4) Show that \( f(x, y) = 1/(x + y)^2 \) satisfies the partial differential equation

\[
\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} = 0
\]

5) Let \( f(x, y) = \frac{x + y}{x^2 y^2} \). Compute all second partials and simplify.

6) Let \( f(x, y) = \frac{x^2 - y^2}{x^2 y^2} \). Does \( f \) satisfy the partial differential equation:

\[
\frac{\partial^2 f}{\partial x \partial y} = 0
\]

7) Find all four second derivatives, simplify, and check that the mixed partials are equal, if \( z = f(x, y) = y/(x + y) \).
14) Let \( w = f(x, y, z); \ x = r \cos \theta \sin \phi, \ y = r \sin \theta \sin \phi, \ z = r \cos \phi. \)
   a) Draw the tree diagram for these quantities
   b) State the chain rule for finding \( \partial f / \partial \theta \)
   c) Use the chain rule to compute \( \partial f / \partial \theta \)
   d) Change the answer in c) to \( x, y \).
   e)) Use part d) to show \( w = x^2 + y^2 \) contains no \( \theta \).

15) Consider the following double integral:

\[
\int_{-1}^{1} \int_{-1}^{-y^2} f \, dx \, dy
\]

a) Sketch the region \( D \).
   b) Write \( D \) as a type two region.
   c) Write the integral as a type two integral.

16) Sketch the region \( D \) in the integral

\[
\int_{-1}^{+1} \int_{0}^{1-x^2} f(x, y) \, dy \, dx
\]

b) Write the limits of integration for \( \int \int_{D} f(x, y) \, dx \, dy \)

17) Write the limits of integration for

\[
\int \int_{D} dx \, dy
\]

where \( D \) is bounded by the curves \( y = 1, \ x = -1, \ y = -x, \ y = \sqrt{x} \)

18) Let \( R = \{(x, y) \mid -1 \leq y \leq 1; \ y^2 \leq x \leq 1\}. \)
   a) Sketch the region \( R \).
   b) Write \( \int \int_{R} f \, dA \) as a type two integral.
   c) Write \( \int \int_{R} f \, dA \) as a type one integral.

19) Let \( R \) be the region bounded by the curves \( y = x + 1, \ y = 1, \ x = -1. \)
   a) Sketch the region \( R \).
   b) Compute the number \( \int \int_{R} y \, dA \).

20) Let \( R = \{(x, y) \mid -1 \leq y \leq 0; \ y^2 \leq x \leq 1\}. \)
   a) Sketch the region \( R \).
   b) Write \( \int \int_{R} f \, dA \) as a type two integral.
   c) Write \( \int \int_{R} f \, dA \) as a type one integral.
21) Let $R$ be the region bounded by the curves $y = x^2 - 1, \ y = x + 1$.
   a) Sketch the region $R$.
   b) Compute the number $\iint_R x \, dA$.

22) Let $R$ be the region bounded by the curves $x = y^{\frac{1}{3}}$ and $y = x^2$. Write the integrals for
   \[ \iint_R f \, dA \]
   in two different ways
   a) As a type one region.
   b) As a type two region.
23) Let \( R = \{(x, y) \mid -1 \leq y \leq 0; y^2 \leq x \leq 1\} \).
   a) Sketch the region \( R \).
   b) Write \( \int \int_R f \, dA \) as a type two integral.
   c) Write \( \int \int_R f \, dA \) as a type one integral.

24) Let \( R \) be the region bounded by the curves \( y = 1, \ x = -1; \ y = -x, \ y = \sqrt{x} \).
    Compute
    \[ \int \int_R x \, dA \]

25) Let \( R \) be the region bounded by
    \( x^2 + y^2 = 1, \ y \geq 0, \ x \leq 0 \). In the integral below,
    \[ \int \int_R x \, dA \]
    a) Sketch the region \( R \).
    b) Use polar co-ordinates to compute \( \int \int_R x \, dA \)

26) Let \( R \) be defined as
    \[ R = \{(x, y) \mid \frac{x^2}{4} + \frac{y^2}{4} \leq 1; \ y \leq 0\} \]
    a) (5 points) Sketch \( R \)
    b) (20 points) Write \( R \) as a type I region.
    c) (10 points) Change to polar co-ordinates to compute the value of
    \[ \int \int_R \sqrt{x^2 + y^2} \, dA \]
2) \( f(x,y) = (x+y)^2 \)

\[
\frac{\partial f}{\partial x} = -2(x+y)^{-3} \frac{\partial}{\partial x} (x+y) = -2(x+y)^{-3} \\
\frac{\partial^2 f}{\partial x^2} = 6(x+y)^{-4} \\
\frac{\partial f}{\partial y} = -2(x+y)^{-3} \quad \text{by symmetry} \\
\frac{\partial^2 f}{\partial y^2} = 6(x+y)^{-4} - \frac{\partial}{\partial y} (x+y) = 6(x+y)^{-4} \\
\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} = 6(x+y)^{-4} - 6(x+y)^{-4} = 0
\]
\[
\frac{Df}{Dx} = \frac{2}{x+u} \left( \frac{y}{x+u} \right) = y \frac{2}{x+u} (x+u)^{-1}
\]

\[
= -y \frac{2}{x+u} (x+u)^{-2} = -y (x+u)^{-2} = \frac{-y}{(x+u)^2}
\]

\[
\frac{D^2 f}{Dx^2} = -y \frac{2}{x+u} \left( \frac{1}{x+u} \right)^2 = -y \left( \frac{2}{x+u} \right) (x+u)^{-2}
\]

\[
= -y (x+u)^{-3} = \frac{2y}{(x+u)^3}
\]

\[
\frac{D^2 f}{DyDx} = \frac{2}{x+u} \left( \frac{-y}{x+u} \right) = - \left[ \frac{\partial^2}{\partial y \partial (x+u)^2} - \frac{\partial}{\partial (x+u)} \left( \frac{2y}{x+u} \right) \right]
\]

\[
= - \left[ \frac{(x+u)^2 - y \frac{2}{x+u} (x+u)}{(x+u)^4} \right]
\]

\[
= -(x+u) \left[ \frac{(x+u)^2 - 2y}{(x+u)^4} \right] = \frac{-(x+u)^3}{(x+u)^4} = \frac{-x}{(x+u)^3}
\]

\[
\frac{Df}{Dy} = \frac{2}{x+u} \left( \frac{y}{x+u} \right) = y \frac{2}{x+u} \left( \frac{1}{x+u} \right)^2
\]

\[
= \frac{(x+u)^2 - y}{(x+u)^2} = \frac{x}{(x+u)^2}
\]
\[
\frac{\partial^2 F}{\partial y^2} = \times \frac{\partial^2}{\partial y^2} (x+y)^2 = -2 \times \frac{\partial}{\partial y} (x+y)^3 \\
= -2 \times \frac{x}{(x+y)^4}
\]

\[
\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial y} \left( \frac{x}{(x+y)^2} \right) = \frac{\partial}{\partial x} \left( \frac{x}{(x+y)^2} \right) = \frac{\partial}{\partial x} \left( \frac{(x+y)^2 - 2x}{(x+y)^4} \right) = \frac{(x+y)^2 - 2x}{(x+y)^4}
\]

\[
= \frac{y-x}{(x+y)^3}
\]

Remark: Note \( \frac{\partial^2 F}{\partial x \partial y} \) \( \frac{\partial^2 F}{\partial y^2} = \frac{\partial^2 F}{\partial x \partial y} \)

for answers to #5, #6, you could try to use Wolfram Alpha!
(4.5)
\[ \frac{df}{\theta} = \frac{df}{dx} \frac{dx}{d\theta} + \frac{df}{dy} \frac{dy}{d\theta} \]

(4.6) \quad \omega = f \quad \frac{df}{dx} \quad \frac{df}{dy}

\[ x = r \cos \theta \sin \phi \quad \frac{dx}{d\theta} = -r \sin \theta \sin \phi \quad \frac{dy}{d\theta} = r \cos \theta \sin \phi \]

\[ y = r \sin \theta \sin \phi \]

\[ \frac{df}{d\theta} = -r \sin \theta \sin \phi \frac{df}{dx} + r \cos \theta \sin \phi \frac{df}{dy} \]

(4.8) \quad \frac{df}{d\theta} = -y \frac{df}{dx} + x \frac{df}{dy}

(4.9) \quad f(x, y, z) = x^2 + y^2

\[ \frac{df}{dx} = 2x \quad \frac{df}{dy} = 2y \]

\[ \frac{df}{d\theta} = -y (2x) + x^2 y = 0 \]

Since \( \frac{df}{d\theta} = 0 \), \( f \) contains no \( \theta \).
15) \[ D = \{(x,y) | -1 \leq y \leq 1, -1 \leq x \leq -y^2\} \]

\[ y = -1, y = 1 \]
\[ x = -1, x = -y^2 \]

\[ \text{This is already a type II region.} \]

\[ \text{This is already a type II integral} \]
\[ \text{(your final won't be this easy :) )} \]
\( D = \{(x, y) \mid -1 \leq x \leq 1; 0 \leq y \leq 1-x^2\} \)

Need first to sketch \( D \)

Type \( \int \int_D \text{ d}x \text{ d}y \) is type II

\( = \{(x, y) \mid c \leq y \leq d; S_l(y) \leq x \leq S_r(y)\} \)

\( c = \text{ lowest } y = 0 \)

\( d = \text{ highest } y = 1 \)

\( x = S_l(y) \text{ left most curve } = -\sqrt{1-y} \)

\( x = S_r(y) \text{ right most curve } = \sqrt{1-y} \)

\( y = 1-x^2 \) so \( x^2 = 1-y \) so

\( x = -\sqrt{1-y} \quad y = -\sqrt{1-y} \)

\( +^\text{ 1st quadrant} \quad -^\text{ 2nd quadrant} \)

right most \quad left most

\[ \int_0^1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x, y) \, dx \, dy \]
If you mean you can do I or II your choice!

Type I: 
- $a =$ left most point  \[ x = -1 \]
- $b =$ right most point \[ x = 0 \]

$y = g_B(x)$ lower curve \[ y = x + 1 \]
$y = g_T(x)$ top most curve \[ y = 1 \]

\[
\int_{-1}^{1} \int_{x+1}^{y} y \, dy \, dx
\]

Inside \[
\int_{x+1}^{y} y \, dy = \left[ \frac{y^2}{2} \right]_{x+1}^{y} = \frac{1}{2} - \frac{(x+1)^2}{2}
\]
\[
= \frac{1}{2} \left[ 1 - (x+1)^2 \right] = \frac{1}{2} \left[ 1 - x^2 - 2x - 1 \right]
\]
\[
= -\frac{1}{2} [x^2 + 2x]
\]

Outside \[
\int_{-1}^{0} -\frac{1}{2} (x^3 + 2x) \, dx = -\frac{1}{2} \left[ \frac{x^3}{3} + x^2 \right]_{-1}^{0}
\]
\[
= +\frac{1}{2} \left[ -\frac{1}{2} + 1 \right] = \left( \frac{1}{2} \right) \left( \frac{1}{3} \right) = \left( \frac{1}{6} \right)
\]
21) Need region R; sketch first

\[ x = y^{1/3} \rightarrow y = x^3 \]
\[ y = x^2 \]

Region is bounded by two dots. Where are they?

\[ x^3 = y^2 \text{ so } x^3 - y^2 = 0 \text{ so } x^2 (x-1) = 0 \]
So \( x = 0, x = 1 \)
\[ y = 0, y = 1 \]
\((0,0) \text{ and } (1,1)\)

2c) Type 1 region:  \( a = \text{leftmost } x = 0 \)
\[ 5 = \text{right most } x = 1 \]
\[ y = S_B(x) \text{ bottom } \quad y = x^3 \]
\[ y = S_T(x) \text{ top } \quad y = x^2 \]

\[ \int_0^1 \int_{x^3}^{x^2} f \, dy \, dx \]

Type II

\[ 0 \quad 1 \]
\[ 0 \quad 1 \]
\[ S_L = y = x^2 : x = \sqrt{y} \]
\[ S_R = x = y^{1/3} \]
\[ c = \text{bottom } y = 0 \]
\[ d = \text{top } y = 1 \]
23c) \( y = 0, y = -1 \); \( x = y^2, x = 1 \)

5) Type II

\[ c = -1 \text{ from } x = y^2 \]
\[ d = 0 \text{ from } y = 0 \]
\[ x = s_L = y^2 \text{ from } x = y^2 \]
\[ x = s_R = 1 \text{ from } x = 1 \]

\[ \int_{-1}^{0} \int_{y^2}^{1} f(x, y) \, dxdy \]
23c) Type I

\[ a \]
\[ \text{left most point} \]
\[ x = a^2 \]
\[ y = 0 \]
\[ x = 0 \]

\[ b \]
\[ \text{right most point from equation } x = 1 \]

\[ S_b = \text{bottom curve from } x = a^2 \]
\[ y = \pm \sqrt{x} \]
\[ \text{but we are in fourth quadrant so } y = -\sqrt{x} \]

\[ S_t = \text{top curve from equation } y = 0 \]

\[ \int \int_{-\sqrt{x}}^{0} f \, dy \, dx \]
25 o)

\[ x^2 + y^2 = 1 \]

circle

(b) Polar region is \( 0 \leq r \leq 1 \) to be inside circle.

\[ \frac{\pi}{2} \leq \theta \leq \pi \] to be in 2nd quadrant.

\[
\iint_{R} x \, dA = \int_{0}^{\pi/2} \int_{0}^{1} r \cos \theta \, dr \, d\theta
\]

\[
= \left( \int_{0}^{1} r^2 \, dr \right) \left( \int_{0}^{\pi/2} \cos \theta \, d\theta \right)
\]

\[
= \left( \left[ \frac{r^3}{3} \right]_{0}^{1} \right) \left( \left[ \sin \theta \right]_{0}^{\pi/2} \right)
\]

\[
= \left( \frac{1}{3} \right) \left( \sin \frac{\pi}{2} - \sin 0 \right)
\]

\[
= \left( \frac{1}{3} \right) \left( 1 \right) = \frac{1}{3}
\]

\[- \frac{1}{3} \]
(a) \[ \frac{x^2}{4} + \frac{y^2}{4} \leq 1 \rightarrow \quad x^2 + y^2 \leq 4 \]
Circle radius 2
\[ 0 \leq y \leq 2 \quad 3^{rd} \text{ and } 4^{th} \text{ quadrant} \]
\[ b = \text{rightmost } x = 2 \]
\[ a = \text{leftmost } x = -2 \]
\[ S_B = \text{bottom curve} \]
\[ x^2 + y^2 = 4 \quad y = \pm \sqrt{4-x^2} \]
\[ 5 \pi + \pi \leq \theta \leq 2 \pi \quad \theta = -\sqrt{4-x^2} \]
\[ S_T = \text{top curve } y = 0 \quad \text{from } y \leq 0 \]

\[
\int_{-2}^{2} \int_{0}^{\sqrt{4-y^2}} f(x, y) \, dx \, dy
\]

(b) Circle radius 2: \[ 0 \leq r \leq 2 \]
3^{rd} and 4^{th} quadrants \[ \pi \leq \theta \leq 2\pi \]
\[ \sqrt{x^2+y^2} = \sqrt{r^2} = r \]

\[
\int_{0}^{\pi} \int_{0}^{2r} r \, r \, d\theta \, dr = \left( \int_{0}^{2} r^2 \, dr \right) \left( \frac{2\pi}{\pi} \right)
\]
\[ = \left( \frac{r^3}{3} \right)_{0}^{2} \left( \frac{\theta}{\pi} \right)_{0}^{2\pi}
\]
\[ = \left( \frac{8}{3} \right)(\pi) = \frac{8\pi}{3} \]