1) (35 points) Let \( z = f(x, y); x = t \cosh(s), y = t \sinh(s). \)

a) For general \( f \), state the chain rule for \( \frac{\partial f}{\partial t}. \)

b) Compute a), for general \( f \).

c) Eliminate the \( s, t \) in b), in favor of \( x \) and \( y \)

d) Now take \( f(x, y) = \frac{xy}{(x^2 - y^2)} \). Compute the partial derivatives; simplify.

e) Use c), d) to show \( f \) has no \( t \) in it.

\[
\begin{align*}
\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial s} \frac{\partial s}{\partial t} + \frac{\partial f}{\partial t} \frac{\partial t}{\partial t} \\
&= \frac{\partial f}{\partial s} \cosh s + \frac{\partial f}{\partial t} \sinh s
\end{align*}
\]

\[
\frac{\partial f}{\partial x} = \frac{1}{t} \left[ \frac{\partial f}{\partial s} \right] x + \frac{\partial f}{\partial t} y
\]

\[
\frac{\partial f}{\partial x} = \frac{y(x^2 - y^2) - xy(2x)}{(x^2 - y^2)^2} = \frac{y \left( x^2 - y^2 - 2x^2 \right)}{(x^2 - y^2)^2} = \frac{-y(x^2 + y^2)}{(x^2 - y^2)}
\]

\[
\frac{\partial f}{\partial y} = \frac{x(x^2 - y^2) - xy(-2y)}{(x^2 - y^2)^2} = \frac{x \left( x^2 - y^2 + 2y^2 \right)}{(x^2 - y^2)^2} = \frac{x(x^2 + y^2)}{(x^2 - y^2)^2}
\]

\[
\frac{\partial f}{\partial t} = \frac{1}{t} \left[ x \left( \frac{-y(x^2 + y^2)}{(x^2 + y^2)^2} \right) \right] + y \left( \frac{x(x^2 + y^2)}{(x^2 + y^2)^2} \right) = \frac{x(x^2 + y^2)}{(x^2 - y^2)^2} \left[ -xy + xy \right] = 0
\]
1) (35 points) Let \( z = f(x, y); x = \cos(\theta) \cos(\phi), \ y = \sin(\theta) \cos(\phi). \)

a) For any \( f \), state the chain rule for \( \frac{\partial f}{\partial \theta} \).

b) Compute a), for general \( f \).

c) Eliminate the \( \theta, \phi \) in b), in favor of \( x \) and \( y \).

d) Now take \( f(x, y) = \frac{x+y}{x-y} \). Compute \( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \) and SIMPLIFY.

e) Use c) and d) to compute \( \frac{\partial f}{\partial \phi} \). Simplify (common denominator; factor)

\[
\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta}
\]

\[
= \frac{\partial f}{\partial x} (-\sin(\theta) \cos(\phi)) + \frac{\partial f}{\partial y} (\sin(\theta) \cos(\phi))
\]

\[
= -y \frac{\partial f}{\partial x} + x \frac{\partial f}{\partial y}
\]

\[
\frac{\partial f}{\partial x} = \frac{(x+y) - (x+y)}{(x-y)^2} = \frac{-2y}{(x-y)^2}
\]

\[
= \frac{(x-y) + (x+y)}{(x-y)^2} = \frac{2x}{(x-y)^2}
\]

\[
\frac{\partial f}{\partial \phi} = -y \frac{2y}{(x-y)^2} + x \frac{2x}{(x-y)^2}
\]

\[
= \frac{2(x^2 + y^2)}{(x-y)^2}
\]
Find the general solution to the PDE:

\[ \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 0 \]

As in lecture, let \( x = \frac{u+v}{2}, \ y = \frac{u-v}{2} \)

But now compute \( \frac{\partial f}{\partial u} \) using the chain rule

\[
\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} = \frac{\partial f}{\partial x} \left( \frac{1}{2} \right) + \frac{\partial f}{\partial y} \left( \frac{-1}{2} \right)
\]

\[
\frac{\partial f}{\partial u} = \frac{1}{2} \left[ \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \right] = \frac{1}{2} \left[ \frac{\partial f}{\partial u} \right] = 0.
\]

So now all we need to do is solve

\[ \frac{\partial f}{\partial u} = 0, \] which has solution

\[ f = g(v). \] What \( v \)?

\[
\begin{align*}
X &= \frac{u+v}{2} \\
\bar{Y} &= \frac{u-v}{2}
\end{align*}
\]

Suggest that \( \frac{2v}{2} = v \)

So \( f(x,y) = g(x-y) \) for

\[ g(y), \]