converges.
3) If $x_n + y_n$ converges and $x_n$ converges then $y_n$ converges.
4) If $x_n$, $y_n$ diverge then $x_n + y_n$ diverges.
5) If $x_n$ converges and $y_n$ diverges then $x_n + y_n$ diverges.
6) If $1/y_n$ diverges then $y_n \to 0$.
7) If $1/y_n \to 0$ then $y_n \to \infty$.
8) If $1/y_n$ diverges and $y_n$ is unbounded then $y_n \to 0$.
9) If $x_n y_n$ converges then at least one of $x_n$, $y_n$ converges.
10) If $x_n y_n$ converges and $y_n$ is unbounded then $x_n \to 0$.
11) If $x_n y_n$ converges and $x_n$ converges and $y_n$ is unbounded then $x_n \to 0$. 