4083 Vector Supplement

Some problems for E3 2008

1) Let \( f(x, y) = \frac{x+y}{x^2+y^2} \). Compute all second partials.

\[
f(x, y) = \frac{x}{x^2+y^2} + \frac{y}{x^2+y^2} = \frac{1}{x^2} + \frac{1}{y^2} \quad \text{note } f(x, y) = f(y, x).
\]

So we will have \( f_x = f_y \) and \( f_{xx} = f_{yy} \).

\[
f_x = -\frac{1}{x^3} y - \frac{2}{x^2 y^2} \quad f_{xx} = \frac{2}{x^3 y} + \frac{6}{x^2 y^3}
\]

\[
f_y = -\frac{2}{x^2 y^3} - \frac{1}{x^2 y^2} \quad f_{yy} = \frac{2}{x^3 y^2} + \frac{6}{x y^4}
\]

\[
f_{xy} = f_{yx} = \frac{2}{x^2 y^3} + \frac{2}{x^2 y^2}
\]

Set \( f(1, 1) = -1 \) or \( -\sqrt{2} \).

2) Sketch the surface

\( z = -(x^2+y^2) \) is the graph of \( z = x^2+y^2 \).
I have to draw the curve $z = f(x, y)$ which requires an auxiliary line $y = \frac{1}{2}$, parallel to the $x$-axis. This is hard to show since the $x$ and $y$ axes are at the top of the surface. So in the pic above, I added fake $x, y$ axes, parallel to the real ones.

Now it's easy to draw a fake $y = \frac{1}{2}$ line with two dots where my curve hits the solid.

All I need is a third point $z = -(x^2 + y^2)$ hits its apex. Then $x = 0$, then $z = -\frac{1}{2}$.

Now connect the three dots in a way that looks kind of circular if you want.
connect the sketch. Then trace \( y = f(x, y) \)

in \( xz \)-plane: \( 3 = f(x, y) = -y^2 \)

d) Compute \( f_x(p) \) where \( p = p(0, \frac{1}{2}) \)

\[ f(x, y) = -x^2 - y^2 \]
\[ f_x(x, y) = -2x \]
\[ f_x(0, \frac{1}{2}) = 0 \]

e) Referring back to the trace in part d), explain why \( f_x(p) \) has the value it does.

The figure is showing the vertex of the parabola \( f(x, y) \) at an absolute maximum so its derivative is 0.

3) Let \( f(x, y) = \frac{x^2 - y^2}{x^2 y^2} \). Does \( f \) satisfy the PDE

\[ \frac{\partial^2 f}{\partial x \partial y} = 0? \]

Rewrite \( f = \frac{x^2 y^2 - x^2 - y^2}{x^2 y^2} \)

\[ \frac{\partial^2 f}{\partial y \partial x} = -\frac{2}{y^3} \frac{\partial^2 f}{\partial x \partial y} = -2 \left( \frac{2x}{y^3} \right) = \frac{2x}{y^3} \left( -\frac{2}{x^2} \right) = 0 \]

So yes, it does.
Let \( f(x, y) = \ln(x^2 - y^2) \). Does \( f \) satisfy the wave equation \( f_{xx} - f_{yy} = 0 \)?

\[
f_x = \frac{2}{dx} \ln(x^2 - y^2) = \frac{1}{x^2 - y^2} \cdot \frac{2x}{dx} (x^2 - y^2) = \frac{2x}{x^2 - y^2}
\]

\[
f_{xx} = \frac{2}{dx} \left( \frac{2x}{x^2 - y^2} \right) = 2 \left( \frac{x^2 - y^2}{x^2 - y^2} \right) - 2x \frac{2x}{(x^2 - y^2)^2} = \frac{-2x^2 - 2y^2}{(x^2 - y^2)^2}
\]

\[
f_y = \frac{2}{dy} \ln(x^2 - y^2) = \frac{-2y}{x^2 - y^2}
\]

\[
f_{yy} = \frac{2}{dy} \left( \frac{-2y}{x^2 - y^2} \right) = \frac{-2y}{x^2 - y^2} \cdot \frac{-2y}{x^2 - y^2} = \frac{-2y^2}{(x^2 - y^2)^2}
\]

\[
\therefore f_{xx} - f_{yy} = 0 \quad \checkmark
\]

**NOTE** \( f = \ln((x-y)(x+y)) = \ln(x-y) + \ln(x+y) \).

The general solution to \( f_{xx} - f_{yy} = 0 \) is

\( g(x-y) + h(x+y) \).