Lecture Extra: Parametric Equations 1

Skill I: change parametric → function

\[ x = x(t), \quad y = y(t) \quad \rightarrow \quad y = f(x) \]

Example 1. Let \( x = \cos t, \quad y = \sin^2 t \)

a) Write this as a function \( y = f(x) \)

b) Give a quick sketch of graph of \( x = \cos t, \quad y = \sin^2 t \)

Solution a) Whenever you see sines and cosines, use Skill I. You should start by thinking:

\[
\cos^2 t + \sin^2 t = 1 \quad \Rightarrow \quad x^2 + y = 1 \quad \Rightarrow \quad y = 1 - x^2
\]

Now you have to worry about domain and range.

x is domain: \(-1 \leq \cos t \leq 1\) so \(-1 \leq x \leq 1\)

y is range: \(0 \leq \sin^2 t \leq 1\) so \(-1 \leq y \leq 1\)

b) \( y = 1 - x^2 \) u a parabola

\[ y = 1 - x^2 \]

but if you use \(-1 \leq x \leq 1\)

\[ 0 \leq y \leq 1 \]

you get:

The answer
Remark. When I did this, let me go to skill 2:

function \rightarrow parametric

\[ y = f(x) \rightarrow x = x(t), \ y = y(t). \]

This is always easy. \[ x = t, \ y = 1 - t^2. \] See?

Grab semester 0. God people doing \[ x = \cos t, \ y = \sin^2 t \]

Because they remembered skill I. Don't mix up skill I and skill II.

Is there something wrong with using \[ x = \cos t, \ y = \sin^2 t? \]

Yes.

\[ y = 1 - x^2 \]

\[ x = \cos t, \ y = \sin^2 t \]

Not the same.

Example. Let \[ x = \cos^2 t, \ y = \sin t. \]

a) convert to a function

b) sketch the graph

Solution: 

...
Solution: \[ \text{again, } \cos^2 t + \sin^2 t = 1 \]
\[ x = \cos^2 t, \quad y = \sin^2 t \quad \Rightarrow \quad x + y = 1 \]

This is not a function \( y = f(x) \); if \( x = \sin t \)
\[ x = 1 - y^2 \]

**IMPORTANT POINT:** Do not write \( y = \sqrt{1-x} \) in an attempt to make it into a function. You will lose points, lower your grade, and a curse will descend on your family for three generations.

**Why not? Reason 1:**

Graph of \( y = \sqrt{1-x}, \quad 0 \leq x \leq 1 \)

Graph of \( y^2 = 1 - x, \quad 0 \leq x \leq 1 \)

Not the same

**Reason 2:** What is \( y' \) at \( x = 1 \)? \( y' = \frac{1}{2\sqrt{1-x}} \)

But if you look at \( x = \cos^2 t, \quad y = \sin t \)

\[ x' = -2\cos t \sin t, \quad y' = \cos t \quad \text{so at } t=0 \quad (x=1) \]

\[ x' = 0, \quad y' = 1 \]

So you're losing important derivative info of you take the square root
Reason 2. The whole point of this section of the course is that \( y = 1 - x^2 \) and \( x = 1 - y^2 \) are just as good, you don't always need \( y = f(x) \). So taking the square root misses the point.

Ok, enough already. Let's just do the problem.

So \( x = 1 - y^2 \) and now domains:

\( x: \quad 0 \leq \cos^2 t \leq 1 \) so \( 0 \leq x \leq 1 \),

\( y: \quad -1 \leq \sin(t) \leq 1 \) so \( -1 \leq y \leq 1 \).

b) sketch. Here's how you figure out \( x = 1 - y^2 \).

\[ \begin{array}{c}
\text{start:} \\
\eta = x^2 \\
\text{flip} \\
\eta = -y^2 \\
\text{now} \\
x = 1 - y^2 \\
\end{array} \]

Now \( x = 1 - y^2 = (-y^2) + 1 \)

\[ \begin{array}{c}
\text{domain, range} \\
\end{array} \]

Note

\[ \begin{array}{c}
\end{array} \]
Example 2. Convert \( x = 1 - y^2 \) to parametric.

Again, this is always easy: \( y = t, \ x = 1 - t^2 \).

Example 2. Convert \( y = t, \ x = \frac{1}{t} \) to a function. Skill I.

One way: solve each for \( t \):

\[
\begin{align*}
\frac{1}{y} &= t \\
\frac{1}{x} &= t
\end{align*}
\]

Remark. Convert the hyperbola \( xy = 1 \) to parametric. Skill I.

Again, easy \( y = \frac{1}{x} \) so if \( x = t, \ y = \frac{1}{t} \).

Example 2. Convert \( x = e^t, \ y = e^{-t} \) to a function. Skill I.

Solution: could solve for \( t \):

\[
t = \ln x, \quad t = -\ln y
\]

But it's even easier than that:

\[
y = e^{-t} = \frac{1}{e^t} = \frac{1}{x} \quad \text{(Why?)}
\]

Domains: \( x = e^t > 0, \ y = e^{-t} > 0 \) so the graph is in the first quadrant.

Remark. Convert the hyperbola \( x^2 - y^2 = 2 \) to parametric.

Solution or, um, er... I'll show you after I do polar coordinates. In the meantime:

Check that

\[
\begin{align*}
\frac{1}{\sqrt{2}} (t + \frac{1}{t}), \ y &= \frac{1}{\sqrt{2}} (t - \frac{1}{t})
\end{align*}
\]

(Why? \( t + \frac{1}{t} = \cos \theta + \sin \theta \). I know, I know: polar co-ord.)