This section is an attempt to get a geometric feeling for the product rule. Now products of functions are pretty hard to deal with, and the product rule will be harder to understand geometrically: for example, it adds "vertices" in locations much more difficult to predict than for the chain rule.

But there a few simple rules that can help visualize products. For example, it’s usually easy to understand the effects of multiplying $g$ by $f$ when $f$ is like $+1$ or $-1$ or zero: the first leaves $g$ alone, the second flips $g$, the third zeroes it out.

Try the case when $f$ has a root at a point $x = c$: look at $f(x) = x$ and $g(x) = \text{anything}$ say $g(x) = \cos(x)$. Note that when $x = 0$, $g(x) = 1$, so $f \cdot g$ is like $f \cdot 1$. So you expect that near $x = 0$, $f \cdot g$ graphs like $f$ alone. That’s what the graph shows: $f(x)g(x) = x \cos(x)$ superimposed on the graph of $f(x) = x$. 

![Graph](image.png)
You see very clearly that $f \cdot g$ looks a lot like plain old $f$, near where $g(x) = 1$.

The product rule explains why they look so alike, and that’s what I want to talk about next: the graphs look alike because for $f \cdot g$, you get $(f \cdot g)(0) = f(0)$ AND ALSO $(f \cdot g)'(0) = f'(0)$. Check it out:

\[
(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x) \quad \text{so, plugging in}
\]
\[
(f \cdot g)'(0) = f'(0)g(0) + f(0)g'(0)
\]
\[
= f'(0) \cdot 1 + 0 \cdot g'(0)
\]
\[
= f'(0)
\]

Here’s two more graphs: $g$ is still just $\cos(x)$ but I’ll go kinda wild with $f$, taking on the left $f(x) = x^3$ and on the right $f(x) = x^{\frac{2}{3}}$ (this being about as wild as mathematicians ever get. alas).
This same trick works even if \( g(0) \) isn’t exactly equal to 1. Ummm . . . say it’s like \( g(0) = -1 \). Then near \( x = 0 \), \( f \cdot g \) is like \( -f \). In the graph below, I check it with \( g(x) = (x - 1)^3 \)

The product rule explains it again:

\[
(f \cdot g)'(0) = f''(0)g(0) + f(0)g'(0) \\
= f''(0) \cdot g(0) + 0 \cdot g'(0) \\
= -f'(0)
\]

The rilly intersting thing here is the second line of the equation: it tells me that whatever \( g(0) \) is, the product \( f \cdot g \) near \( x = 0 \) is gonna look like \( f(x) \cdot g(0) \).
There’s some morals to be drawn from all this. First, it helps get a grip on the product rule. People walking cold into calculus think, oh the product rule is gonna come out as 
\[(f \cdot g)' = f' \cdot g'.\]
The actual terms are like \[f' \cdot g\] and this gives a geometric explanation.

The idea here, that \(f \cdot g\) near \(x = 0\) acts like \(f \cdot g(0)\), is called freezing. You fix one of the \(x'\)'s, let the other vary. The idea kind of comes from Taylor series, and it’s a good analytic exercise to work it out. Take two terms of the Taylor for \(f\), with two terms of that for \(g\), and you get

\[
[f(c) + f'(c)(x - c)] \cdot [g(c) + g'(c)(x - c)] \\
= f(c)g(c) + \{f'(c)g(c) + f(c)g'(c)\}(x - c) \\
+ f'(c)g'(c)(x - c)^2
\]

The first order terms, the coefficients multiplying \(x - c\), are \(\{f'(c)g(c) + f(c)g'(c)\}(x - c)\) which, hey, is the product rule.

But I won’t go without a fight