We expect \( f_{xy} = f_{yx} \), but this actually needs \( f \) to have continuous second partial derivatives, and we've seen how weird things can get without it. So here's a case where \( f_{xy} \neq f_{yx} \):

\[
f = \frac{xy(y^2-x^2)}{x^2+y^2}, \quad f_{xy}(0,0) \neq f_{yx}(0,0)
\]

\[
\frac{df}{dx} = y \frac{\partial}{\partial x} \left( \frac{y^2-x^2}{x^2+y^2} \right) = y \sqrt{\frac{4x^2(y^2-x^2)}{(x^2+y^2)^2}}
\]

\[
= y \left[ \frac{2(x^2-y^2)(x^2+y^2) - (y^2-x^2)2x}{(x^2+y^2)^2} \right]
\]

Now, here's the first trick: We won't use \( f_{xy}(0,0) \), but our next derivative is \( \frac{\partial^2 f}{\partial y \partial x} \), so I don't even need the \( x \)'s around any more. So set \( x = 0 \):

(Three happy thoughts for this trick)

\[
\frac{df}{dx}(0,y) = y \left[ \frac{2(4^2-0^2)-(4^2-0^2)(0)}{(4^2+0^2)^2} \right] = y \left[ \frac{32}{65} \right] = \frac{16}{65} 
\]

So now it's easy: \( \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (y) = 1 \). So \( f_y(0,0) = 1 \).

Here's the second trick: When I switch \( x, y \) (in \( f \)),

\[
f(x,y) = \frac{xy(y^2-x^2)}{x^2+y^2} \rightarrow \frac{yx(x^2-y^2)}{y^2+x^2} = -\frac{xy(x^2-y^2)}{x^2+y^2} = -f(x,y)
\]
So it isn't symmetric in x, y - it's off by a "-" sign.

This means I can compute \( f_y(x, 0) \) by interchanging \( x \) and \( y \) in \( f_x(0, 0) \) - and throwing in a "-".

You can check it by doing the derivatives long-hand:

\[
\begin{align*}
    f_x(0, 0) &= y \\
    f_y(x, 0) &= -x \\
    f_{yx}(x, 0) &= -1 \\
    f_{xy}(0, 0) &= 1 \\
    f_{yx}(0, 0) &= -1
\end{align*}
\]

Clearly not very equal!
Lecture Supplement
Traces and Change of Variables

Let \( f(x, y) = \sqrt{1 - x^2 - y^2} \), \( P = P(\frac{1}{2}, \frac{1}{2}) \).

1) Sketch the surface, locate \( P, f(P) \).

\[ f(P) = \sqrt{1 - \frac{1}{4} - \frac{1}{4}} = \frac{1}{2} \]

2) Sketch the trace \( z = f(\frac{1}{2}, y) \) in the \( yz \) plane.

Locate \( P, f(P) \).

\[ z = f(\frac{1}{2}, y) = \sqrt{1 - \frac{1}{4} - y^2} = \sqrt{\frac{3}{4} - y^2} \]

is a half-circle with radius \( \frac{\sqrt{3}}{2} \approx 0.86 \)

\( P = P(\frac{1}{2}, 0) \) so \( y \approx 0.5 \)

\[ f(P) \]
c) Sketch the trace $z = f(x, y)$ on the surface.

$3 = f(x, y)$ is parallel to the line $x = \frac{1}{2}$, which is parallel to the $y$-axis. That curve intersects the hemisphere of the dot in $e = \theta \Rightarrow \theta = \sqrt{\frac{2}{3}}$ reaches its maximum at $y = 0$, where $3 = f(x) = \sqrt{\frac{2}{3}} \approx 0.76$. While $P$ is at $f(\frac{1}{2}, 0) = \sqrt{\frac{2}{3}} \approx 0.7$. So the max should be shown higher than $f(P)$.

d) Compute $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial x}(P)$, $\frac{\partial f}{\partial y}(P)$.

Note $f(x, y) = \sqrt{1-x^2-y^2} = \sqrt{1-x^2-x^2} = f(y, x)$

so the function is symmetric. So all I have to do is $\frac{\partial f}{\partial x}$. Similarly $\frac{\partial f}{\partial y}$ is symmetric. So all I have to do is $\frac{\partial f}{\partial x}(P)$. 
\[ \frac{df}{dx} = \frac{2}{3} \left( 1 - x^2 - y^2 \right)^{\frac{1}{2}} = \frac{1}{3} \left( 1 - x^2 - y^2 \right)^{-\frac{1}{2}} 2 \left( 1 - x^2 - y^2 \right) \]
\[ = \frac{1}{2} \left( 1 - x^2 - y^2 \right)^{-\frac{1}{2}} (-2x) = \frac{-x}{\sqrt{1 - x^2 - y^2}} \]
\[ \frac{df}{dy} = \frac{y}{\sqrt{1 - x^2 - y^2}} \] just switch x and y

\[ \frac{dy}{dx} (P) = \frac{df}{dy} (P) = \frac{-1}{\sqrt{1 - x^2 - y^2}} = \frac{-1}{\sqrt{3}} = -\frac{1}{\sqrt{3}} \]

(c) Why does \( \frac{df}{dy} (P) \) have the sign it does?

Note the trace. The \( \frac{df}{dy} \) asks about the slope of \( y = f(x,y) \). Look at the graph in the yz plane. The curve is decreasing for P, so the slope is negative.
Lecture

Changing Coordinates

If $f = f(r, \theta)$ write the equation $\frac{\partial f}{\partial \theta} = 0$ in terms of derivatives in $x$ and $y$

Solution: according to the chain rule

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta}$$

But $x = r \cos \theta$ so $\frac{\partial x}{\partial \theta} = -r \sin \theta$

$y = r \sin \theta$ so $\frac{\partial y}{\partial \theta} = r \cos \theta$

So the equation is $\frac{\partial f}{\partial x} (-r \sin \theta) + \frac{\partial f}{\partial y} (r \cos \theta) = 0$

Note that this is $det \begin{bmatrix} r \cos \theta & r \sin \theta \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = 0$

If $F = (x, y)$, this can be written as $\nabla \times \nabla f = 0$

or, $\nabla f$ and $\nabla f$ are parallel.

Why does this make sense?

If $\frac{\partial f}{\partial \theta} = 0$, then $f$ does not change in the $\theta$ direction.

So all change is in the $r$ direction. But this gives the direction of maximum change. So $\nabla f$ must be parallel to $F$.

Another way to view $\nabla \times \nabla f = 0$ is that $\theta$ torsion is zero - if $f$ contains no $\theta$, it can't rotate.
Let \( u = \frac{x+y}{\sqrt{2}} \), \( v = \frac{x-y}{\sqrt{2}} \) (45° rotation)

Express \( \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \) in terms of \( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \).

Well, \( \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} \)

\( \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} \)

so I need \( x \) and \( y \) as functions of \( u, v \)

\( u = \frac{1}{\sqrt{2}} (x+y) \quad \Rightarrow \quad u+v = \frac{\sqrt{2}}{2} \cdot x \quad \Rightarrow \quad x = \sqrt{2} (u+v) \)

\( v = \frac{1}{\sqrt{2}} (x-y) \quad \Rightarrow \quad u-v = \frac{\sqrt{2}}{2} \cdot y \quad \Rightarrow \quad y = \frac{\sqrt{2}}{2} (u-v) \)

\( \frac{\partial x}{\partial u} = \frac{1}{\sqrt{2}} \quad \frac{\partial y}{\partial u} = \frac{1}{\sqrt{2}} \quad \frac{\partial x}{\partial v} = \frac{\sqrt{2}}{2} \quad \frac{\partial y}{\partial v} = \frac{\sqrt{2}}{2} \)

\( \frac{\partial f}{\partial u} = \frac{1}{\sqrt{2}} (\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}) \)

\( \frac{\partial f}{\partial v} = \frac{\sqrt{2}}{2} (\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}) \)

Now, the equation \( \frac{\partial^2 f}{\partial u \partial v} = 0 \) becomes \( \frac{\partial^2 f}{\partial u \partial v} = 0 \)

\( \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \left( \frac{\sqrt{2}}{2} \left( \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \right) \right) = 0 \quad \Rightarrow \quad \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \right) + \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \right) = 0 \quad \Rightarrow \quad \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial y \partial x} - \frac{\partial^2 f}{\partial x \partial y} = 0 \)

\( \Rightarrow \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} = 0 \quad \text{which is the wave equation.} \)
Solve the wave equation: \( \frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0 \)

Change to \( u, v \), to get \( \frac{\partial}{\partial u} \left( \frac{\partial f}{\partial v} \right) = 0 \)

Now, \( \frac{\partial}{\partial u} = 0 \) means \( f \) has no \( u \)'s in it, so \( f = g(v) \). So

\[ \frac{\partial f}{\partial v} = g(v) \] so \( f = \int g(v) dv + c \) the constant is constant in \( u \), but it could be any function of \( u \)!

So \( f(x, u) = g_+ (v) + g_- (v) \)

\begin{align*}
&= g_+ \left( \frac{x-u}{v} \right) + g_- \left( \frac{x+u}{v} \right) \\
&= h_+ (x-u) + h_- (x+u)
\end{align*}

\( h_+ \) is a right-moving wave, \( h_- \) is a left-moving wave.