2) (20 points) \( C \ f(x) = \frac{\left(x^4 - x - 1\right)}{\left(x^2 + x\right)} \).

a) Find a \( g \) to which \( f \) is asymptotic.

b) Show \( f \) is asymptotic to \( g \) by computing a limit.

\[
\begin{align*}
g &= x^2 - x + 1 \\
\end{align*}
\]

\[
\begin{align*}
\lim_{x \to \pm \infty} \ln f(x) &= \lim_{x \to \pm \infty} \frac{-2x - 1}{x^2 + x} \\
&= \lim_{x \to \pm \infty} \frac{-2 - \frac{1}{x}}{1 + \frac{x}{x^2}} \\
&= 0 + 0 = 0
\end{align*}
\]
4) (40 points) Let
\[ f(x) = x^3(x^2 - 1)^2 \]

a) Use the product and chain rules to differentiate
b) Simplify
c) Find all critical points

\[ f'(x) = 3x^2(x^2 - 1)^2 + 2x^3 \cdot 2(x^2 - 1)(2x) \]

\[ = 3x^2(x^2 - 1)^2 + 4x^3 \cdot (x^2 - 1) \]

\[ = x^2(x^2 - 1)(7x^2 - 2) \]

\[ x = 0, \pm \frac{1}{\sqrt{7}} \]
2) (20 points) Let

\[ f(x) = \frac{x^3 - x^2 - 1}{x^2 + x} \]

a) Find a \( g \) to which \( f \) is asymptotic.

b) Show \( f \) is asymptotic to \( g \) by computing a limit.

\[ \lim_{x \to \pm \infty} \left[ \frac{x^2 + x}{x^3 - x^2 - 1} \right] \]

\[ \lim_{x \to \pm \infty} \left[ \frac{x^3}{x^2} \right] = \lim_{x \to \pm \infty} \left[ x \right] = \infty \]

\[ \lim_{x \to \pm \infty} \left[ \frac{-x^2}{x} \right] = \lim_{x \to \pm \infty} \left[ -x \right] = \mp \infty \]

\[ \lim_{x \to \pm \infty} \left[ \frac{2x + x^2}{x^2 + x} \right] = \lim_{x \to \pm \infty} \left[ \frac{2 + \frac{x^2}{x^2}}{1 + \frac{x}{x}} \right] = \frac{2}{1} = 2 \]
4) (35 points) Use the product and chain rule to differentiate, simplify, and find all critical points:

\[ y = x(x^2 - 1)^2 \]

\[ y' = 2x(x^2 - 1) + x \cdot 2(x^2 - 1) \cdot 2x \]

\[ = (x^2 - 1) \left[ (x^2 - 1) + 4x^2 \right] \]

\[ + (x^2 - 1) \cdot 5x^2 - 1 \]

\[ y' = 0 \text{ when } x = \pm 1, x = \pm \sqrt{\frac{1}{5}} \]

CP: \( x = \pm 1, x = \pm \sqrt{\frac{1}{5}} \)
2) (15 points) Compute the limit:

\[
\lim_{x \to -1} \frac{(x^3 + 6x^2 + 11x + 6)}{(x^2 - x - 2)}
\]

\[
= \frac{-1 + 6 - 11 + 6}{-1 + 1 - 2} = 0
\]

Factor and cancel \( x - (-1) = x + 1 \)

\[
x^2 - x - 2 = (x + 1)(x - 2)
\]

\[
\frac{x + 1}{x^2 + 6x + 6}
\]

\[
\frac{x^2 + 11x + 6}{-2x^2 + 11x + 6}
\]

\[
\frac{6x + 6}{0}
\]

\[
\lim_{x \to -1} f = \lim_{x \to -1} \frac{x^2 + 6x + 6}{x - 2}
\]

\[
= \frac{1 - 5 + 6}{-1 - 2} = \frac{2}{-3} = -\frac{2}{3}
\]
4) (45 points) Let
\[ f(x) = \frac{(x + 1)^2}{x^3} \]

a) Use the product and chain rules to differentiate
b) Simplify
c) Find all critical points

\[ f'(x) = \frac{2(x + 1) \cdot 2(x + 1) - (x + 1)^2 \cdot \frac{2}{3} x^{-\frac{1}{3}}}{x^3} \]

\[ = \frac{2(x + 1)^3 x^{\frac{1}{3}} - 2(x + 1)^{\frac{5}{3}}}{x^3} \]

\[ = \frac{2(x + 1)^{\frac{2}{3}} [3x - (x + 1)]}{x^{\frac{13}{3}}} \]

\[ f'(x) = 0 \text{ when } x = -1, \frac{1}{2} \]

\[ f'(x) \text{ due to } x = 0 \]

\[ f'(x) = \frac{\frac{1}{2}}{2} = \frac{1}{4} \text{ due to } \]

Critical Points:
\[ x = -1, x = \frac{1}{2} \]
2) (20 points) Compute

\[
\lim_{{x \to -1}} \frac{{x^3 - x^2 + 2}}{{x^2 + x}} = \frac{{-1 - 1 + 2}}{{1 - 1}} = \frac{0}{0}
\]

Factor and cancel a \(x - (-1) = x + 1\) from numerator and denominator:

\[
x^2 + x = x(x + 1)
\]

\[
x + \frac{x^2 - 2x + 2}{x^2 - x^2 + 2}
\]

\[
-x^3 + x^2
\]

\[
-x^2 + 2
\]

\[
-2x^2 - 2x + 2x + 2
\]

\[
2x + 2
\]

\[
2 - 2 = 0
\]

\[
\lim_{{x \to -1}} f = \lim_{{x \to -1}} \frac{{x^2 - 2x + 2}}{x} = \frac{{1 + 2 + 2}}{-1} = \boxed{-5}
\]
3) (15 points) Use freezing to sketch the graph of \( y = x^2(x - 1)^{\frac{2}{3}} \).

Show your work beneath here

Freezing points: \( x = 0, 1 \)

Near \( x = 0 \) is \( x^2 \), \( x^2(-1)^{\frac{2}{3}} = x^2 \)

Near \( x = 1 \) is \( (x - 1)^{\frac{2}{3}} = (x - 1)^{\frac{2}{3}} \)

Use the space beneath here to draw your graph
4) (35 points) Use the quotient and chain rule to differentiate, simplify, and find all critical points:

\[ y = \frac{(x + 1)^2}{x^{\frac{1}{3}}} \]

\[ y' = \frac{2(x+1)x^{\frac{1}{3}} - (x+1)^2 \cdot \frac{1}{3} x^{-\frac{2}{3}}}{(x^{\frac{1}{3}})^2} \]

\[ = \frac{2(x+1)x^{\frac{1}{3}}}{1} - \frac{(x+1)^2}{3} x^{-\frac{2}{3}} \]

\[ = \frac{(x+1)}{3^{\frac{1}{3}}} \left[ \frac{6x - (x+1)}{3} x^{-\frac{2}{3}} \right] = \frac{(x+1)(5x-1)}{3^{\frac{1}{3}}} x^{-\frac{2}{3}} \]

\[ y' = 0 \quad \text{when} \quad x = -1 \quad \text{or} \quad x = \frac{1}{5} \]

\[ y' \text{ due when} \quad x = 0. \]

\[ f(0) = \frac{(0+1)^2}{0^{\frac{1}{3}}} = y_0 \quad \text{due not a CP} \]

\[ \text{CP:} \quad x = -1 \quad \frac{1}{5} \]