1) (20 points) Let \( z = f(x, y); x = u + v, y = u - v. \)

a) Draw the tree diagram for these variables.

b) Without knowing \( f \), state the chain rule for \( \frac{\partial f}{\partial v}. \)

c) Now take \( f(x, y) = (x + y)^2 \) and use b) to show \( f \) has no \( v \) in it.

\[
\begin{align*}
\frac{\partial f}{\partial v} &= 2(x + y) \frac{\partial x}{\partial v} + 2(x + y) \frac{\partial y}{\partial v} \\
\frac{\partial f}{\partial u} &= 2(x + y) \frac{\partial x}{\partial u} \quad \frac{\partial f}{\partial v} = -1 \\
\frac{\partial f}{\partial u} &= 2(x + y) \quad \frac{\partial f}{\partial v} = 0 \\
\text{since the change is zero, } f \text{ has no } v
\end{align*}
\]
2) (40 points) Let $D = \{(x, y) \mid x^2 + y^2 \leq 1; \ x \leq 0\}$
   a) Sketch $D$; use about an eighth of the page.
   b) Write $D$ as a type II domain.
   c) Use polar co-ordinates to compute:

   \[\int_D \int \sqrt{1-x^2-y^2} \, dA\]

   \[
   \int_{\pi/2}^{3\pi/2} \int_0^1 \frac{r}{\sqrt{1-r^2}} r \, dr \, d\theta
   \]

   Inside:
   \[
   \int_0^1 \frac{r}{\sqrt{1-r^2}} r \, dr = \frac{1}{2} \int_0^1 \sqrt{u} \, du = \frac{1}{2} \left[ \frac{2}{3} u^{3/2} \right]_0^1 = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}
   \]

   Outside:
   \[
   \int_{\pi/2}^{3\pi/2} \frac{1}{2} \, d\theta = \left[ \frac{1}{2} \theta \right]_{\pi/2}^{3\pi/2} = \left[ \frac{1}{2} \theta \right]_{\pi/2}^{3\pi/2} = \frac{1}{2} \pi
   \]
3) (40 points) Let \( z = f(x, y) = \sqrt{1 - (x^2 + y^2)} \), and \( P = P(\frac{1}{2}, 0) \).

a) Sketch the surface. **Use half the page.**

b) On the graph in a), sketch the location of \( P \) and \( f(P) \).

c) Sketch the trace \( z = f(\frac{1}{2}, y) \), in the \( yz \) plane and on the surface in a).

\[
\sqrt{2/4} = \sqrt{1/2} \approx 0.86
\]

b) \( P = \frac{1}{2}, 0 \) \( f(P) = f(\frac{1}{2}, 0) = \sqrt{1 - \frac{1}{4}} = \sqrt{3/4} \approx 0.86 \)

c) \( f(\frac{1}{2}, y) = \sqrt{2/4 - y^2} \)

\[\max \text{ at } y = 0\]

ii) See a)
1) (20 points) Let \( z = f(x, y); x = r \cosh(s), \ y = r \sinh(s) \).

a) Draw the tree diagram for these variables.

b) Without knowing \( f \), state the chain rule for \( \frac{\partial f}{\partial r} \).

c) Now take \( f(x, y) = \frac{x}{y} \) and use b) to show \( f \) has no \( r \) in it.

\[
\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r}
\]

\[
f = \frac{r}{s}, \quad \frac{\partial f}{\partial x} = \frac{1}{y}, \quad \frac{\partial f}{\partial r} = \cosh s
\]

\[
\frac{\partial f}{\partial y} = -\frac{x}{y^2}, \quad \frac{\partial f}{\partial r} = \sinh s
\]

\[
\frac{\partial f}{\partial x} = \frac{1}{y} \cosh s - \frac{x}{y^2} \sinh s = \frac{\cosh s}{\sqrt{s}} - \frac{r \cosh s}{r^2 \sinh s}
\]

\[
= \frac{1}{r} \left( \tanh s - \coth s \right) = 0
\]

Since the denominator is zero, \( f \) has no \( r \).
2) (40 points) Let \( D = \{(x, y) \mid x^2 + y^2 \leq 1; \ x \geq 0, \ y \leq 0\} \)

a) Sketch \( D \); use about an eighth of the page.

b) Write \( D \) as a type II domain.

c) Use polar co-ordinates to compute:

\[
\int_D \int \sqrt{x^2 + y^2} \, dA
\]

\[\text{a) } \]

\[\text{b) } \]

\[
D = \{(x, y) \mid -1 \leq y \leq 0, \quad 0 \leq x \leq \sqrt{1-y^2}\}
\]

\[\text{c) } \]

\[
\iint_D \sqrt{x^2 + y^2} \, dA = \int_0^{\sqrt{1/2}} \int_0^{2\pi} \sqrt{r^2} \, r \, dr \, d\theta
\]

\[
= \int_0^{\sqrt{1/2}} r^2 \, d\theta \Bigg|_{\theta = 0}^{\theta = 2\pi} = \frac{\pi}{2} r^2
\]

Inside

\[
\int_{2\pi/2}^{2\pi} r \, r \, d\theta = r^2 \left[ \theta \right]_{2\pi/2}^{2\pi} = \frac{\pi}{2} r^2
\]

Outside

\[
\int_0^{\pi/2} r^2 \, dr = \left( \frac{r^3}{2} \right) \bigg|_0^{\pi/2} = \frac{\pi}{8} \cdot \frac{1}{2} - 0 = \frac{\pi}{16}
\]
3) (40 points) Let \( z = f(x, y) = 1 - (x^2 + y^2) \), and \( P = P(\frac{1}{2}, 0) \)

a) Sketch the surface. **Use half the page.**

b) On the graph in a), sketch the location of \( P \) and \( f(P) \).

c) Sketch the trace \( z = f(\frac{1}{2}, y) \), in the \( yz \) plane and on the surface in a).

\[ f(P) = 1 - \left( \frac{1}{2} \right)^2 - 0^2 = \frac{3}{4} \]

\[ c) b = f(\frac{1}{2}, y) = \frac{3}{4} - y^2 \]

\[ c) c \]
1) (20 points) Let \( z = f(x, y); x = r \cos(\theta) \sin(\phi), \ y = r \sin(\theta) \sin(\phi) \).

a) Draw the tree diagram for these variables.

b) Without knowing \( f \), state the chain rule for \( \frac{\partial f}{\partial \theta} \).

c) Now take \( f(x, y) = x^2 + y^2 \) and use b) to show \( f \) has no \( \theta \) in it.

\[
\begin{align*}
    \frac{\partial f}{\partial \phi} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \phi} \\
    \frac{\partial f}{\partial x} &= 2x \\
    \frac{\partial f}{\partial y} &= 2y \\
    \frac{\partial f}{\partial \theta} &= -r \sin \theta \sin \phi \\
    \frac{\partial f}{\partial r} &= -2x \sin \theta \cos \phi + 2y \cos \theta \sin \phi \\
    &= 2r \sin \phi \left[ -x \sin \theta + y \cos \theta \right] \\
    &= 2r \sin \phi \left[ -x \cos \phi \sin \theta + r \sin \phi \cos \theta \right] \\
    &= 2r \sin \phi \left[ 1 \right] = 0
\end{align*}
\]

Since the partial is zero, \( f \) has no \( \theta \).
2) 40 points. Let \( D = \{(x, y) \mid x^2 + y^2 \leq 1; \ y \leq x, \ x \geq 0\} \)

a) Sketch \( D \); use about an eighth of the page.

b) Write \( D \) as a type II domain.

c) Use polar co-ordinates to compute:

\[
\int_D \int (x^2 + y^2) \, dA
\]

a) 

b) \( D = \{ (x, y) \mid 0 \leq y \leq \frac{1}{\sqrt{2}} \} \quad y \leq x \leq \sqrt{1-y^2} \}

C) \[
\int_0^{\pi/4} \int_0^1 r^2 \, r \, dr \, d\theta
\]

Inside \[
\int_0^{\pi/4} \int_0^1 r^3 \, dr = \int_0^{\pi/4} \left[ \frac{r^4}{4} \right]_0^1 = \frac{1}{4}
\]

Outside \[
\int_0^{\pi/4} \int_0^1 \frac{1}{u} \, dc = \frac{1}{u} \int_0^{\pi/4} \, dc = \frac{1}{u} \cdot \pi
\]

\( \int_0^{\pi/4} \)
3) (40 points) Let \( z = f(x, y) = 1 - (x^2 + y^2) \), and \( P = P(0, \frac{1}{2}) \)

a) Sketch the surface. **Use half the page.**

b) On the graph in a), sketch the location of \( P \) and \( f(P) \).

c) Sketch the trace \( z = f(x, \frac{1}{2}) \), in the \( zx \) plane and on the surface in a).

b) \( P = P(0, \frac{1}{2}) \)  \( f(P) = 1 - 0^2 - \frac{1}{4} = \frac{3}{4} \)

c) \( z = f(x, \frac{1}{2}) = 1 - x^2 - \frac{1}{4} = \frac{3}{4} - x^2 \)

\( \text{hence at } x = 0 \)

(1) **See a)**