Series: Root and Ratio Test

Background: look at $ \sum \frac{1}{k^r}$. Since it's a sum, we can do it by limit comparison; BCT would work too. Of course, only LCT would work on $\sum \frac{1}{e^{k}}$.

Now let's choose it: $\sum \frac{1}{k!}$. Now, since it's a product, LCT doesn't work. But we think this is like $\sum \frac{1}{k!} = \sum \frac{1}{(k)}$ comes from series $|ab| < 1$.

So we need the product of BCI: $a_k \leq c_k$ or $\frac{1}{k!} \leq \frac{1}{k^n}$ or $k \leq 1$ or $k < 1$, yes, $k > 1$ yes $k \leq 0$.

Then, now choose it again $\sum \frac{1}{k!}$. Again, LCT is not likely to work; try BCT: $\frac{1}{k^n} \leq \frac{1}{k^n}$ NO $k \leq 0$ $k > 1$ yes

So LCT and BCT both fail.

(But I need us a test that goes directly from $\sum \frac{1}{k^n}$ to $|ab| < 1$. There are actually two such tests.

1st test: Root Test compute $\lim_{k \to \infty} |a_k|^{\frac{1}{k}} = R$

- If $R < 1$ $\sum$ conv
- If $R > 1$ $\sum$ div
- If $R = 1$ test fails.

To do this, I'll need tricks about $1^k$ and $1^k$.
1) \( |z|^k = 1 \) if \( a > 0 \), \( |a|^k = a \)

2) \( |a|b|^k = |a|^{\frac{k}{2}} |b|^\frac{k}{2} \)

3) \( |a| + |b|^k \neq |a|^{\frac{k}{2}} + |b|^{\frac{k}{2}} \) (Do not do this)

4) \( |a|^k |b|^k = |a| \)

5) \( \lim_{k \to 0} k^{-\frac{k}{2}} = 1 \)

6) If \( C \) is a constant, (no k limit)

\( \lim_{k \to 0} |c|^k = 1 \)

Now let's do \( \sum \frac{k}{e^k} \).

\[ \left| \frac{k}{e^k} \right|^k = \left( \frac{k}{e^k} \right)^k \text{ since } k \geq 0 \]

\[ = \frac{k^k}{(e^k)^k} = \frac{k^k}{e} \]

\[ \lim_{k \to \infty} |c|^k = \lim_{k \to \infty} \frac{k^k}{e} = \frac{1}{e} \]

\( e = \frac{1}{e} < 1 \) so \( \sum c_k \) converges by root test.
How do I know when to use root test

a) you see \( C^k \)
b) it isn't geometrically

c) there aren't any sums.

Easy example: \( \sum \frac{(-1)^k k^2}{(3k!)} \)

\[
\lim_{k \to \infty} \left| a_k \right|^\frac{1}{k} = \lim_{k \to \infty} \frac{(-1)^k k^2}{(3k!)^k} \cdot \frac{1}{k}
\]

\[
= \frac{(-1)^k}{(3k!)^{\frac{1}{k}}} \cdot \frac{k^2}{k} = \frac{(-1)^k}{(3k!)^{\frac{1}{k}}} \cdot k
\]

Since \( k^2 > 0 \) and \( 3k! > 0 \)

\[
\frac{k^2}{(3k!)^{\frac{1}{k}}} = \frac{\left( \frac{k^2}{3k!} \right)^k}{(3k!)^{\frac{1}{k}}}
\]

\[
= \frac{\left( \frac{k}{3} \right)^k}{(3k!)^{\frac{1}{k}}}
\]

Since \( (a^b)^c = a^{bc} = (a^c)^b \)

\[
\Rightarrow \quad \text{can switch the } 2 \text{ out}
\]

to outside.

\[
\lim_{k \to \infty} \frac{|a_k|^\frac{1}{k}}{l_k} = \lim_{k \to \infty} \frac{(\frac{k^2}{3k!})^k}{l_k} = \frac{12}{\infty} = 0
\]

\[ L = \frac{1}{2} < 1 \] so \( \sum a_k \) converges by root,

When to not use root: When you have just \( k^5, k^2, k^3 \) --- watch this ---
we'll try using root on

\[ \sum \frac{1}{k} \] and \( \sum \frac{1}{k^2} \)
\[ a_n^{1/n} = \frac{1}{k^{1/n}} = \frac{1}{e^{k^{1/n}}} \]

\[ \lim_{k \to \infty} \sqrt[n]{c_k} = \frac{1}{1} = 1 \]

So \( R = 1 \), and we'll use \( R = 1 \) for \( \sum k^{1/k} \). Which can't have \( R = 1 \) for \( \sum k^{1/k} \) which can.

So it looks like root test can tell the difference between \( \sum k^{1/k} \) and \( \sum k^{1/k} \). That's why we say the test fails!

How to use root test with sums:

1) Try using algebra to eliminate the sum

2) Try using another test like limit comp to eliminate the sum

Examples:

\[ \sum \frac{k+1}{e^k} = \sum \frac{k}{e^k} + \sum \frac{1}{e^k} \]

Use root here

Use limit comp here.
This will do some of the sums in numeracy, except maybe $\sum \frac{\sqrt{k^2+k}}{e^k}$.

So first I'll use L'Hopital to make this into $\frac{\sqrt{k^2}}{e^k}$

$\sum \frac{k}{e^k}$, then root to finish it off.

$\lim_{k \to \infty} \frac{\sqrt{k^2+k}}{e^k} = \lim_{k \to \infty} \frac{\frac{k^2+1}{2\sqrt{k^2}}}{e^k} = \lim_{k \to \infty} \frac{\sqrt{1+k}}{\sqrt{k}} = \sqrt{1} = 1$.

$L = 1$ so $L < 0$ so $\sum \frac{\sqrt{k^2+k}}{e^k}$ and $\sum \frac{\sqrt{k}}{e^k}$ both converge.

Now use root to show $\sum \frac{\sqrt{k}}{e^k}$ converges, so by L'Hopital, $\sum \frac{\sqrt{k^2+k}}{e^k}$ also converges.

People just love to avoid this! Here's a great favorite:

$\sum \frac{\sqrt{k^2+k}}{\sqrt{k^2} e k}$. 
one thing they can do is:

\[
\int \frac{3^k + 4^k}{1 + 5^k} \, dk = \frac{(2^k)^k + (4^k)^k}{\frac{1}{5}(5^k)^k}
\]

\[10 + 51^k \neq a^k + s^k\]

But let's see where the zeros: \( b = \frac{3 + 4}{1 + 5} = \frac{7}{6} \)

\( 2 = \frac{7}{6} > 1 \) so 1 can only by root

except it doesn't. Two ways

1. \( \lim_{k \to \infty} \sum \frac{2^k + 4^k}{1 + 5^k} \) is like \( \sum \frac{a^k}{5^k} = \sum \left(\frac{2}{5}\right)^k \)

convergent series \( |\frac{2}{5}| < 1 \).

\[
\lim_{k \to \infty} \frac{2^k + 4^k}{1 + 5^k} = \lim_{k \to \infty} \frac{\frac{2^k}{5^k} + \frac{4^k}{5^k}}{\frac{1 + 5^k}{5^k}}
\]

\[
= \lim_{k \to \infty} \frac{\left(\frac{2}{5}\right)^k + 1}{\left(\frac{5}{5}\right)^k + 1} = \frac{0 + 1}{1 + 1} = 1 \text{ when } |\frac{2}{5}| < 1
\]

\( \therefore \) \( 0 < a < 0 \) so \( \sum a_k \) conv by LCT \( a/\sum (\frac{2}{5})^k \)
you could also write this as
\[
\sum_{k=1}^{\frac{3}{5}} \frac{3^k}{1+5^k} + \sum_{k=1}^{\frac{4}{5}} \frac{4^k}{1+5^k} \quad \text{and do a BCT on each of these, since}
\]
\[
\frac{3^k}{1+5^k} \leq \frac{1}{5^k} \quad \text{divide by } 3^k
\]
\[
\frac{1}{1+5^k} \leq \frac{1}{5^k} \quad \text{cross multiply}
\]
\[
5^k \leq 1+5^k \quad \text{constant}
\]
\[
\Rightarrow \quad \text{why, yeah!}
\]

hence another thing people love to try
\[
\left(3^k+4^k\right)^\frac{1}{k} = \left[4^k \left(1+\frac{3^k}{4^k}\right)\right]^\frac{1}{k}
\]
\[
= \left(4^k\right)^\frac{1}{k} \left[1+\left(\frac{3}{4}\right)^k\right]^\frac{1}{k}
\]
\[
= 4 \left[1+\left(\frac{3}{4}\right)^k\right]^\frac{1}{k}
\]

and now \(\lim_{k \to \infty} \left(3^k+4^k\right)^\frac{1}{k}\)
\[
= \lim_{k \to \infty} \left[4 \left(1+\left(\frac{3}{4}\right)^k\right)^\frac{1}{k}\right]
\]
\[
= 4 \left(1+0\right)^0 = 4 \cdot 1 = 4.
\]

except somehow I don’t remember
\[
\lim_{k \to \infty} \left(1+\left(\frac{3}{4}\right)^k\right)^\frac{1}{k} \quad \text{Jelly o de term note form!}
\]
The idea here is
"I can make up anything I want because it's true."

Yeah, sure you can. Remember the
\[ \frac{(1+x)^x}{x^x} \approx \frac{x^x}{x^x} = 1 \] except it isn't true.

There's another way to take \[ \sum \frac{b_k}{a_k} \] and set
\[ \frac{1}{|b|} \leq 1 \] called the ratio test.

**Ratio Test:** Compute \[ \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = R \]

If \[ R < 1 \] \[ \sum a_k \] converges, if \[ R > 1 \] \[ \sum a_k \] diverges.
If \[ R = 1 \] test cofactors.

You can see the idea in a geometric series:
\[ \sum a_k \] where \[ a_k = r^k \]
\[ \frac{a_{k+1}}{a_k} = \frac{r^{k+1}}{r^k} = \frac{r^{k+1}}{r^k} = r = r \]
So \[ \left| \frac{a_{k+1}}{a_k} \right| = |r| \] and geometric series says:
\[ \sum r^k \] converges if \( |r| < 1 \) and diverges if \( |r| > 1 \).
So let's use it on $\sum \frac{k}{e^k}$

\[
\left| \frac{c_{k+1}}{c_k} \right| = \frac{c_{k+1}}{c_k} \quad \text{since these terms are positive}
\]

\[
= \frac{\ln(k+1)}{e^{k+1}} - \frac{1 + \frac{1}{k}}{e^k} \rightarrow \frac{1}{e}
\]

So $\lim_{k \to \infty} \left| \frac{c_{k+1}}{c_k} \right| = \ln - \frac{1 + \frac{1}{k}}{e} \rightarrow \frac{1}{e}$

$\sum \frac{k^2}{e^k}$

\[
\frac{c_{k+1}}{c_k} = \frac{(k+1)^2}{e^{k+1}} - \frac{k^2 + 1}{e^k} = \frac{c_{k+1}}{c_k} \frac{1 + \frac{1}{k} + \frac{1}{k^2}}{e^k}
\]

\[
= \frac{(1 + \frac{1}{k})^2 + \frac{1}{k}}{1 + \frac{1}{k}} \cdot \frac{1}{e}
\]

$\sum \ln \left( \frac{1 + \frac{1}{k} + \frac{1}{k^2}}{1 + \frac{1}{k}} \right) \cdot \frac{1}{e} = (1 + O)^2 + \frac{1}{e}$

$= \frac{1}{e}$
and this would have worked just as easily with\[ \sum \frac{\sqrt{h_n}}{ek_n}. \]

So there's a great deal of overlap between root and ratio. But there's one place ratio really excels and root flops: factorials

\[
(k+1)! = \frac{(k+2)k!}{k+1},
\]

Similarly, \((k+2)! = \frac{(k+3)(k+2)!}{k+2}\).

Except does \[ \sum \frac{k^2e^k}{(k!)^2} \] come as dw?  

\[ \frac{|a_{kn}|}{a_{m}} = \frac{\frac{(k+2)^2 e^{k+1}}{(k+1)^2}}{\frac{k^2 e^k}{(k!)^2}} = \frac{(1+\frac{1}{m})^2 e}{[\frac{k e^k}{k!}]^2} = \frac{(1+\frac{1}{m})^2 e}{(k+1)^2}. \]

So \[ \frac{e^m}{k=0} \frac{a_{kn}}{a_{m}} \]

is \[ \frac{\ln}{k-0} \frac{\frac{k^2 e^k}{\omega^2}}{=0} \]

\[ R = 0 < 1 \] so \[ \sum e_k \] converges by ratio.