Last section, I saw that I needed 'limits' but I didn’t know what kind of thing they are. Of course secretly I do know: limits are what you get from the $\epsilon$, $\delta$ definition of a limit;

$$\lim_{x \to c} f(x) = L$$

means for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$0 < |x - c| < \delta \; \text{implies} \; |f(x) - L| < \epsilon$$

If every you’ve messed with this definition, you know it is no picnic. But in addition to developing this definition, the founders of calculus evolved a collection of tricks so that the computation of limits could be made almost algebraic. That’s what I’ll present.

First, some simple algebra facts:

$$\lim_{x \to c} x = c$$

if $\lim_{x \to c} f(x) = F$ and $\lim_{x \to c} g(x) = G$ then

$$\lim_{x \to c} [f(x) + g(x)] = F + G$$

$$\lim_{x \to c} [f(x) \cdot g(x)] = F \cdot G$$

$$\lim_{x \to c} \left[ \frac{f(x)}{g(x)} \right] = \frac{F}{G}$$

this last if $G \neq 0$

In practice, this leads to things like

$$\lim_{x \to 3} \frac{x^2 - x + 1}{x^2 + 2} = \frac{3^2 - 3 + 1}{3^2 + 2} = \frac{7}{11}$$
This lets me compute limits for *algebraic* expressions. It works with square roots, cube roots, as well, so

$$\lim_{x \to 2} \frac{\sqrt{x + 1}}{x} = \frac{\sqrt{2 + 1}}{2} = \frac{\sqrt{3}}{2}$$

There are only two types of cases where this won’t work:

1) *Cases where I don’t already know the limit.*

I got $\lim_{x \to 3} x = 3$, which I used to build up $\lim_{x \to 3} x^2 = 3^2$, but I don’t get $\lim_{x \to 3} \cos(x) = \cos(3)$. Ditto for all the other trig functions, and logs, and exponentials and . . . well any other type of functions. Those all have to be done separately, one by one.

2) $\lim_{x \to c} \left[ \frac{f(x)}{g(x)} \right]$ where $\lim_{x \to c} g(x) \neq 0$.

I’m a bit luckier with those, because there are only a few types of things that can happen, so I’ll be able to do a general theory for them.

Here’s an example:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \frac{0}{0}$$

which means that I have no idea what the limit really is – certainly I can’t use the algebra limit laws. What I can do is – check this out – notice:

$$\frac{x^2 - 1}{x - 1} = \begin{cases} 
  x + 1 & \text{if } x \neq 1 \\
  \text{dne} & \text{if } x = 0
\end{cases}$$

But limits ask what $f$ is like near a point: when I ask $\lim_{x \to c}$ all that matters is $f$ near $c$ but never $f$ exactly at $c$. So:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} (x + 1) = 2$$
So that’s a plan: say I have

\[ \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{??}{0} \]

What I do is move away from \( x = c \), and then I can use algebra to simplify \( \frac{f}{g} \) – typically, I factor an \((x - c)\) from \( f \) and \( g \), then cancel common factors – all away from \( x = c \). Like to compute

\[ \lim_{x \to 2} \frac{x^2 - 3x + 2}{x^2 - x - 2} = \frac{0}{0} \]

I factor an \((x - 2)\) from numerator and denominator, getting

\[ \frac{x^2 - 3x + 2}{x^2 - x - 2} = \frac{(x - 2)(x - 1)}{(x - 2)(x + 1)} \]

Then since I was extra-careful to stay away from \( x = 2 \), I can cancel numerator and denominator, to get

\[ \frac{(x - 2)(x - 1)}{(x - 2)(x + 1)} = \frac{(x - 1)}{(x + 1)} \]

and now

\[ \lim_{x \to 2} \frac{x^2 - 3x + 2}{x^2 - x - 2} = \lim_{x \to 2} \frac{(x - 1)}{(x + 1)} = \frac{2 - 1}{2 + 1} \]

That’s the trick: stay away from the zero in the denominator, do your algebra, then use the simplified version to do the limits by just plugging
This works great with other limits, besides \( \frac{0}{0} \). You can use it in salads, as an appetizer, . . .

\[
\lim_{x \to 0} \frac{1 + \frac{1}{x^2}}{1 - \frac{1}{x^2}} = \frac{1 + \frac{1}{0}}{1 - \frac{1}{0}} = \text{whoops!}
\]

The \( \frac{1}{0} \) aren’t acceptable so I try to clear them away. They come from \( \frac{1}{x^2} \), so I start by clearing \( x^2 \)'s from denominators:

\[
\frac{1 + \frac{1}{x^2}}{1 - \frac{1}{x^2}} = \frac{x^2 \left(1 + \frac{1}{x^2}\right)}{x^2 \left(1 - \frac{1}{x^2}\right)} = \frac{x^2 + 1}{x^2 - 1}
\]

Now,

\[
\lim_{x \to 0} \frac{1 + \frac{1}{x^2}}{1 - \frac{1}{x^2}} = \lim_{x \to 0} \frac{x^2 + 1}{x^2 - 1} = \frac{0^2 + 1}{0^2 - 1} = 1
\]

Or try it here, with one sided limits and an absolute value:

\[
\lim_{x \to -1^+} \frac{x^3 + 1}{|x + 1|} = 0
\]

Again, whoops; this time we use the right-handed limit \( x \to 1^+ \) to simplify: \( x > -1 \) so \( x + 1 > 0 \) so \( |x + 1| = x + 1 \) and

\[
\lim_{x \to -1^+} \frac{x^3 + 1}{|x + 1|} = \lim_{x \to -1^+} \frac{(x + 1)(x^2 - x + 1)}{x + 1}
\]
\[
= \lim_{x \to -1^+} (x^2 - x + 1) = (-1)^2 - (-1) + 1
\]

For algebraic functions of \( x \), I’ve succeeded in making the computation of limits an algebraic matter. This has a geometric consequence: I can tell which of the four types of discontinuities occurs when there’s a zero in a denominator.
The discontinuities I’m concerned about all have a zero in a denominator—say when \( x = c \). So they all have denominators; we’ll make them \( \frac{f(x)}{g(x)} \).

Note this doesn’t even begin to look at functions of the form \( \sin\left(\frac{1}{x}\right) \)!

Anyway: what can happen?

1) \( \lim_{x \to c} f(x) \neq 0 \) and \( \lim_{x \to c} g(x) = 0 \). Then \( \frac{f(x)}{g(x)} \) has a vertical asymptote at \( x = c \) AND the left and right limits each tend to a positive or negative infinity.

An other possibilities are: \( \lim_{x \to c^+} \frac{f(x)}{g(x)} \) exists and is some finite number \( R \), while \( \lim_{x \to c^-} \frac{f(x)}{g(x)} \) exists and is some finite number \( L \).

2) \( L = R \). Then \( \frac{f(x)}{g(x)} \) has a removeable discontinuity at \( x = c \); less precisely, \( \frac{f(x)}{g(x)} \) has a hole in its graph at the point \((c, L)\).

3) \( L \neq R \). Then \( \frac{f(x)}{g(x)} \) has a jump discontinuity at \( x = c \); the ”jumping-off points” are \((c, L)\) on the left and \((c, r)\) on the right.

The last possibility is that either or both of the limits \( \lim_{x \to c^+} \frac{f(x)}{g(x)} \), \( \lim_{x \to c^-} \frac{f(x)}{g(x)} \) fail to exist (and are not infinite). Then \( \frac{f(x)}{g(x)} \) has an oscillatory discontinuity at \( x = c \).

I do want to say that this theory doesn’t cover everything. Again, \( \sin\left(\frac{1}{x}\right) \) isn’t a quotient like \( \frac{f(x)}{g(x)} \). And a function like \( \frac{1}{x} + \frac{1}{|x|} \) isn’t a single quotient either. When you bring it to a common denominator, you get \( \frac{|x| + x}{x|x|} \), and if you check, \( \lim_{x \to c} f(x) = 0 \) and \( \lim_{x \to c} g(x) = 0 \), so you can’t use the asymptote theorem. Nonetheless \( \frac{1}{x} + \frac{1}{|x|} \) has a one-sided asymptote, and taking left or right-handed limits would bring that out.