In the previous sections, we’ve seen the effects of algebra on graphs; used algebra to talk about graphs. In the theory of limits, we explore what happens when algebra fails. It turns out, amazingly, that we can *still* understand what the effect of failed algebra is on the graph.

The basic way (until the log chapter), that algebra can fail is for a denominator to be zero. It’s important to emphasize that once a denominator has become zero, there’s no rescuing it. The function \( \frac{x}{x} \) is *not* equal to one; it’s equal to one except at \( x = 0 \). At \( x = 0 \), you can no longer do the algebra to cancel numerator and denominator.

One way to say it is that \( \frac{1}{x} \) does not exist at \( x = 0 \), and that multiplying by \( x \) to get \( x \cdot \frac{1}{x} = \frac{x}{x} \) cannot magically bring it into existence. My partner and I have four imaginary children (the oldest is Zawadi; she’s six. The twins, Sage Heather and Sierra Lisette are four and the baby, Chen Li, is six months). We have ‘em but we don’t brush their teeth or feed ‘em breakfast. It’s the same with \( x \cdot \frac{1}{x} \): you can’t start pretending it’s really there at \( x = 0 \).

So what’s the graphical effect of this non-algebra? Pick a function like \( y = \frac{x^2-1}{x-1} \). Normally, I’d say \( \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = (x + 1) \), but now I have to be careful of the denominator. Best I can do is,

\[
\frac{x^2-1}{x-1} = \begin{cases} 
  x + 1 & \text{if } x \neq 1 \\
  \text{dne} & \text{if } x = 1 
\end{cases}
\]

On a graph, the "dne" is usually represented by an open circle:
The open circle symbolizes a point that isn’t actually on the graph, an enlargement of a microscopic gap in the graph. Algebraically, $f(x)$ fails to exist at $x = 1$; geometrically the graph is broken into two pieces, one to the left of $x = 1$, one to the right (blue and green, below). And this happens every time there’s a zero in a denominator: the graph always gets broken.

The idea that a zero in a denominator of a function always causes a break in the corresponding graph is the origin of the idea of ”continuity”. Intuitively, a continuous function is a function whose graph can be drawn in a single, fluid motion. If that’s too vague, you could say a function is continuous if its graph can be drawn without lifting the pen from the paper. A graph which is broken into pieces requires a lift to get across the break, hence is not continuous.

Fact is, whenever a function has a zero in its denominator, its graph has a break, into left and right pieces. In the graphs following I colored the pieces red and blue, to emphasize the break. Here they are:

$$
y = \frac{x}{|x|} \quad y = \frac{1}{x} \quad y = \sin \left( \frac{1}{x} \right)
$$
What is extraordinary about these graphs is that they show, essentially, *everything* that can happen when a denominator is zero. I don’t mean that literally: if I take \( y = \frac{x^2-1}{|x-1|} \) or \( y = \frac{\tan x}{|x|} \) I get graphs like these two:

And if, instead of \( y = \frac{1}{x} \), I take \( y = \frac{1}{x^2} \) or \( y = \frac{1}{x} + \frac{1}{|x|} \), I get these two graphs:
I don’t mean that there are only four graphs one can draw; I mean: if there’s a zero denominator, there are only four **types** of graph that can be drawn.

A function visually with a hole in it, called a function with a **removeable discontinuity**. The discontinuity is removeable: that means, if the hole were simply filled in, the function would have no breaks at all.

A function whose y-values go to infinity, negative infinity, or both. This is called a function with a **vertical asymptote**. To join the two halves of the function, you’d have to drag one or the other piece up or down an infinite amount.

A function visually which suddenly jumps from one level to another. This is called a function with a **jump discontinuity**. To join the two pieces of the function, one or the other has to be dragged up or down a finite amount.

A function visually which wiggles up and down an infinite number of times in a finite length. This is called an **oscillatory discontinuity**.
Fine: I can recognize a discontinuity when I see one. What I want next is an algebraic understanding of discontinuities. And this isn’t gonna happen.

Fact is, limits are not algebraic or geometric things. They’re a new kind of thing. It took European mathematicians several centuries to deal with that, because – well, you’ll see: limits certainly look like they ought to be algebraic. I’ll do my best to make it seem that way, but it’s all a cover-up.

I start back at \( f(x) = \frac{x^2-1}{x-1} \), my removable discontinuity. Plug in \( x = 1 \) and ya get \( \frac{0}{0} \), which is not a number. There’s no place to plot it; at \( x = 1 \) on the graph, instead of a point on the graph, you get a hole.

BUT, check out below:

\[
\begin{array}{c}
\text{The hole has a y-height, a y-coordinate, namely } y = 2. \text{ That’s almost as good as having a value at } x = 1. \text{ You can’t say } f(1) = 2 \text{ but you can say, “the graph of } f(x) = \frac{x^2-1}{x-1} \text{ has a hole at } P(1,2)”. \text{ Locating the height of the hole is a substitute for } f \text{ having a value at } x = 1. \\
\end{array}
\]
Also, it’s not like the hole is just anywhere, like in this graph:

Filler? Me?!
Instead, the hole is attatched to the graph. When you think about what that means, it means you can get the height of the hole by looking at the height of \( f \) for \( x \) near \( x = 1 \). For example, here’s a table that you’d use to graph \( f \) reallyreally close to \( x = 1 \):

\[
\begin{array}{ccc}
  x = 1 - \frac{1}{10^n} & y = \frac{x^2 - 1}{x - 1} & x = 1 + \frac{1}{10^n} & y = \frac{x^2 - 1}{x - 1} \\
  x = .9 & y = 1.9 & x = 1.1 & y = 2.1 \\
  x = .99 & y = 1.99 & x = 1.01 & y = 2.01 \\
  x = .999 & y = 1.999 & x = 1.001 & y = 2.001 \\
  x = .9999 & y = 1.9999 & x = 1.0001 & y = 2.0001 \\
  x = .99999 & y = 1.99999 & x = 1.0001 & y = 2.0001 \\
\end{array}
\]

What do you get from this? First: the actual height of the hole, \( y = 2 \), never appears as a \( y \) value in the table. But that’s OK: the \( x \) value \( x = 1 \) never appears either.

Second, as the \( x \) values get closer to \( x = 1 \), the \( y \) values get closer to \( y = 2 \).

It’s like They Are Listening,and \( x = 1 \) and \( y = 2 \) are the Things That Cannot Be Said, because then they Get You. But even if I can’t say those words, I can talk \emph{around} them, plug in \emph{near} \( x = 1 \). And then \( y \) is near:

Oops I can’t say. But I sure can guess; look at the table to guess what \( y \) has to be.
Third: this isn’t algebraic. The whole business of ”I won’t say but look and see what you guess” sure isn’t any kind of algebra ever known to man woman or beast. This is what’s so tricky. You want to say, there’s this clear geometric meaning, ”the hole is connected to the graph”.

But what exactly does ”connected” really mean?

Or if not geometric, then try algebraic: ”if you plug in x’s closer and closer to one, forever, then you’ll get y’s closer and closer to 2”.

But what’s it really mean to plug in ”forever?”

Zip. Nada. Nuthin’. And that’s why limits aren’t geometric or algebraic things. They are some sort of mixed alg-geo-info-connecto . . . Thing.

Stay tooned . . .