Chapter 1: Numbers

Section 7 Part 2: Orders of Magnitude

I first heard about orders of magnitude in 1968, at a public astronomy seminar, Cornell University. In 1967, the radio astronomer Jocelyn Bell had detected a signal with a peak repeating every 1.33 seconds (see Figure 72 and p87). The signal followed the rotation of the earth, so she and her advisor Anthony Hewish concluded it was extraterrestrial; they called the signal LGM-1 (for ‘Little Green Men’: a signal so regular could be from an extraterrestrial civilisation). The signals were later understood to come from stars, and the stars were named pulsars.

When Frank Drake returned from the Aricebo radio telescope in Puerto Rico (see Figure 81 and p87) with data suggesting a new pulsar in a different part of the sky, Cornell opened a seminar to discuss what might cause these signals.

Memory fades after fifty years. I recall Hans Bethe was there, the Nobel laureate who explained the nuclear reactions that produce the energy of stars. Drake opened, discussing how the telescope needed to be modified to receive these signals; he had a slide with a graduate student doing the updates, hanging upside down 150m above the valley floor. Edwin Salpeter was there (he’d explained the production of carbon in stars). He discussed the energy required to generate the signal; for the Crab pulsar (Figure 73) the energy is order of magnitude $10^{32}$ joules/year. Salpeter explained for the non-physicists that a joule was about the order of magnitude of the energy released by a tomato falling one foot onto a floor.

Carl Sagan stood and remarked the signal was about five orders of magnitude greater than the entire energy output of the sun, and therefore, if the signal came from an intelligent civilization, ‘it was a very stupid intelligent civilization’ – there are many ways of sending signals requiring orders of magnitude less energy.

Finally, Tommy Gold (who had developed the theory of the magnetosphere surrounding the earth) stood and proposed that pulsars were highly magnetized rotating neutron stars. As the star rotated, its magnetic field would accelerate particles to near the speed of light, and those would then emit synchrotron radiation like the blue light seen in the Crab nebula, Figure 73. Gold’s explanation is the accepted explanation today (see p88).
It was amazing to see these brilliant scientists doing science, thinking about results, doing computations off the top of their heads. It was clear I had a lot to learn, and not just about orders of magnitude.

The powers of ten \(10^0, 10 = 10^1, 100 = 10^2\ldots\) form standards for the way we think about numbers – we think of sandwiches as costing about ten dollars, tickets to a theme park about a hundred (see the research earlier referenced as cognitive reference points, p28 and the references there). The idea of orders of magnitude make this precise.

What we want to do, roughly, is say the pair 4, 7 are the same size, as are the pair 60, 80. In scientific notation, we’d have \(4 \times 10^0, 7 \times 10^0\) versus \(6 \times 10^1, 8 \times 10^1\). The different exponents tell us the two pairs are different orders of magnitude.

But what about 9.9999999 versus 10.00000001? It’s misleading to say one is substantially larger/smaller than the other. Should I be rounding? Where’s the cut off from one order of magnitude to the next? For example, if 9 should belong with 11, what about 8? Maybe 5 is the cut off, since it’s half-way?

Once again, it’s about setting standards: we have to establish an unambiguous system. One way to do this is think about what we said, "different exponents tell us the two pairs are different orders of magnitude". Exponents can always be accessed as logarithms of the numbers; more precisely, if \(x = 4.768 \times 10^8\), \(\log_{10}(x) = 8.5761\ldots\), and it’s the 8 we want to pick out (this is called the characteristic of \(x\)). But even this is a bit tricky. \(5.0 = 5 \times 10^1\) and the log is 1.6989\ldots, so we’d give it order of magnitude 1. On the other hand, .05 = 5 \times 10^{-2} but the log is \(-1.3010299957\), so do we give it order of magnitude \(-1\) or \(-2\)?

The standard we set is this (see p88):

**Order of Magnitude:** To find the order of magnitude of a number \(x\), take \(\log_{10} x\), and correctly round it.

For example, \(x = 2.78\) has \(\log_{10} x = 0.4440447959\). Since we have 0.4440447959 < .5, this is rounded down to zero and 2.78 has order of magnitude zero. For \(x = 3.78\), \(\log_{10} x = 0.5774917998 > .5\), so this number is rounded up to one, and 3.78 has order of magnitude one.

Most people can’t compute logs in their head. Without logs, the cut off between order of magnitude \(n\) and \(n + 1\) is \(10^{n+\frac{1}{2}} = 10^n \cdot 10^{\frac{1}{2}} = 3.1622776602 \cdot 10^n\). Then \(2.78 < 3.1622776602 < 3.78\), though likely we’d use 3.16 instead of 3.1622776602.
Returning to Carl Sagan and pulsars, a large order of magnitude was used to show something was unlikely or impossible. The next example is from biology: the mode of action of drugs on cells.

This has an antique sound; in fact it’s the title of a book by the British physiologist/mathematician A.V. Hill (Nobel Prize 1922), who started his work in 1909. Some drugs were known to work specifically on certain diseases – for example, quinine on malaria. These were called ‘specifics’, but how they worked was a mystery. In 1797 Caddel and Davies wrote Medical, Philosophical and Vulgar Errors of Various Kinds Considered and Refuted:

"...supposing an admiral sent down channel, across the Bay of Biscay, and up to the Mediterranean, with express orders to attack the Maltese, but with the strictest charge not to molest any other state whatever; I cannot conceive any medicine such a specific as to conform most punctually with such orders, to act vigorously against one particular gland or humour of the body, without in the least affecting or disturbing any other."

One theory was that a given drug coated the surface of the cells affected by the drug. Hill worked at the beginning of quantitative physiology and pharmacology; he measured the amount of drug necessary to produce a response, and calculated it was several orders of magnitude too small to coat a cell. Drugs must be doing something else.

We can imaging the drug getting inside a cell and doing its work there, but this again is orders of magnitude off. The cell membrane is composed of phospholipid molecules; Figure 74. But,

"...hormones, being mostly hydrophilic (or lipophobic) substances, are unable to pass through membranes, so that their influence must somehow be exerted from outside. The membranes of cells, although very thin (3 to 6 nm) are effectively impermeable to ions and polar molecules. Although $K^+$ ions might achieve diffusional equilibrium over this distance in water in about 8 ms, they would take some 12 days (280 hrs) to equilibrate across a phospholipid bilayer [...] Likewise, the permeability of membranes to polar molecules is low."

Here the order of magnitude difference is 280 hours, or $10^6$ seconds, verses $10^{-3}$ seconds. This vast gap tells us drugs cannot work that way. The receptor theory of drug interaction explains these discrepancies. The drug binds to a molecule called a receptor that spans the cell membrane. The chemical interaction on the outside of the cell changes the receptor’s shape on the interior, where it interacts with other molecules to have an effect on the cell. Cells that don’t produce the particular receptor are not affected by the drug, hence the specificity.

This next example is about precision: here, order of magnitude will tell us how accurately we need to be in building machines, in this case, large modern airplanes. The spar of an airplane wing is a support beam; see Figure 75. Many different kinds of structures attach directly onto it – for example, the airplane engines and the ailerons that control turning. One consequence is that the spar has to be a precise size, so that each piece will fit in exactly the right place. For a modern 27 meter long airplane wing, the spar has to be constructed to .3mm accuracy, a difference of five orders of magnitude. Such extreme accuracy is needed because of the strong forces acting on the wing during flight; differences between the wings could cause instability, or help cracks form and spread, causing the loss of the wing during flight.

Our second example comes from square-cube ideas. It’s inspired by recent work on environmental remediation. At the height of the space race in the 1960’s launch complexes were used to clean and degrease rocket engines; the chemical used was trichloroethylene (TCE), which is now known to be toxic and carcinogenic. It seeps into the ground and, over long periods of time, can contaminate groundwater. See Figures 76 and 77.

To de-contaminate the soil, metal particles are injected into the ground; these combine with TCE to form new compounds that are no longer toxic. The question is how the metals should be delivered. If we think of these as small spheres, the TCE will act on the surface of the sphere; for maximum efficiency, the surface area should be as large as possible. We’ll need a couple of formulas: A sphere of radius $r$ has surface area $S = 4\pi r^2$ and volume $V = \frac{4}{3} \pi r^3$. Now let’s compare two sizes of metal balls.

In the first, our spheres are about the size of a cell: the radius is 100µm or $10^{-4}$m. This gives it a volume of $\frac{4}{3} \pi 10^{-12}$m, and a surface area of $4\pi 10^{-8}$m.

Now, instead of using cell-sized metal balls, let’s use virus-sized par-
particles. Their radius is 100nm or $10^{-7}$m, with a volume of $\frac{4}{3}\pi 10^{-21}$m, and a surface area of $4\pi 10^{-14}$m. This is worse – smaller surface area – but, since we’re using smaller metal balls, we’ll have more of them. How many more? Divide the volume of the large balls by that of the small. Cancelling the $\pi$ and the $\frac{4}{3}$, it’s $10^{-12}$ divided by $10^{-21}$, or, $10^9$.

Now I have $10^9$ particles, each with surface area $4\pi 10^{-14}$m, so my new surface area is $4\pi 10^{-5}$m. The single larger ball had a surface area of only $4\pi 10^{-8}$m, deactivation of TCE by three orders of magnitude. It’s nano-particle remediation.

Our last example is digestion: we’ll follow a piece of potato as it’s converted into sugars and fats in the body (see p88 for references). We’ll see a range of eight orders of magnitude, from organs like the stomach, that we can easily see, to molecules we can only understand with complex scientific instruments. The lesson here is about the slow progress of medicine; what we now know required hundreds of years of work, and hundreds of years of development of scientific instruments.

We’ll be taking a journey from organs we can see with our eyes and feel with our hands, to smaller and smaller structures, all the way down to single molecules. We can no longer see or manipulate these tiny structures; the closer we zoom in on these micro-structures, the more exact understanding we get on how food is processed.

Digestion actually starts in the mouth, goes on to the esophagus, stomach and small intestine. Some of this is mechanical: food is broken into small pieces. Size does matter here: a variety of enzymes (for example proteases and lipases) are responsible for breaking down proteins and fats. These enzymes attach to the surface of the food, so the larger the surface area presented to them, the faster and more thoroughly they work.

If we look at a piece of food – say a sphere 1cm in radius – the surface area $S$ is about 12.6 cm$^2$, order of magnitude 1. If the food is broken down to a thousand smaller spheres, $S$ is about 270 cm$^2$, order of magnitude 3. Chewing processes food to be digested three orders of magnitude more efficiently. Snakes and alligators gulp, and it can take days for a snake to digest a mouse.

Our small particles enter the intestine next. An average small intestine is about 6m long – say order of magnitude two. The intestine is lined with circular folds shown in Figure 78. Each fold is roughly 8mm or $8 \times 10^{-2}$m high: order of magnitude $-2$, a jump of five orders of magnitude.

Each circular fold is has a surface coated with thousands of tubes called villi, about 1.5mm or $1.5 \times 10^{-3}$m high (see Figure 79); we’ve
jumped down one more order of magnitude. The villi increase the area that can absorb nutrients by a factor of thirty: one order of magnitude.

The villi themselves are coated with a brush border, made up of further wormy tubes called microvilli (Figure 79 again), 1µm or 10⁻⁶m high, they serve much the same purpose as the villi. For a decrease of three orders of magnitude in size, they increase the surface are by a factor of 600, or three orders of magnitude.

And there it ends – almost. The folds and villi and microvilli can grab food from the flow through the intestines, but fats still need to be broken down by lipases, proteins by proteases; complex carbohydrates need to be hydrolyzed to monosaccharides. Finally, all these molecules have to get to the bloodstream.

All this occurs in a thin layer covering the microvilli called the glyco-calyx. This is made up of actin filaments, a protein that usually forms the contractile filaments of muscle cells, but in this case forms a protein layer that can contract to keep fluid moving. The actin molecules themselves are about 5 nm or 5 × 10⁻⁹m, another two orders of magnitude smaller. The brush border holds digestive enzymes, and the resulting sugars, etc, are transferred from the interior of the intestine to the bloodstream by the cells comprising the villi, shown in Figure 80.

The cells have transport molecules located in their membrane; one transport might carry selected sugars through the cell wall: glucose, for example. This has a molecular diameter (see p89) of 9 Angstroms or 9 × 10⁻¹⁰m, another order of magnitude. These are small enough to get from the microvilli to the blood through cell pores (actual gaps in the cell wall) about 500 to 800 angstroms or 5 × 10⁻⁸m to 8 × 10⁻⁸m – certainly large enough to let the glucose flow freely into the blood.

Let’s close with an example from neuroscientist:

"Single large neurons have physical dimensions observable at low optical magnification, that of a tenth of a millimeter. That is big enough to be dissected by hand with pins, using a good magnifying glass. Moving just two orders of magnitude down to the micrometer level, which requires a good microscope, one is at the scale of synaptic transmission. One may observe synapses at the union between nerve and muscle, for example. Two orders of magnitude further down, at tens of nanometers, with the aid of electron microscopy, we find the realm of single ion channels and of signal transduction and molecular biology. (Rodolfo R. Llinas I of the Vortex: From Neurons to Self. The MIT Press, 2001.)
The remarkable nature of these signals at first suggested an origin in terms of man-made transmissions which might arise from deep-space probes, planetary radar, or the reflection of terrestrial signals from the moon. None of these interpretations can, however, be accepted because the absence of any parallax shows that the source lies far outside the solar system. [...] A tentative explanation of these unusual sources in terms of the stable oscillations of white dwarf or neutron stars is proposed.

The Aricebo telescope consists of a reflecting mesh hung above shade-tolerant vegetation near the town of Aricebo, Puerto Rico (also known as La Villa del Capitan Correa); see Figure 8.1. The mesh is about 1000m in diameter, and reflects radio waves to a detector about 150m above the dish. Construction of the telescope was an enormous engineering and political undertaking.

Aricebo was conceived by DARPA (the Defense Advanced Research Projects Agency) as a means to detect incoming guided missiles: high speed objects entering the upper atmosphere ionize it, and this can be detected on radar. As the upper atmosphere was poorly understood, the telescope was designed to carry out atmospheric research; W.E. Gordon of Cornell Engineering wrote in his proposal:

The discovery that free electrons in the Earth’s ionosphere incoherently scatter signals that are weak but detectable [...] makes possible the exploration of the upper atmosphere [...] that the radar components [...] are all within the state of the art means that the exploration can begin as soon as the radar is assembled. From “Design study of a radar to explore the earth’s ionosphere and surrounding space” by W. E. Gordon, H. G. Booker, & B. Nichols, Cornell.

Ward Low at DARPA realized the telescope could also be used to study the behavior of radio waves, and to intercept Soviet communications. The proposed atmospheric telescope would be built as a general-purpose radio telescope. The perfect location would have to be in the tropics, so that all planets passed over the telescope; a natural valley would save on construction costs ... and Braulio Dueno from the University of Puerto Rico at Mayaguez was studying at Cornell. Aricebo was chosen, and the telescope opened in 1964. The first big scientific result was the discovery that the rotational period of Mercury was 59 days, not 88 days as previously thought. For military intelligence, it detected Soviet radar waves reflected off the moon,
and gave the location of the radar station. For the history, see Daniel R. Altschuler, The National Astronomy and Ionosphere Center’s (NAIC) Arecibo Observatory in Puerto Rico, http://www.astro.wisc.edu/sstanimi/Students/daltschuler_2.pdf


‘At a conference in London in 1951 I had argued that dense, collapsed stars would be ideally suited to emit strong radio signals, since their magnetic fields may be enormously strengthened by the collapse and extend out into low density space.

‘The pulsars now seemed to represent just the stellar objects I had discussed then. Calculations existed for the collapsed “neutron stars” that indicated approximately their size, as small as a few kilometers, and their mass, on the order of a solar mass .4 Astronomers generally thought that even if they existed, they could never be discovered. However hot, a star so small could not be seen at astronomical distances. But they had not considered the energy concentration resulting from the collapse: enormous magnetic field strengths and a spin energy quite comparable with the entire content of nuclear energy of the star before its collapse. With these considerations it was not unreasonable to expect them to be observable.

‘I had another clue to the nature of the new objects: While there was some irregularity in the pulse-to-pulse timing, the long-term accuracy of these “clocks” was enormously better than a statistical addition of the pulse irregularities would have allowed.

‘I proposed the model of a rapidly spinning neutron star, which, as a result of some asymmetries, sent out a strong beam of radiation from one region of longitude. This beam would sweep over the Earth once in each rotation period and would be seen as a short pulse. The underlying long-term accuracy would then be produced by the spin of the object, while short-term timing fluctuations would just reflect details of the emitting mechanism. I compared it to the rotating beacon of a lighthouse, whose lamp, hanging from the rotating shaft, could wobble a little; the long-term accuracy would still be that of the rotation of the shaft.’


p85 This discussion of digestion follows Thomas Fischer Weiss’ Cellular Biophysics: Transport, MIT Press 1996.
A single molecule is an object in constant motion, spinning, contracting, expanding, etc; it doesn’t have any single size. Instead, quantum mechanics assigns a probability that the molecule has a given size. The idea of ‘molecular diameter’ is a substitute. There are many ways to do the computation; see for example ‘kinetic diameter’ at https://en.wikipedia.org/wiki/Kinetic_diameter; atomic radius at https://en.wikipedia.org/wiki/Atomic_radius; hard spheres models, http://www.kayelaby.npl.co.uk/general_physics/2_2/2_2_4.html, and more.