Lecture Extra Another gradient Example

We did one of these examples in class; here's another.

The gradient theorem tells you that the of you are on the surface \( z = f(x, y) = P \), and you move in the direction \( \mathbf{u} = \nabla f(\mathbf{x}) \), then this gives the (geometric) slope you can have, and the slope is \( |\nabla f| \).

So let this problem does test this.
We'll find a curve that moves in \( \mathbf{u} \), and see what the slope is, and see if it matches \( |\nabla f(\mathbf{p})| \). Also we'll move in a different direction \( \mathbf{v} \) and see the slope is smaller.

1) Set \( z = f(x, y) = y^2 - x^2 \), \( P = P(0, 1) \).
2) Find \( \nabla f \), \( \nabla f(\mathbf{p}) \), \( \mathbf{u} \) and \( \nabla f(\mathbf{p}) \).

\[
\nabla f = (-2x, 2y) \quad \nabla f(\mathbf{p}) = (0, 2) \quad |\nabla f(\mathbf{p})| = \sqrt{0^2 + 2^2} = 2
\]

\[
\mathbf{u} = \frac{\nabla f}{|\nabla f(\mathbf{p})|} = (0, 1) \quad \text{(0, 1)}
\]

3) Find a line \( F(t) \) that started \( P \) and moves in the direction \( \mathbf{u} \).

This is going:
\[
F(t) = P + t\mathbf{u} = (0, 1) + t(0, 1) = (0, t + 1)
\]

4) Find \( F(\mathbf{t}(t)) \).

\[
f(x, y) = y^2 - x^2 \quad \Rightarrow \quad f(\mathbf{t}(t)) = (t + 1)^2 \]

\[
F(t) = (0, t + 1)
\]
\[ \text{and } \frac{df}{dt} = \frac{df}{dy} (\varepsilon + \hat{y})^2 = 2(\varepsilon + \hat{y}) \]
\[ \frac{df}{dt} (0) = 2(0 + \hat{y}) = 2 \]

4) show \( \frac{df}{dy} (0) = |DF (y)| \)
\[ 2 = 2 \quad \checkmark \]

5) repeat steps b), c), d) using \( \vec{u} = (1, 2) / |(1, 2)| \)
instead of \( \vec{u} \).

f(b) \[ \vec{u} (\alpha) = \vec{p} + a \vec{u} = (0, 1) + a \left( \frac{1, 2}{\sqrt{5}} \right) \]
\[ = \left( \frac{4}{\sqrt{5}}, \frac{4 + 2a}{\sqrt{5}} \right) \]

f(c) \[ f(u, \beta) = y^2 - x^2 \]
\[ f(\alpha) = \sqrt{5} (\alpha) = \left( \frac{1}{\sqrt{5}}, \frac{1 + 2a}{\sqrt{5}} \right) \]

\[ f(\alpha (\alpha)) = \frac{(1 + 2a)^2}{\sqrt{5}} - \frac{a^2}{\sqrt{5}} \]

f(d) \[ \frac{df}{dt} = \frac{2}{\sqrt{5}} (1 + 2a)(2) - \frac{2a}{\sqrt{5}} \]
\[ \frac{df}{dt} (0) = \frac{2}{\sqrt{5}} (2) - \frac{2}{\sqrt{5}} (0) = 4 (\sqrt{5}) \]

f(e) note \( \frac{df}{dt} (0) = 4(\sqrt{5}) < 2 = |DF (0)| \)
so indeed going in \( \vec{u} \) direction gives greater slope.
2) Repeat problem 1 with \( f(x,y) = x^2 + y^2 \), \( P = P(2,2) \), \( \overrightarrow{v} = (1,1) \)

a) \( \nabla f = (2x, 2y) \quad \nabla f \mid P = \nabla f(2,2) = (2, 4) \)

b) \( \nabla f(0) = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5} \)

\( \overrightarrow{u} = \nabla f \bigg|_{(0,0)} = \frac{(2, 4)}{2\sqrt{5}} = \left( \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) \)

c) \( f(x,y) = x^2 + y^2 \)

\( f(P) = (1 + \frac{x}{\sqrt{5}})^2 + 2 \left( \frac{2}{\sqrt{5}} \right) (\frac{2}{\sqrt{5}}) \)

\( f(P) = (1 + \frac{x}{\sqrt{5}})^2 + 2 \left( \frac{2}{\sqrt{5}} \right) \left( \frac{2}{\sqrt{5}} \right) \)

d) \( \frac{\partial f}{\partial x} = 2 \left( 1 + \frac{x}{\sqrt{5}} \right) (\frac{1}{\sqrt{5}}) + 2 \left( 2 + \frac{2}{\sqrt{5}} \right) (\frac{2}{\sqrt{5}}) \)

\( \frac{\partial f}{\partial x} = \frac{2 \cdot 5 \sqrt{5}}{5} = 2 \sqrt{5} = 1 \sqrt{f(P)} \)

e) \( \frac{\partial f}{\partial y} = 2 \cdot \frac{5 \sqrt{5}}{5} = 2 \sqrt{5} = 1 \sqrt{f(P)} \)

\( \overrightarrow{v} = (1,1) \), \( |\overrightarrow{v}| = \sqrt{1^2 + 1^2} = \sqrt{2} \)

\( \overrightarrow{a} = P + 4 \frac{\overrightarrow{v}}{|\overrightarrow{v}|} = (2,2) + (4 \sqrt{2}, 4 \sqrt{2}) \)

f) \( f(\overrightarrow{a}) = (1 + 4 \sqrt{2})^2 + (2 + 4 \sqrt{2})^2 \)

g) \( \frac{\partial f}{\partial y} = 2 \left( 1 + 4 \sqrt{2} \right) (\frac{4}{\sqrt{2}}) + 2 \left( 2 + 4 \sqrt{2} \right) (\frac{4}{\sqrt{2}}) \)

\( \frac{\partial f}{\partial y}(0) = \frac{2 \sqrt{2} + 4 \sqrt{2}}{\sqrt{2}} = 6 \sqrt{2} \)

h) \( \frac{\partial f}{\partial y}(0) = \frac{6}{\sqrt{2}} = 3 \sqrt{2} = \sqrt{12} = 1 \sqrt{f(P)} \)
3) Reflect with \( f(x, y) = \sqrt{1 - x^2 - y^2}, \quad P = P(0, 1) \) 
\[ \overrightarrow{v} = (1, -1) \]

4) Reflect with \( g = f(x, y) = (y - 3(x^2))^2, \quad P = P(0, 1) \) 
\[ \overrightarrow{v} = (1, 1) \]