Lecture Extra 5: Tangent Planes

1. Let \( f(x, y) = (y - \cos x)^2 \). \( P = P(\pi/2, 1) \)

2. Find an implicit tangent plane to \( f \) at \( P \)

The form of this is \( \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \) where

\[
\mathbf{n} = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (y - \cos x, -2y(y - \cos x)) \]

\[
\mathbf{r}_0 = \left( \frac{\pi}{2}, 1, f(\pi/2, 1) \right)
\]

So, let's go:

\[
\frac{\partial f}{\partial x} = 2(y - \cos x) \frac{\partial}{\partial x} (y - \cos x) = 2y(y - \cos x)(\sin x)
\]

\[
\frac{\partial f}{\partial y} = 2(y - \cos x) \frac{\partial}{\partial y} (y - \cos x) = 2y(y - \cos x)(-1)
\]

\[
\frac{\partial f}{\partial x}(\pi/2) = \frac{\partial f}{\partial y}(\pi/2) = 2(1 - \cos(\pi/2)) = 2(0) = 0
\]

\[
\frac{\partial f}{\partial y}(\pi/2) = 2(1 - \cos(\pi/2)) = 2
\]

\[
f(\pi/2, 1) = f(\pi/2, 1) = (1 - \cos(\pi/2))^2 = 1
\]

\[
\mathbf{r}_0 = \left( \frac{\pi}{2}, 1, 1 \right), \quad \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0
\]

\[
(\pi/2, 1, 1) \cdot \left( x - \frac{\pi}{2}, y - 1, z - 1 \right) = 0
\]

\[
(2\pi + 2y - 2) = 0
\]

\[
2\pi + 2y - 2 = 0
\]

\[
2\pi - 2 = 0
\]

\[
\pi = 1
\]

\[
\pi = 2\pi - 2
\]

6. Find a parametric curve \( F = (x(t), y(t), z(t)) \)

That parameterizes the \( x \)-trace \( y = f(x, 1) \), and \( t \) to

with \( F(t_0) = (\pi/2, 1, 1) \).

The \( x \)-trace by itself is \( z = f(t, 1) = (1 - \cos t)^2 \).

In 3D, \( y = 1 \) so \( \mathbf{r}(t) = (\pi/2, t, (1 - \cos t)^2) \)

and \( \mathbf{r}_0 = \pi/2, t, (1 - \cos t)^2 \)
\[ \mathbf{F}'(0) = (1, 0, 2(1 - \cos t)) \]
\[ \mathbf{F}'(0) = (1, 0, 2(1 - 0)) = (1, 0, 2) \]

1) \text{Show that } \mathbf{F}'(0) \text{ is parallel to the tangent plane of } P.
   \text{To do this, you'd just show } \mathbf{n} \cdot \mathbf{F}' = 0. \text{ Because } 
   \mathbf{n} \text{ is 1 to the vector in the tangent plane.}
   \mathbf{n} \cdot \mathbf{F}' = (2, 2, -1) \cdot (1, 0, 2) = 2 + 0 - 2 = 0

2) \text{Let } Z = (x + y^2)^2; \ P = P(1, \phi). \text{ Find the tangent plane to } P \text{ at } P.

   \text{As before, } \frac{\partial F}{\partial x} = 2(x + y^2)(x + y^2); \ \frac{\partial F}{\partial y} = 2(x + y^2)(2y).

   \frac{\partial F}{\partial x} (1, \phi) - \frac{\partial F}{\partial y} (1, \phi) = 2(1 + \phi) + 4 \phi = 2(\phi + 2) = \mathbf{V}.

   \mathbf{F}(1, \phi) = F(1, 1) = (1 + 1)^2 = 4.

   \mathbf{n} = (2, 4, -1); \ \mathbf{F}_0 = (1, 1, 4).

   \mathbf{n} \cdot (\mathbf{r} - \mathbf{F}_0) = (2, 4, -1) \left[ (x - 1, y - 1, z - 4) \right]

   = 4(x - 1) + 8(y - 1) - (3 - 4) = 0

   |4x + 8y - z = 8|

3) \text{Parameterize the surface } Z = f(1, y), \text{ and find } \mathbf{F} \text{ with }
   \mathbf{F}(1, y) = (1, 1, 4). \text{ So } 3 = f(1, y) = (1 + y^2)^2 \text{ so }
   \mathbf{F}(t) = (1, t, (1 + t^2)) \text{ and } t_0 = 1
And \( \vec{r}'(t_0) \).
\[
\vec{r}' = (0, 1, 2(1+t^3)/(2t))
\]
\( \vec{r}'(t_0) = \vec{r}'(1) = (0, 1, 8) \).

2) Show \( \vec{r}'(t_0) \) is parallel to the tangent plane \( \mathbf{P} \).
\[
\vec{r} \cdot \vec{r}' = \begin{vmatrix}
4 & 8 - 1 & 0.48
\end{vmatrix} \cdot (8, 1, 8)
\]
\[
= 0 + 8 - 8 = 0.
\]