Volumes

Why?

Check out Figure 1: it’s the Cook County Courthouse (in Chicago; I was at the University of Chicago before UT). Lower floors are courtrooms, higher floors jail cells. You can’t escape by jumping out a window, and no helicopter can land on the roof.

If you’re an environmental engineer working with the architecture firm that designed this building, you’d need to decide what the HVAC (heating, air-conditioning) requirement are. And to do that, you need to know the volume of air. How do you compute that?

We’re going to look at volumes for a very limited number of objects; they will all be volumes under a surface \( z = f(x, y) \) and above a Domain \( D \) in the \( xy \)-plane. Figure 2 shows a cut cylinder: it’d be like a can of beans, except someone took a chain saw and cut off the top in a slanting way (trained professional, do not attempt at home). The function \( f(x, y) \) gives the top (it would be something like \( z = 3 - x - y \)); the domain \( D \) is the \( D \) in the \( xy \) plane, and it’s a circle because bottoms of cans are circular.

Figure 4 is another typical example; the false-colored surface is \( z = 3 + y + x^2 \); the domain \( D \) is a rectangle.

We have some experience with functions \( z = f(x, y) \), and none with domains \( D \). So the first issue is how to describe, mathematically, those domains. We’ll work through some examples; there will be more in the 14U problems.

We’ll start with a domain like Figure 4 – a rectangle. Let’s take an example:

\[
D = \{(x, y) \mid 1 \leq x \leq 2; 1 \leq y \leq 3\}
\]

This how we’ll describe many domains; it’s pronounced "the collection of all pairs in the plane" ("\{(x, y)\}" where ("\mid") "\(x\) is between \(1\) and \(2\) and \(y\) is between \(1\) and \(3\)."

Usually, you’ll be given a formula like this, and be asked to draw the domain, Figure ???. Here’s how to do it. Start by converting \(1 \leq x \leq 2\) into two equations for lines: \(x = 1\) and \(x = 2\). Draw those with dotted lines. Now do the same for \(y, y = 1\) and \(y = 3\). Now you have a rectangle, and the domain \(D\) is what’s inside. It’s the part inside because \(D\) was described with equations like \(1 \leq x \leq 2\). That means \(x\) is confined to a small region: the inside.
Worked Examples

Sketch $D = \{(x,y) \mid 0 \leq x \leq 1; \ x \leq y \leq 1\}$

As before, we get lines $x = 0$ and $x = 1$ (that first is hard to draw because it’s right on the $y$-axis), and $y = x$ and $y = 1$. Figure 5 looks like there are two triangles, a top and a bottom, but that’s an illusion. A triangle has three sides, which we are drawing in blue. The "? triangle" doesn’t have a blue bottom! I’ve seen this happen on exams; you have to be careful here: the correct domain is drawn in Figure 6.

This next one is even sneakier, so I love to put it on exams:

$$D = \{(x,y) \mid x^2 + y^2 \leq 1; \ 0 \leq y \leq x\}$$

Changing that into equations, $x^2 + y^2 = 1$, which gives a circle; since we have $\leq$, it’s the inside of a circle (called a disc). For the second pair of inequalities, er get $x = 0$ and $y = x$, all drawn in Figure 7. My program won’t draw dotted lines on circles, alas. Anyway, the two lines $x = 0$ and $y = x$ divide the inside of the circle into four pieces, and again I have to figure out which one is my $D$.

This time, the inequalities save us: Since $x \geq 0$, $D$ has to be in either the first or fourth quadrants, so regions I and IV are wrong. Additionally, $y \geq x \geq 0$ so $y$ has to be positive, and therefore $D$ has to be in either the first or second quadrant. The only region satisfying both the $x$ and $y$ restrictions is region I, in Figure 8.

And now for something completely different: Sketch the region bounded by $y = x^2$ and $y = x + 2$. Then write it as $D = \{ \ldots \}$

The sketch is easy; I can draw a parabola and a straight line: Figure 9. But to do $D = \{ \ldots \}$, I need $x \leq y \leq ?$ and ditto for $y$. Here, the $y$ is easiest; $y$ goes between the parabola on the bottom and the line on the top. Or, just take the equations for $y$ and put them back into an inequality: $x^2 \leq y \leq x + 2$. It’s the $x$ that’s hard. What we used to look for was vertical $x =$ lines; we’ll have to think. $a \leq x \leq b$ would tell you how far left or right $x$ can go in Figure 9. Those are indicated by the green dots in the figure; you could just see what the co-ordinates are.

A better way for this kind of problem is to notice the dots are placed where the line hits the parabola: $x + 2 = x^2$. It’s a quadratic, so if you just solve for $x$, you’ll get the left and right $x$:

$$x^2 = x + 2; \ x^2 - x - 2 = 0; \ (x + 1)(x - 2) = 0; \ x = -1, x = 2$$

$$D = \{(x,y) \mid -1 \leq x \leq 2; \ x^2 \leq y \leq x + 2\}$$
The $D = \{(x, y) \mid -1 \leq x \leq 2; \ x^2 \leq y \leq x + 2\}$ is a mathematical
description of some curves; this assumes that the curves are all given
by $y = f(x)$. Well, of course! What isn’t?

What isn’t is Figure 10; it shows the domain $D$ bounded by curves
$x = y^2$, $y = 1$. And, as we discussed when we did parametric
equations, there’s no easy way out of this; we have to accept that
some curves simply aren’t $y = f(x)$.

This means we’re going to have to rethink how we describe do-
mains. And we’ll need terminology for that. A description like
$D = \{(x, y) \mid -1 \leq x \leq 2; \ x^2 \leq y \leq x + 2\}$ is called a Type I
or $T_1$ description of $D$. In general, this is what a $T_1$ description looks
like:

$$D = \{(x, y) \mid a \leq x \leq b; \ b(x) \leq y \leq t(x)\}$$

What this means is shown in Figure 11. Let’s sort this out:
1) $a$ is the left-most $x$ in $D$; $b$ is the right-most $x$ in $D$
2) $y = b(x)$ is the bottom curve of $D$; $y = t(x)$ is the top curve of $D$.

Side note: If you’re looking at my old exams or quizzes, the old $g_B(x)$
is now $b(x)$; the old $g_T(x)$ is now $t(x)$. With that sorted out, what do
we do with the domain bounded by the curves $x = y^2$, $y = 1$, Figure
10? We can’t write it as $D = \{(x, y) \mid a \leq x \leq b; \ b(x) \leq y \leq t(x)\}$.

A Type II or $T_2$ description of $D$ is given by

$$D = \{(x, y) \mid c \leq y \leq d; \ l(y) \leq x \leq r(y)\}$$

What this means is shown in Figure 12. Let’s sort this out:
1) $c$ is the bottom-most $y$ in $D$; $d$ is the top-most $y$ in $D$
2) $x = l(y)$ is the left curve of $D$; $r = r(y)$ is the right curve of $D$.

WORKED PROBLEM

Write the domain bounded by the curves $x = y^2$, $y = 1$, Figure 10, as
a $T_2$ domain.

$x = l(y)$ and $x = r(y)$ are easy from Figure 12: $x = l(y) = 1$ and
$x = r(y) = y^2$. As for $c, d$, they are the points where the curves $x = y^2$
and $x = 1$ intersect, so $y^2 = 1$, just as before, a quadratic, with two
solutions: $y = \pm 1$. So:

$$D = \{(x, y) \mid -1 \leq y \leq 1; \ y^2 \leq x \leq 1\}$$

And it’s time to try the 14U.