We want to show that $r = \cos \theta$ gives a circle—more precisely, a circle of radius $\frac{1}{2}$ centered at $(0, \frac{1}{2})$.

Recall a circle centered at $(0, 0)$ of radius $R$ has equation $(x-a)^2 + (y-b)^2 = R^2$. So we're going for

$$(x-\frac{1}{2})^2 + y^2 = (\frac{1}{2})^2$$

Since $r = \cos \theta$ then $r \cdot r = r \cdot \cos \theta$ or $r^2 = r \cdot \cos \theta$

or $x^2 + y^2 = x$ or $x^2 - x + y^2 = 0$. Now we do

$$x^2 - x + \frac{1}{4} + y^2 = \frac{1}{4} \quad \text{or} \quad (x-\frac{1}{2})^2 + y^2 = (\frac{1}{2})^2$$

Now we do as we did in class that $r = \cos \theta$ goes around the circle twice. Why is that? Let's compare and to do that we're gonna parameterize this circle.

We go around a circle once if we have

$$x = \cos \theta \quad 0 \leq \theta \leq 2\pi \quad x = \frac{1}{2} \cos \theta$$

$$y = \sin \theta \quad y = \frac{1}{2} \sin \theta$$

Circle radius 1

Circle radius $\frac{1}{2}$

We can center the circle by adding $\frac{1}{2}$ to $x$

$$x = \frac{1}{2} + \frac{1}{2} \cos \theta \quad \text{check: is } (x-\frac{1}{2})^2 + y^2 = (\frac{1}{2})^2 ?$$

$$y = \frac{1}{2} \sin \theta$$

$$(x-\frac{1}{2})^2 + y^2 = \left(\frac{1}{2} + \frac{1}{2} \cos \theta - \frac{1}{2}\right)^2 + (\frac{1}{2} \sin \theta)^2$$

$$= \left(\frac{1}{2} \cos \theta + \frac{1}{2}\right)^2 \sin^2 \theta = (\frac{1}{2})^2$$

So this is a good parameterization of the circle—one time around.
Now let's see what we have:

\[ r = \cos \theta \]

So

\[ x = r \cos \theta = \cos \theta \cos \theta = \cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta) \]

\[ y = r \sin \theta = \cos \theta \sin \theta = \frac{1}{2} \sin 2\theta \]

So

\[ x = \frac{1}{2} + \frac{1}{2} \cos 2\theta \]

\[ y = \frac{1}{2} \sin 2\theta \]

Instead of

\[ x = \frac{1}{2} + \frac{1}{2} \cos \theta \]

\[ y = \frac{1}{2} \sin \theta \]

The 20 makes us go around the circle twice.
On polar coordinates \((x, y)\) is represented by \((r \cos \theta, r \sin \theta)\) where \(\theta\) means something geometric: \(\theta\) is the angle that \((x, y)\) makes with the \(x\)-axis - well, more like in the picture.

\[ (x, y) \quad \rightarrow \quad (r \cos \theta, r \sin \theta) \]

I can move \((x, y)\) around the circle (of radius \(r\)) by moving the angle. Increasing the angle moves \((x, y)\) \textit{clockwise}.

\[ (r \cos \theta, r \sin \theta) \quad \rightarrow \quad (r \cos \theta, r \sin (\theta + \Delta \theta)) \]

and decreasing the angle moves \textit{clockwise}.

\[ \text{(just flip the picture around)} \]

So this motion of \((x, y)\) along the circle is called a \textit{rotation}. (from the Latin 'rota' meaning 'wheel' \(\circ\circ\))
6b: clockwise rotation of \((r,\theta)\) by 45°
\[(r \cos \theta, r \sin \theta) \rightarrow (r \cos (\theta + 45°), r \sin (\theta + 45°))\]
and counterclockwise would be \(\pm 114°.

Clockwise rotation of \((x,y)\) by 90°
\[(r \cos \theta, r \sin \theta) \rightarrow (r \cos (\theta - 90°), r \sin (\theta - 90°))\]

Let's look at a 90° clockwise rotation of the parabola \(y = x^2\). The positive y axis is \((0, r) = (r \cos (\pi/2), r \sin (\pi/2))\)

Thus, get rotated to \((r \cos (\theta), r \sin (\theta)) = (r, 0)\) = the positive x-axis

\[\uparrow \rightarrow \uparrow\]

Similarly, the positive x-axis \((r, 0) = (r \cos (\pi/2), r \sin (\pi/2))\) becomes \((r \cos (-\pi/2), r \sin (-\pi/2)) = (0, -r)\) = the negative y-axis

\[\downarrow \rightarrow \downarrow\]

So \(y \rightarrow x\) and \(x \rightarrow -y\). Then the parabola \(y = x^2\) becomes \(x = (-y)^2\) or \(x = -y^2\)

\[\uparrow \rightarrow \downarrow\]
A 45° clockwise rotation is messier:

\[ (\cos \theta, \sin \theta) \rightarrow (\cos(\theta - \pi/4), \sin(\theta - \pi/4)) \]

Again this is easiest to see if we look at the axes:

- \( y \)-axis: \( (\cos \pi/4, \sin \pi/4) = (0, r) \)
  \[ \rightarrow (\cos \pi/4, \sin \pi/4) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} r) \]

So \( y \rightarrow \frac{\sqrt{2}}{2} (x + y) \).

Similarly, the \( x \)-axis: \( (\cos 0, \sin 0) = (r, 0) \)
  \[ \rightarrow (\cos(-\pi/4), \sin(-\pi/4)) = (\frac{\sqrt{2}}{2} r, -\frac{\sqrt{2}}{2} r) \]

So \( x \rightarrow \frac{\sqrt{2}}{2} (x + y) \).

Let's see what happens to a hyperbola \( y = \frac{1}{x} \)

I'll rewrite it as \( xy = 1 \). Then:

The \( xy \) rotated to:

\[ \frac{\sqrt{2}}{2} (x - y) \frac{\sqrt{2}}{2} (x + y) = 1 \]

or \( \frac{1}{2} (x^2 - y^2) = 1 \) or \( x^2 - y^2 = 2 \)

Hey! What's with the \( \frac{\sqrt{2}}{2} \)? look at it this way.
Now \( P \) is on the hyperbola where \( x = y \), so \( xy = x^2 = 1 \), \( x = 1 \), \( y = 1 \), so \( P = (1, 1) \). Then \( P \) is like

\[
\begin{align*}
x &= 0 \quad r^2 = 1^2 + 1^2 \\
\frac{x}{1} &\quad r = \sqrt{2} \\
1 &\quad r = \sqrt{2}.
\end{align*}
\]

Whereas \( Q \) is on the \( x - \alpha y \), so \( y = 0 \), so \( x^2 - \alpha^2 = 2 \) or \( x = \sqrt{2} \).

So \( P (1, 1) \rightarrow \Phi (\sqrt{2}, 0) \). But they are each \( \sqrt{2} \) units from the origin; rotations don't change lengths.

**Devous Trick**: Parameterize \( xy = 1 \) as \( x = e^{-t}, y = e^t \).

Then \( x \rightarrow \frac{1}{2} (e^{-t} - e^t) = \frac{1}{2} (e^{-t} - e^t) \frac{1}{2} = -\sqrt{2} (e^t - e^{-t}) \)

\( y \rightarrow +\sqrt{2} \sinh t \)

So the hyperbola \( x^2 - y^2 = \sqrt{2} \) is parameterized by \( \sqrt{2} \cosh t, -\sqrt{2} \sinh t \).

Thus, the hyperbolic functions $\text{coth}$.