Extra Lecture: Polar Areas

Let's do the world's dumbest area: the region
bounded by the lines $y=0$, $y=x$ and $x=1$

1. Since $y=x$ and $x=1$, these meet when $y=1$. $P(1,1)$
2. Since $y=x$ and $y=0$, $x=0$ $Q(0,0)$
3. $x=1$ and $y=0$ = $R(1,0)$.

These are the sides of the triangle. The height is 1.
The base is 1, so $A = \frac{1}{2}bh = \frac{1}{2}$

Now you do it with polar co-ordinates.

$y=x$ has equation $\theta = \pi/4$; $y=0 \rightarrow \theta = 0$
and $x=1$? Well, $x = r \cos \theta = 1$ so $r = \frac{1}{\cos \theta}$.

If I went to use our polar formula

$$\frac{1}{2} \int_{\alpha}^{\beta} \left[f(\theta)ight]^2 d\theta$$

We take $\alpha = 0$, $\beta = \pi/4$ and $f(\theta) = \frac{1}{\cos \theta}$.

So the area is $\frac{1}{2} \int_{0}^{\pi/4} \left[\frac{1}{\cos^2 \theta}\right] d\theta$

The integration troubles here—which I don't expect you to remember— are that

$$\int \sec^2 \theta d\theta = \tan \theta$$

So, Area $= \frac{1}{2} \int_{0}^{\pi/4} \sec^2 \theta d\theta = \frac{1}{2} \left[\tan \theta\right]^{\pi/4}_{0}$

$= \frac{1}{2} \left[\tan \pi/4\right] - \frac{1}{2} \left[\tan 0\right] = \frac{1}{2} (1) - \frac{1}{2} (0) = \frac{1}{2}$
now I'll do an area I don't already know —
the area inside \( r = \sin(2\theta) \). Remember it looks like
four pieces, each of which is like

\[
\int_0^{\pi/2} r^2 \, d\theta
\]

and we saw these angles were \( 0 \leq \theta \leq \pi/2 \).

So area is 4 times
\[
\frac{1}{2} \int_{\pi/4}^{\pi/2} (\sin^2 2\theta) \, d\theta
\]

\[
= 4 \cdot \frac{1}{2} \int_0^{\pi/2} \sin^2 2\theta \, d\theta = 2 \int_0^{\pi/2} \sin^2 2\theta \, d\theta
\]

as before, \( \sin^2 x = \frac{1 - \cos 2x}{2} \), so

\[
\sin^2 (2\theta) = 1 - \cos (2 \cdot 2\theta) = \frac{1}{2} (1 - \cos 4\theta)
\]

Then Area =
\[
2 \int_0^{\pi/2} \frac{1}{2} (1 - \cos 4\theta) \, d\theta
\]

\[
= \int_0^{\pi/2} 1 - \cos 4\theta \, d\theta
\]

Now, again, I have to remember some integrals —

\[
\int \cos kx \, dx = \frac{1}{k} \sin kx + C. \text{ So, set }
\]

\[
\int \cos 4\theta \, d\theta = \frac{1}{4} \sin 4\theta + C \quad \text{and}
\]

\[
\int_0^{\pi/2} (1 - \cos 4\theta) \, d\theta = \left[ \theta - \frac{1}{4} \sin 4\theta \right]_0^{\pi/2}
\]

\[
= \left[ \frac{\pi}{2} - \sin (4\pi/2) \right] - \left[ 0 - \frac{1}{4} \sin 0 \right]
\]

\[
= \left[ \frac{\pi}{2} - \sin 0 \right] - 0
\]

\[
= \frac{\pi}{2}
\]