I usually start with algebra-graphing review, because I’ve got a hidden agenda: besides simply reviewing the algebra I need, I want to cast it in a different light, emphasizing first, the connection between algebra moves and graphing, and, second, introducing examples of what I’ll be concerned with in calculus.

I start with \( y = x^2 \), because almost everyone knows the graph and knows the graph has a vertex at \( x = 0 \). I be a bit cagey about what a vertex is, but one thing I do say is that you can not only see it on the graph, you can see it algebraically. Since \( y = x^2 \) is a square, you have \( y \geq 0 \), so the graph lies completely above the \( x \)-axis, except at the vertex, where it touches. The touch of the vertex and the axis is, algebraically, the intersection of the curves at \( y = 0 \) and \( y = x^2 = 0 \) or \( x = 0 \). So, I got the vertex located at \( P(0,0) \). A really simple example of algebra and geometry together, true, but I got plans, I’m goin places.

Next I mess with it; check out \( y = x^2 + 1 \). This moves the vertex, and because \( y = x^2 + 1 \geq 1 \) I can use the same cheap algebra trix to locate the new vertex at \( P(0,1) \). I draw the moral that changing \( f(x) \to f(x) + 1 \) pushes graphs up one unit. Ditto, \( f(x) \to f(x) - 1 \) pulls graphs down one unit, and in particular, \( x^2 \to x^2 - 1 \) drags \( y = x^2 \) so that it cuts the \( x \)-axis, at \( x = \pm 1 \).

This is an important first example, because the vertex is still at \( x = 0 \), which means that the vertex is half-way between the roots. Its part of one of the great general results of calculus, the Mean Value Theorem: for a continuous function, between two roots there’s always a vertex, and this is my first example.

Meanwhile, back at \( y = x^2 \), my next algebra move is gonna be to change \( x^2 \to (x + 1)^2 \). Again, it’s easy to track the vertex; \( (x + 1)^2 \geq 0 \) so the graph touches the axis at \( x + 1 = 0 \) or \( x = -1 \). And again, \( f(x) \to f(x + 1) \) shifts the graph left one unit.

This is paradoxical, since positive numbers are to the right on the number line, so I expect + should move me right. Turns out, its a complete analogy with daylight savings time: anyone who thinks that setting the clock one hour ahead is gonna get them more sleep, is in for a big surprise.
Side note: so why does \( f(x) \rightarrow f(x) + 1 \) shift in the positive direction? Because of the asymmetric way we choose to treat \( x \) and \( y \). Think of changing \( y = x^2 \) to \( y + 1 = x^2 \) and you see adding one to \( y \) shifts the graph down, just as adding one to \( x \) shifts it left.

By the way, all this has one bright side: the effect of adding one to \( x \) is the opposite of what I expected, which shows why it’s good to have algebra around, to check my intuitions.

I usually talk about \( f(x) \rightarrow -f(x) \) flipping \( f \) about the \( y-\text{axis} \), then start sketching combinations like \( y = 1 - x^2 \), but the interesting case is something like \( y = x^2 - x \), which you can’t get by any combination of adding ones, except that you can; that’s what completing the square is about. I know \( x^2 - x = (x - \frac{1}{2})^2 - \frac{1}{4} \) and having just practiced graphing multiple translations of vertices . . . square completed, I’ve checked again that between two roots is a vertex. The half’ that I get here is the same half’ as in vertexes are half-way between roots’ and likely a lot of students have bad memories of completing squares and halvsies. But I make the promise that calculus will give me a much better way of finding vertices than completing the square ever could, so that’s actually good news.

Also by the way, \( y = x^2 - x \) is a great example to show that \( f(x) \rightarrow f(-x) \) flips \( f \) about the \( y \)-axis, if you’re gonna do that one.

The next function I like to talk about, very important for what’s to come, is \( y = |x| \). It has a vertex, again at an absolute minimum, but this vertex is pointy. You can see that’s it’ll be pointy, because the vertex, at \( x = 0 \), is a place where the lines \( y = x \) and \( y = -x \) meet. The \( 45^\circ \) line meets the \(-45^\circ \) line, and this means the two meet at an angle of \( 90^\circ \), a sharp \( 90^\circ \). There’s no rounding off; you can’t look closer and see rounding off; it’s interesting to graph \( y = x^2 \) and \( y = |x| \) together, like this:
You can see the difference between the rounded-off and the sharp, and can make the promise that calculus will be the way to talk precisely about that difference. My students have usually had calculus before, so it’s possible to say, that what you see here is the difference between the first derivative being zero and the first derivative failing to exist.

If you want you can pursue that a bit, by talking about ”angle of approach” for a line, the way it approaches the x-axis, that’s just it’s slope. For the parabola $y = x^2$ there’s no one angle of approach, but as the enlargements show, the approach gets flatter and flatter as you focus in towards $x = 0$. It makes sense to talk about a zero angle of approach here.

Between -.5 and .5

Between -.1 and .1
If you do angle of approach, there’s an interesting pair of functions to graph, say using a graphing calculator: compare the axis approach of \( y = \sin x \) on \([-\pi, \pi]\) with those of \( y = x\sqrt{1-x^2} \) on \([-1, 1]\). At \( x = 0 \) they appear to cross in much the same manner, but check out the approaches at \(-1, -\pi\), say. \( y = \sin x \) approaches at a nice, sensible angle, but \( y = x\sqrt{1-x^2} \) is vertical. Same deal at \( 1, \pi \) too. This demonstrates the type of information angle of approach’ can give.
Next, time for a quick talk about \( y = \sqrt{x} \). I like it here because it brings in the whole domain issue, but also because its approach is at ninety degrees what I call the nose-dive, or, when I'm in a tacky mood, the ValueJet approach. I’m seeing a different way a function can have a vertex, and again there’s a promise, that derivatives will sort it all out, and give me some much classier names than ’pointy place’, ’nose dive’ their parents didn’t spend the big bucks to hear phrases like that; you can sound so much cooler if you say ”the derivative has a discontinuity of the second type.” Cool is important.

I quickly do \( y = x^3 \), \( y = x^{\frac{1}{3}} \) and especially \( y = x^{\frac{2}{3}} \), this last because I think the graph is very neat, I call it ’gull wings’ or maybe even ’DeLorean’ because of the car in Back To The Future. Technically, I tell them the description is ’a cusp’ and that the name comes to mathematics from astrology. But secretly it’s about derivatives again. The rounded vertices are about derivatives being zero; the angular ones about derivatives failing to exist; the nose-dives and cusps about derivatives being infinite. Hidden in this is a kind of a promise: although it looks like there’s a lot of different vertexes, there’s actually only a few different kinds, and calculus is gonna give me a complete list of the kinds of vertexes that can actually happen. All that wide variety, all the different functions in the world and I’ll be able to say, what can happen and what can’t.

Amazing.
The next part of the review is composition of functions. Natch, I’ve been doing compositions, and their effects informally, when I did \( f(x) \to f(x+1) \), which means I’ve already got a bunch of examples to practice composing on.

Now I’m gonna be like a magician. What I’m up to is, I want to see the geometric effect of doing a composition; I want to take a harmless function say, \( g(x) = 1 - x^2 \), then start composing it, doing the \( f(g(x)) \) thing, with functions \( f(x) = |x|; = x^2; = \sqrt{x}; = x^{\frac{2}{3}} \). And I want to see the effect on the graphs. When I do the composition, I’ll be tracking what happens to old vertices and how new vertices come into being.

The magicians secret is, I’m doing the chain rule. For simple functions \( f \) like these, \( f(x) = x^a \), and the chain rule states that

\[
[g^a(x)]' = a [g^{a-1}(x)] [g(x)]'
\]

As vertexes come from numbers where a derivative is zero or fails to exist, I can use the factorization of \( [g^a(x)]' \) to see that vertices of \( g^a \) happen when \( g' \) is zero or fails to exist, or when \( g \) is zero. Geometrically, the vertices of \( g^a \) will be either at the same position as those of \( g \), or at the axis-crossings of \( g \).

Well to work, starting with \( y = x^2 \). I explain that we’re really starting with the \( y = x \) graph, then applying the square’ operation to it, and when we do that, we get the line transformed to the parabola, like so:
Ummm . . . I knew that. But for us, it’s more important to describe in words what happened, which is, three things:
a) Any part below the axis gets reflected above the axis.
b) Axis crossings get flattened, into rounded little vertices;
c) Straight lines get bent’upwards

I tell them that all of this is intuitive and very pictorial, for what I’ll later use derivatives to make precise and very algebraic.

Now I try it for real, on something like \((x^2 - 1) \rightarrow (x^2 - 1)^2\). Check out the pics below; which are labeled:
At point 1, the axis crossing at \(x = -1\) has been changed into a vertex of \((1 - x^2)^2\).
At point 2, the vertex at \(x = 0\) has been flipped around the axis, but there’s still a vertex at \(x = 0\).
At point 3, the axis crossing at \(x = 1\) has been changed into a vertex too.
It’s instructive to try the same reasoning on $y = x^2$ itself, as it gets transformed from $x^2 \rightarrow x^4$. Check out the graph below, where $y = x^2$ is superimposed with $y = x^4$. Note that $x^4$ is flatter at the origin, and also bent more upwards than $x^4$. It also don’t hurt to see what squaring might do to a trig function something like $y = \sin x$.

Next I try square roots, and again I think of the line $y = x$, then applying the square root’ operation to it, and when we do that, we get the line transformed to the anti-parabola, like so:

This time,

a) Any part below the axis gets the Jimmy Hoffa treatment: it’s disappeared.

b) Axis crossings get changed into vertical tangents;

c) Straight lines get bent’ downwards.
It’s fun to try this on \( y = x + 1 \) and then compare that to the graph you’d get just by shifting the graph of \( y = \sqrt{x} \) left one unit; ditto for \( y = 1 - x \), which students usually have a hard time conceptualizing because of the application of the transformation \( \sqrt{x} \rightarrow \sqrt{-x} \). It seems much easier for them to see the line \( y = 1 - x \) chopped off then bent.

Now try it for real, this time \( 1 - x^2 \rightarrow \sqrt{1 - x^2} \). The graph shows five parts:
1) The region \( x < -1 \) is wiped from the face of the earth, gone.
2) The axis crossing at \( x = -1 \) is made vertical.
3) The vertex at \( x = 0 \) is still there.
4) The axis crossing at \( x = 1 \) is made vertical.
5) The region \( x > 1 \) is also gone.

\[ \text{Part 1} \quad \text{Part 2} \quad \text{Part 3} \quad \text{Part 4} \quad \text{Part 5} \]

It’s all as predicted: the old vertices stayed where they were, but I picked up new vertices from the zero-crossings of the function \( g(x) = 1 - x^2 \).

Now to try it with something a bit ummy: absolute value functions.
The next function along is $y = |x|$. Again, ya got yer comparison graphs, and regions:

a) Any part below the axis gets reflected above the axis.
b) Axis crossings are ”sharpened”, into pointy vertices;
c) Anything above the axis is left alone

One point to make here is that all I’m doing in parts a) and c) is describing in geometric terms the definition of the absolute value function. For example, in ”$|x| = -x$ if $x < 0$” You translate ”$x < 0$” as ”below the axis” and the part ”$|x| = -x$” really reads ”flip the graph about the axis.”
Again I try it on something real like $1 - x^2 \rightarrow |1 - x^2|$. In regions 1 and 3, $1 - x^2$ is below the axis, so the graph there gets flipped above. In region 2, $1 - x^2$ is above the axis, so the graph gets left alone. The axis crossing at point a), $x = -1$, has been changed into a vertex of $|1 - x^2|$. At point b, the vertex at $x = 0$ has been left where it is. At point c, the axis crossing at $x = 1$ has been changed into a vertex too.

A couple of other interesting experiments to try would be $1 - x^2 \rightarrow (1 - x^2)^{2/3}$, or $1 - x^2 \rightarrow (1 - x^2)^3$. 