We have the root test for $(c)^k$ problems—
tor example $\sum \frac{(\ln k)^k}{k^3}$. Very $(c)^k$

So $|a_k|^{\frac{1}{k}} = \left[ \frac{(\ln k)^k}{k^3} \right]^{\frac{1}{k}} = \frac{\ln k}{k^{\frac{1}{k}} 3}$

So $\lim_{k \to \infty} |a_k|^{\frac{1}{k}} = \lim_{k \to \infty} \frac{\ln k}{k^{\frac{1}{k}} 3} = \frac{\infty}{\infty}$

$R = \infty > 1$ so $\sum a_k$ diverges by root.

But—now try $\sum \frac{\ln k}{2k}$

and $|a_k|^{\frac{1}{k}} = \ln \left[ \frac{\ln (k+1)^{\frac{1}{2}}}{2k} \right]^{\frac{1}{k}}$

$= \left[ \frac{\ln (k+1)}{2k} \right]^{\frac{1}{k}}$ But $[k+1]^{\frac{1}{k}}$ doesn't simplify algebraically—and we have to use $\ln$ and

I have to deal with it. So—another test:

**Ratio Test**

Compute $\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = R$

If $R < 1$, $\sum a_k$ conv
If $R > 1$, $\sum a_k$ div.
If $R = 1$, $\sum a_k$ needs a totally different test—this one fails.
Let's start by computing some ratios:

\[
\frac{a_k}{a_{k+1}} = \frac{k+1}{k} = 1 + \frac{1}{k}
\]

\[
\frac{a_k}{\alpha_k} = \frac{3^k}{3^k} = \frac{3^{k+1}}{3^k} = 3
\]

\[
\frac{a_k}{\alpha_k} = \frac{(k+1)!}{k!} = k+1
\]

Now something complicated:

\[
\frac{a_{k+1}}{a_k} = \frac{(k+1)^k}{k^k} = \left(1 + \frac{1}{k}\right)^k = e^k
\]

So first a very easy example:

\[
\sum (-1)^k \frac{3^k}{k!}
\]

This is:

\[
\frac{3^{k+1}}{(k+1)!} = \frac{3^{k+1}}{3^k \cdot k!} = \frac{3}{k+1}
\]

So:

\[
\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} \frac{3}{k+1} = 0
\]

\[
\Rightarrow 0 < 1 \Rightarrow \sum a_k \text{ converges by ratio}
\]
By the way—this is a general thing: if a problem has $k!$ you have to use factorials (almost always).

Now back to the problem where root didn't work very well: $\sum \frac{\sqrt{k+1}}{3^k}$.

$$\left| \frac{a_{k+1}}{a_k} \right| = \frac{\sqrt{k+2}}{3^{k+1}} = \frac{\sqrt{k+2}}{\sqrt{k+1}} \cdot \frac{3}{3} = \frac{\sqrt{k+2}}{3}$$

$$= \sqrt{\frac{1+\frac{2}{k}}{1+\frac{1}{k}}}$$

so $\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = \lim_{k \to \infty} \sqrt{\frac{1+\frac{2}{k}}{1+\frac{1}{k}}} = \frac{1}{3}$

$R = \frac{1}{3} < 1$ so $\sum a_k$ converges by ratio.

Now—both tests: root and ratio fail (when $R=1$, why is that? after all).

If you have a geometric series:

$$\sum r^k = \begin{cases} \text{converge} & |r| < 1 \\ \text{diverge} & |r| > 1 \\ \text{diverge} & |r| = 1 \end{cases}$$

so why don't root and ratio also diverge when $R = 1$?
The key is what S says "geometric series".

If $\sum a_k$ doesn't have geometric series, then's no reason to expect them to work. Here's two.

\[
\sum \frac{1}{k^2} \text{ converges}
\]

\[
\frac{1}{k^2} = \left| \frac{c_{k+1}}{c_k} \right| = \frac{1}{(k+1)^2} = \frac{1}{k^2 + 2k + 1}
\]

\[
= \frac{1}{1 + \frac{2}{k}} = \frac{1}{(1 + \frac{2}{k})^2}
\]

\[
1 = \frac{1}{1} = \lim_{k \to \infty} \left| \frac{c_{k+1}}{c_k} \right| = \frac{1}{1} = 1
\]

So $R = 1$ for both a convergent and a divergent series.

What have I learned?

For series like $\sum \frac{1}{k^2 + 1}$, use comparison tests and $p$-test.

For series with $(\cdot)^k$ use root or ratio.

For series with $k!$, use ratio.
two harder ratio tests; the first arises from \( \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \)

\[
= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}
\]

\(\text{Let } \sum \frac{1}{(2k)!} \mid \mid \frac{a_{k+1}}{a_k} \mid = \frac{(2k+1)2!}{(2k)!}
\]

\[
\frac{1}{(2k+2)!} \quad \text{or right - - - let's do an example}
\]

with \( k = 2 \) then \( 2k+2 = 2 \cdot 2 + 2 = 6 \)

\( 2k = 2 \cdot 2 = 4 \)

\[
\frac{(2k+2)!}{(2k)!} = 6! = \frac{6!}{4!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} = 5 \cdot 6
\]

so \( \frac{(2k+2)!}{(2k)!} \) has two results -

\[
\frac{(2k+2)!}{(2k)!} = \frac{(2k+2)(2k+1)}{(2k+1)(2k)!} = \frac{(2k+2)(2k+1)(2k)!}{(2k)!} = (2k+2)(2k+1)(2k)!/(2k)!
\]

\[
= (2k+2)(2k+1) [\text{check: if } k = 2, +8 \cdot 5] \\
[2 \cdot 2 + 2)(2 \cdot 2 + 1)] = 6 \cdot 5 \checkmark
\]

so \( \mid \frac{a_{k+1}}{a_k} \mid = \frac{1}{(2k+2)(2k+1)} \) and so

\[
\lim_{k \to \infty} \mid \frac{a_{k+1}}{a_k} \mid = \lim_{k \to \infty} \frac{1}{(2k+2)(2k+1)} = 0 \to 0
\]

\( R = 0 < 1 \) so \( \sum a_k \) converges by ratio
So one: \[ \sum \frac{k^k}{k!} \]

\[ \left| \frac{C_k h}{C_m} \right| = \frac{(k+1)^k h}{(k+1)!} = \frac{(k+1)^k h}{k!} \]

\[ = \frac{(1 + \frac{1}{h})^k (k+1)}{k!} = (1 + \frac{1}{h})^k \]

We did these ratios on page 2.

So \[ \lim_{k \to \infty} \left| \frac{C_k h}{C_m} \right| = \lim_{k \to \infty} (1 + \frac{1}{h})^k = 4 \]

\[ \ln(A) = \lim_{k \to \infty} \ln(1 + \frac{1}{h}) = \lim_{h \to 0} \ln(1 + \frac{1}{h}) = \infty \]

So do the \( \frac{\ln}{\ln} \) trick.

\[ = \lim_{h \to \infty} \ln \left( \frac{1}{h} \right) = \ln 1 = 0 \]

Switch to \( x \)

\[ = \lim_{x \to \infty} \ln(1 + \frac{1}{x}) \]

Change variable \( y = \frac{1}{x} \)

\[ = \lim_{x \to \infty} \frac{\ln(1 + y)}{y} = 0^+ \]

\[ = \lim_{y \to 0^+} \frac{\ln(1 + y)}{y} = 0 \]
\[ \lim_{k \to 0^+} \frac{1}{1+\log k} = 1. \]

\[ \log A = 1 \quad \text{so} \quad A = e \]

\[ R = e > 1 \quad \text{so} \quad \sum \frac{k^t}{k!} \text{ diverges by ratio} \]

Remark: \[ k! = \int_0^\infty x^k e^{-x} \, dx \text{ so } k! \text{ isn't an integer..} \]

Stirling's approximation for \( k! \):
\[ k! = \sqrt{2\pi k} \frac{k^k}{e^k} \left( 1 + \frac{1}{12k} + \frac{1}{288k^2} + \cdots \right) \]

so \( \frac{k^t}{k!} \) will be like \( \frac{k^t}{\sqrt{2\pi k} \frac{k^k}{e^k}} = \frac{e^k}{\sqrt{2\pi k}} \)

so that's where the \( e^k \) comes from in the ratio test.

Note: What is \( \pi \) doing there?!!

If you work hard on (1/2)^n, you can show \( (1/2)! = \frac{\sqrt{\pi}}{2} \)

This is just for fun, not some level of math fun.

So we won't need to know this stuff on this page.
Alternating Series

Those ones because all the core series have them

\[
\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots
\]

\[
\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots
\]

\[
\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots
\]

On alternating series is a series of the form

\[
\sum (-1)^k b_k \quad \text{or} \quad \sum (-1)^k b_k \quad \text{or} \quad \sum (-1)^{k-1} b_k
\]

where \( b_k \geq 0 \)

Note \( \sum (-1)^k b_k = - \sum (-1)^{k-1} b_k \)

So the three examples are the same, except for a "-" sign - so we'll talk about just one of them, \( \sum (-1)^k b_k \quad b_k \geq 0 \) or \( \sum (-1)^{k-1} b_k \)

\[
= b_1 - b_2 + b_3 - b_4 + \ldots
\]

Now - alternating series do something very special, and that's why we study them - let's look at an alternating vs. a non-alternating
<table>
<thead>
<tr>
<th>$n$</th>
<th>$\sum_{k=1}^{n} \frac{1}{\sqrt{k}}$</th>
<th>$\sum_{k=1}^{n} (-1)^{k-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>5.02</td>
<td>1.45</td>
</tr>
<tr>
<td>100</td>
<td>18.6</td>
<td>1.55</td>
</tr>
<tr>
<td>1,000</td>
<td>64.8</td>
<td>1.59</td>
</tr>
<tr>
<td>10,000</td>
<td>198.5</td>
<td>.6</td>
</tr>
</tbody>
</table>

These results have been correctly rounded.

This suggests $\sum \frac{1}{\sqrt{k}}$ diverges (we know that by $p$-test, $p = \frac{1}{2} \leq 1$)

$$\sum_{n=1}^\infty \frac{(-1)^{k}}{\sqrt{k}}$$ converges

\[ \text{so - try it again} \]

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\sum_{k=1}^{n} \frac{k}{\sqrt{k}}$</th>
<th>$\sum_{k=1}^{n} (-1)^{k+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>2.9</td>
<td>.65</td>
</tr>
<tr>
<td>100</td>
<td>5.2</td>
<td>.69</td>
</tr>
<tr>
<td>1,000</td>
<td>7.5</td>
<td>.69</td>
</tr>
<tr>
<td>10,000</td>
<td>9.8</td>
<td>.69</td>
</tr>
</tbody>
</table>

In fact - we know $\sum \frac{1}{\sqrt{k}}$ diverges

by $p$-test $p = 1 \leq 1$

But we'll see $\frac{\Theta}{k=1} \frac{(-1)^{k+1}}{\sqrt{k}} = \sum_{k=1}^\infty \frac{(-1)^{k+1}}{\sqrt{k}} = .693$

converges

So - (unfortunately) the $(−1)^{k}$ can make a divergent series into a convergent series.
Why? How does it do this? Let's look

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \cdots + \frac{1}{29} + \frac{1}{30} + \cdots$$

Set

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = (1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{5} - \frac{1}{6}) + \cdots$$

$$= \frac{1}{2} + \frac{1}{12} + \frac{1}{30} + \cdots$$

Compare with $\sum_{k=1}^{\infty} \frac{1}{k}$

$$1 + \frac{1}{2} + \left[ \frac{1}{3} + \cdots + \frac{1}{11} \right] + \frac{1}{12} + \left[ \frac{1}{13} + \cdots + \frac{1}{29} \right] + \frac{1}{30} + \left[ \frac{1}{31} + \cdots \right]$$

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ doesn't have any of this stuff, which is why $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges - it has less to add.

And exactly why does it have less stuff to add? The key is in the

$$\left( \frac{1}{3} - \frac{1}{4} \right) = \frac{1}{12} \quad \left( \frac{1}{5} - \frac{1}{6} \right) = \frac{1}{30} \quad \cdots$$

Subtractions! Subtractions make things smaller.

So does $\sum_{k=1}^{\infty} b_k$ always converge? No -

Let's see an example
\[
\sum (-1)^k b_k = -1 + 2 - 3 + 4 - 5 + 6 - \ldots
\]
\[
= 1 + 1 + 1 + \ldots
\]
and \(\sum 1\) diverges by divergence test.

Oh wait — remember from lecture

If \(\lim b_k \neq 0\) then \(\lim (-1)^k b_k\) diverges due to divergence test, \(\sum (-1)^k b_k\) diverges.

So this suggests

**The Kathy Alternating Series Test ≠ TA DA!**

Take \(\sum (-1)^k b_k\)

1. Check \(b_k \geq 0\) for all \(k\)
2. Check \(\lim_{k \to \infty} b_k = 0\)

Then \(\sum (-1)^k b_k\) converges.

Bzzzz! That test is wrong, wrong, wrong.

Here's an example of the wrongness

\[
1 - \frac{1}{2} + \frac{2}{3} - \frac{1}{4} + \frac{2}{5} - \frac{1}{6} + \ldots
\]

\[
= \sum (-1)^k b_k\]

\[
\text{diverges, } p_{\text{test}} p = 1 \leq 1.
\]

\(\sum\) is very sad Kathy test.
Why is it wrong? Look at \(-\frac{2}{3} + \frac{2}{5}\). 
Yes, \(-\frac{2}{3}\) subtracts, so it makes things small. 
But the very next thing I do is odd back 
\underline{ doubk, what I took off! That just makes 
things bigger.
}

so - when I take off \(b_k\), the next term \(b_{k+1}\) I add on has to be not bigger,

\(b_k\) not bigger than \(b_k - um\) -- \(b_{k+1} \leq b_k\)

so - now we have

our book: alternating series test
other books: ratio test

If you have \(\sum (-1)^k b_k\) or \(\sum (-1)^k b_n\),
check
1) \(b_k \geq 0\)
2) \(\lim_{k \to \infty} b_k = 0\)
3) for all \(k\), \(b_{k+1} \leq b_k\)

Then \(\sum (-1)^k b_k\) and \(\sum (-1)^k b_n\) converge

Example \(\sum_{k=2}^{\infty} \frac{k+1}{k!}\)
1) \( b_k = \frac{1}{\ln k} \geq 0 \) if \( \ln k > 0 \) for \( k \geq e^0 = 1 \) yes, \( k \geq 2 \). 

2) \( \lim_{k \to \infty} \frac{1}{\ln k} = \lim_{k \to \infty} \frac{1}{\infty} = \frac{1}{\infty} = 0 \). 

3) \( \frac{1}{\ln(k+1)} \leq \frac{1}{\ln k} ? \)

\( \ln k \leq \ln (k+1) ? \) 
\( k = k+1 ? \) 
\( k \geq 1 \) - uh, yeah, \( k \geq 1 \).

So \( \sum_{k=2}^{\infty} \frac{1}{\ln k} \) converges by comparison test.

<table>
<thead>
<tr>
<th>n</th>
<th>( \sum_{k=2}^{n} \frac{1}{\ln k} )</th>
<th>( \sum_{k=2}^{\infty} \frac{1}{\ln k} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>6.1</td>
<td>1.14</td>
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<tr>
<td>100</td>
<td>30</td>
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<td>1000</td>
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<td>1.00</td>
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</table>