14. Volumes

1) A volume lies above the triangle in the xy plane, defined by \( y = x, \quad y = -x, \quad x = 1 \). Each \( x \)-section is a half-circle with diameter lying touching the ends of the triangle. What is the volume?

2) Sketch the region in the xy plane.

\[
\begin{array}{c}
y = x \\
y = -x \\
x = 1
\end{array}
\]

3) Set up the volume-by-sections

\[
V = \int_{a}^{b} A(x) \, dx = \pi \ell
\]

4) Start filling \( \frac{\pi}{2} \). The picture in (2) shows \( a = 0, \ell = 1 \)

5) Find \( A(x) \). It's half a circle, so \( A(x) = \pi \frac{x^2}{2} \).
But what is the radius for each \( x \)?
So each to the picture:

\[
\text{diameter } \rightarrow \frac{1}{2} \text{ diameter } = \frac{\text{radius}}{2}
\]
but \( r(x) \) goes from \( \text{zero} \) to \( \text{one} \). So \( r(x) = x \)!!

\[
V = \int_0^1 \frac{\pi}{2} x^2 \, dx = \frac{\pi}{2} \int_0^1 x^2 \, dx = \frac{\pi}{2} \left[ \frac{x^3}{3} \right]_0^1 = \frac{\pi}{2} \left[ \frac{1}{3} \right] = \frac{\pi}{6}
\]

2) What is the volume of the object \( y^2 + z^2 = x \), \( 0 \leq x \leq 1 \)?

This is of the form \( V = \int_a^b A(x) \, dx \) where we were flat-out given \( a = 0, \ b = 1 \).

\[ A(x) = \pi \text{ (radius) }^2 = \pi (\sqrt{x})^2 = \pi x \]

\[
V = \int_0^1 \pi x \, dx = \pi \left[ \frac{x^2}{2} \right]_0^1 = \pi \left[ \frac{1}{2} - \frac{0}{2} \right] = \frac{\pi}{2}
\]
Given the volume of the right-circular cone, base radius $R$, height $H$.

Immediately, I see $0 \leq x \leq H$ so I have

$$\int_0^H A(x) \, dx.$$ Each $x$-section is a circle of radius $r$, so $A(x) = \pi x^2$.

So $V = \pi \int_0^H x^2 \, dx$.

Now, what is $r$? I'll look in the $xz$ plane.

So $\frac{r}{x} = \frac{R}{H}$ or $r = \frac{R}{H} \cdot x$. Also, $V^2 = \frac{R^2}{H^2} x^2$. 
So \[ V = \pi \int_0^H \frac{r^2}{H^2} x^2 \, dx = \frac{\pi r^2}{H^2} \int_0^H x^2 \, dx \]

\[ = \frac{\pi r^2}{H^2} \left[ \frac{x^3}{3} \right]_0^H = \frac{\pi r^2}{H^2} \cdot \frac{H^3}{3} = \frac{1}{3} \pi r^2 H \]

This is the standard formula for volume of a cone.