1) (35 points) Let \( w = f(x, y, z); x = \cos \theta \cos \phi, y = \cos \theta \sin \phi, z = \sin \theta \).

a) Draw the tree diagram for these variables.

b) For any \( f \), state the chain rule for \( \frac{\partial f}{\partial \phi} \).

c) Compute the values in b)

d) Eliminate the \( \theta, \phi \) in terms of \( x \) and \( y \)
e) Now take \( f(x, y) = 1/(x^2 + y^2) \) and use d) to show \( f \) has no \( \phi \) in it.

\[ f = -y \]

\[ \frac{\partial f}{\partial x} = -2 \frac{x}{(x^2 + y^2)^2} \]

\[ \frac{\partial f}{\partial y} = -2y \frac{y}{(x^2 + y^2)^2} \]

\[ -y \frac{\partial f}{\partial x} + x \frac{\partial f}{\partial y} = \frac{x^2 y - 2x y}{(x^2 + y^2)^2} \]

\[ = 0 \]
2) (35 points) Let \( D = \{ (x, y) \mid x^2 + y^2 \leq 1; \ x \geq y \geq 0 \} \)

a) Sketch \( D \); use about an eighth of the page.

b) Write \( D \) as a type II domain.

c) Use polar co-ordinates to compute:

\[
\int \int_D \frac{1}{\sqrt{x^2 + y^2}} \, dA
\]

\[
\begin{align*}
(1) \quad & \qquad = \\
(2) \quad D & = \{ (x, y) \mid 0 \leq y \leq \sqrt{2}; \quad y \leq x \leq \sqrt{1 - y^2} \} \\
(3) \quad D & = \{ (r, \theta) \mid 0 \leq r \leq \sqrt{2}; \quad 0 \leq \theta \leq \pi/4 \} \quad \text{so} \quad x^2 + y^2 = r^2 \quad \text{and} \quad \sqrt{x^2 + y^2} = r \\
\int \int_D f \, dA & = \int_0^{\pi/4} \int_0^{\sqrt{2}} \frac{1}{r} r \, d\theta \, dr = \left( \int_0^{\sqrt{2}} r \, dr \right) \left( \int_0^{\pi/4} d\theta \right) \\
& = \pi/4
\end{align*}
\]
1) (35 points) Let \( z = f(x, y); x = r \cosh(t), \ y = r \sinh(t) \).

a) Draw the tree diagram for these variables.

b) For any \( f \), state the chain rule for \( \frac{\partial f}{\partial t} \).

c) Compute the values in b)

d) Eliminate the \( r, t \) in terms of \( x \) and \( y \) e) Now take \( f(x, y) = \ln(y^2 - x^2) \) and use d) to show \( f \) has no \( ti \) in it.

\[
\frac{\partial r}{\partial t} = x, \quad \frac{\partial y}{\partial t} = r \cosh(t)
\]

\[
\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}
\]

\[
= \frac{\partial f}{\partial x} x + \frac{\partial f}{\partial y} r \cosh(t)
\]

\[
\frac{\partial f}{\partial x} = \frac{1}{y^2 - x^2} \frac{\partial}{\partial x} (y^2 - x^2) = \frac{2x}{y^2 - x^2}
\]

\[
\frac{\partial f}{\partial y} = \frac{1}{y^2 - x^2} \frac{\partial}{\partial y} (y^2 - x^2) = \frac{2y}{y^2 - x^2}
\]

\[
\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = \frac{y(-2x)}{y^2 - x^2} + \frac{x(2y)}{y^2 - x^2}
\]

\[
= \frac{-2x + 2yx}{y^2 - x^2} = C
\]
2) (35 points) Let \( D = \{(x, y) \mid x^2 + y^2 \leq 4\} \)

a) Sketch \( D \); use about an eighth of the page.

b) Write \( D \) as a type two domain.

c) Use polar co-ordinates to compute:

\[
\iint_D \sqrt{x^2 + y^2} \, dA = \iint_D \sqrt{x^2 + y^2} \, dA
\]

\[
= \left( \int_0^2 r^2 \, dr \right) \left( \int_0^{2\pi} \, d\theta \right) = \left[ \frac{r^3}{3} \right]_0^2 \int_0^{2\pi} \, d\theta
\]

\[
= \left( \frac{8}{3} \right) (2\pi) = \frac{16\pi}{3}
\]
1) (35 points) Let \( f(x, y) = \frac{x-y}{x+y} \). Check whether \( f \) is a solution to the PDE:

\[
\frac{\partial^2 f}{\partial x^2} + y \frac{\partial^2 f}{\partial y^2} = 0
\]

\[
\frac{\partial^2 f}{\partial x^2} = \frac{2}{(x+y)^2} \left( (x-y) (x+y) + (x-y) \frac{\partial}{\partial x} (x+y) \right)
\]

\[
= \frac{2y}{(x+y)^2} = 2y (x+y)^{-2}
\]

\[
\frac{\partial^2 f}{\partial y^2} = 2y (-2)(x+y)^{-3} (1) = \frac{-4y}{(x+y)^3}
\]

\[
\frac{\partial^2 f}{\partial y \partial x} = \frac{2}{(x+y)^2} \left( (x-y) (x+y) - (x-y) \frac{\partial}{\partial y} (x+y) \right)
\]

\[
= \frac{-2x}{(x+y)^2} = -2x (x+y)^{-2}
\]

\[
\frac{\partial^2 f}{\partial y^2} = -2x (x-2) (x+y)^{-3} (1) = \frac{-4x}{(x+y)^2}
\]

\[
x \frac{\partial^2 f}{\partial x^2} + y \frac{\partial^2 f}{\partial y^2} = \frac{-4xy}{(x+y)^3} + \frac{4xy}{(x+y)^3} = 0
\]
2) (35 points) Let \( D = \{ (x, y) \mid x^2 + y^2 \leq \frac{1}{4}, y \leq 0 \} \)
   
a) Sketch \( D \); use about an eighth of the page.
   b) Write \( D \) as a type two domain.
   c) Use polar co-ordinates to compute:

\[
\int_D \int xy \, dA
\]

\( D = \{ (x, y) \mid -1 \leq y \leq 0, -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2} \} \)

\( D = \{ (r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi \} \).

\[
xy = (r \cos \theta)(r \sin \theta) = r^2 \cos \theta \sin \theta.
\]

\[
\iint_D f \, dA = \int_0^{2\pi} \int_0^1 r^2 \cos \theta \sin \theta \, r \, dr \, d\theta
\]

\[
= \left( \int_0^{2\pi} \cos \theta \sin \theta \, d\theta \right) \left( \int_0^1 r^3 \, dr \right)
\]

\[
= \left( \int_0^{2\pi} u \, du \right) \left( \int_0^1 \frac{r^4}{4} \, dr \right)
\]

\[
= \left( \left. u \right|_0^{2\pi} \right) \left( \left. \frac{r^5}{20} \right|_0^1 \right) = \left. 0 \right|_0^{2\pi} = 0
\]