Chapter 1: Numbers

Section 3: The Problem of Fractions

Compared to the integers, fractions are more like a necessary evil. To a modern mathematician, they’re easily defined, but historically have been difficult to compute with.

Much of what we think of as modern mathematics was developed in civilizations, which used bureaucracies as an organizational tool in what is now India, China, the Middle East and Egypt. All of these developed a class of administrators – individuals who could read, write, and perform mathematical computations. The mathematics was important; the bureaucracies regulated land use, paid or forced labor, wages, livestock, agriculture, military affairs, construction, trade . . . . Often, these professional administrators would be concerned with what we think of as science – for example, astronomy. All of this involved a great deal of counting, measuring, and computing; administrators developed techniques for doing computations efficiently (and techniques for training the next generation). These are the computational techniques we’ll examine.

We’ll start with Egypt, where our literate administrators were called scribes. Workers (or forced laborers) were paid in standard-sized units of grain, or bread, or beer. The difficulty was in dividing, say three standard loaves among four workers. Each worker gets \( \frac{3}{4} \) of a loaf, but how? Do you cut each loaf in half, then half again, to get fourths? Then each worker gets three little slices? Let’s not even try jugs of beer. The scribes needed more than the ability to write fractions like \( \frac{2}{3} \); they needed to compute with fractions, and relate those computations to other kinds of numbers. We have much the same problem, for example, in food rationing during wars or other crises. Let’s say everyone gets two and a third ounces of cooking oil per week. Multiply that by 23 million people and you need . . . . well, you need an efficient way to do that kind of computation.

Figure 41 shows Egyptian fractions. They were were rewritten as sums of unit fractions. Unit fractions are fractions with one as the numerator, so, instead of writing \( \frac{3}{4} \), the Egyptian system was to write \( \frac{3}{4} = \frac{1}{2} + \frac{1}{4} \). This method also makes the division of the loaves of bread more practical: everyone gets a half loaf, then a quarter loaf.

Going further back in history, numbers, written language, and accounting seem to have originated in the Middle East (historical Mesopotamia – see p47). The professional administrators developed mixed systems (like our \( \frac{9}{4}, 2\frac{1}{4} \) and 2.25), as well as mixed units (pints,
quarts, liters). Mesopotamian scribes also used base sixty 'decimals'; they’d write $\frac{1}{12}$ as $\frac{5}{60}$, and $\frac{4}{45} = \frac{1}{12} + \frac{1}{120} = \frac{5}{60} + \frac{20}{600}$. Or, in a modern decimal-like notation 5, 20.

Base sixty is very convenient for dealing with unit fractions: $\frac{1}{2} = \frac{30}{60}$; $\frac{1}{3} = \frac{20}{60}$ (try that with decimals!). The outcast here is $\frac{1}{7}$; in decimal notation, $\frac{1}{7} = 0.142857142857142857$ and so on. The “and so on” originally had no acceptable definition, though it can be thought of as a shorthand for the phrase “if you continue to divide, you will continue to get blocks of 142857”. This is one of the main problems with fractions.

Historically, Mesopotamian scribes wrote something like $\frac{1}{7} = 0.142857$ and then warned “approximation given since 7 does not divide”. This leaves quite a bit out: for example, if $\frac{1}{7}$ is the amount of tax on a piece of land, and you’re a government, you want the largest number you can get away with (rounding up). If you’re the one paying that tax, you want the smallest (rounding down). Writing $\frac{1}{7} = 0.142857$ doesn’t say where you are.

You could argue that this is a basic deficiency of decimal notation, and for this reason, fractions simply aren’t the same as decimals.

Mesopotamians took a different view: at some point, an unknown scribe wrote

$$8, 34, 16, 59 < \frac{1}{7} < 8, 34, 18$$

The decimal version is

$$0.14285640 < \frac{1}{7} < 0.1428611$$

Writing $0.14285640 < \frac{1}{7} < 0.1428611$ tells you the largest and smallest value you could take, but now the problem is, it doesn’t leave you with just one number. Our practical scribes had a solution: take the average, 0.14285787. Now we have a single number to use, and we know the largest and smallest variations.

Moderns think about this differently: we’d say 0.14285787 is an approximation to the real value of $\frac{1}{7}$, but that it isn’t the real value. The way to talk about approximate versus real is to introduce the idea of error: error = (real value - approximation). Here, the error is $\frac{1}{7}$ - 0.14285875. This doesn’t seem to help, because we don’t know the real value of $\frac{1}{7}$. But we do know how large and how small $\frac{1}{7}$ could be:

$$0.14285640... < \frac{1}{7} < 0.1428611$$

Now subtract the average (the approximation) from all three sides:

$$0.14285640... - 0.14285875 < \frac{1}{7} - 0.14285875 < 0.1428611 - 0.14285875$$
Rewriting, and noticing the middle is now real-approximation = error, 

\[-0.00000235 < \text{error} < 0.00000235\]

or as we’d write it today, 

\[|\text{error}| < 0.00000235\]

Actually, we’d write \[|\text{error}| < 2.35 \times 10^{-6}\]. This might look familiar; it hints at the modern idea of a limit. What’s missing is the \textit{epsilon} ‘\(\epsilon\)’, which controls how small the error gets. We’ll deal with that in Section 3, on real numbers.

How does this help our scribe? Imagine some lowly scribe presenting the taxes to his boss. The boss remarks, "You have taken the seventh part; there is an error. Perhaps the tax is too small?" But now our scribe can bow low and say, "Oh Shining One, the tax on this land is ten bushels of barley, and the error is but a part of one grain". I’d probably hate being a scribe.

There’s point to this silly story: when we talk about whether fractions are really decimals, or whether infinite decimals are limits, we’re exporting our own twenty-first century beliefs back thousands of years, to a place they don’t belong. The historian Eleanor Robson makes this point explicitly:

On the constructivist historical view, the emphasis is on difference, localism, and choice: why did societies and individuals \textit{choose} to describe and understand a particular mathematical idea or technique one particular way as opposed to any other? How did the social and material world in which they lived affect their mathematical ideas and praxis?


Robson takes a specific example from a Mesopotamian "problem set":
A square is \(\frac{1}{3}\) cubit and \(\frac{1}{2}\) finger on each side; what is its area? (the answer should be \(9\frac{1}{2}\) grains).

Here we have to deal with conversion of units and mixed decimal/fraction notation. The scribe first converted the numbers to base sixty notation, squared the number (using the same kinds of techniques we’d use to do a multiplication), and then converted the answer back to decimal/fraction notation – in different units. The actual answer the scribe gave, though, was \textit{not} \(9\frac{1}{2}\); the scribe converted the true area to one simpler to write in mixed notation. Base sixty, to the scribe, was just a computational tool to make certain conversions and computations easy. There wass no issue of whether fractions were "really" decimals.
But, historical anachronisms aside, we are after a modern understanding of fractions. Let’s start. First, notation: fractions are quotients of integers, \( \frac{p}{q} \), so we write the collection of all fractions as \( \mathbb{Q} \) (Quotients). Getting these into decimals takes work.

Let’s take a simple fraction, \( \frac{237}{10} \), and convert it to 23.7. To start, the fraction \( \frac{237}{10} \) has a piece, 23, to the left of the decimal point. What’s left over is the fractional part, the \( \frac{7}{10} \). You can access the fractional part by taking away the first part, to get 23.7 – 23 = .7. Now multiply by 10: 10 \cdot .7 = 7, and you have the part to the right of the decimal point.

How do I know to not multiply by 100 to get 70? Why does multiplying by 10 seem to be just right? If we’d had \( \frac{2307}{100} \), multiplying by 100 = 10² would have been ‘just right’, and multiplying by 10 would be ‘not enough’. Let’s translate this into mathematics: the part to the left of the decimal point is in \( \mathbb{N} \); the part to the right of the decimal point is the fraction \( f \), where 0 ≤ \( f \) < 1.

The mathematical way to say 10 is ‘just right’ for the fractional part \( f = .7 \) is that \( \frac{1}{10} \leq .7 < \frac{1}{10^2} \). In contrast, for \( f = .07 \), the fact that 100 = 10² is ‘just right’ and 10 = 10¹ is ‘too small, gets rewritten as \( \frac{1}{10} \leq .07 < \frac{1}{10^2} \). The general idea is:

**The Archimedean Principle for Rationals:** If \( r = \frac{p}{q} > 1 \) is a rational number, then there is always a power of ten, \( 10^m \), with \( m \geq 0 \) and \( 10^m \leq n < 10^{m+1} \). If instead \( 0 < \frac{p}{q} < 1 \), there’s a negative power of ten, \( 10^{-m} \), with \( m \geq 1 \) and \( 10^{-m} \leq \frac{p}{q} < 10^{-m+1} \)

To check, multiply both sides of the inequality by \( 10^m \), and then you get \( 1 \leq 10^m \cdot (\frac{p}{q}) < 10 \). You’ve now got a number between one and ten; that’s the next decimal place of \( \frac{p}{q} \) (see p47 for a sketch of the proof). For a finite decimal, repeating the process will bring out all the decimal places, one by one.

Repeating decimals don’t fit this scheme very well – there’s always a fractional part left over. And again we have to ask: does this mean that decimals are the wrong idea to understand fractions?

The Chinese scholar Lui Hui (Figure 42) expressed similar ideas when calculating the value of \( \pi \); he used the approximation \( \pi \approx 3 \) and warned that this was not the true value, but was good enough for most practical purposes (see p47).
Lui Hui also said how he’d estimated \( \pi \): he computed the area of a 96-agon inscribed inside a circle (a 96-agon is a 96-sided figure; for comparison, a triangle is a 3-agon). He also gave a formula for going from one approximation to a better one: see Figure 43, where he goes from the area of a 6-agon to that of a 12-agon. He then goes to a 24-agon, a 48-agon, and finally a 96-agon.

Something new happened: for \( \frac{1}{7} \), we got a decimal approximation. But Lui Hui generates not just one number, but a whole collection of numbers. Technically, the collection of numbers Lui Hui gave for \( \pi \) is called a sequence (see p47). The sequence is a collection of better and better approximations. Again, we think ‘limit’ and ‘numbers like \( \frac{1}{7} \) are limits of actual finite decimals.’ This is teleology: we know how all these issues turned out, so we’re imagining that the path mathematics took was somehow preordained. Limits, etc, may not be at all what Lui was thinking. And perhaps at some point, it might have been argued that decimals really were a poor choice, because, like the ancient scribe wrote, in \( \frac{1}{7} \), ‘7 does not divide.’

These kind of ideas were used in the work of Ptolemy, an astronomer living in Alexandria, Egypt about 100 CE. Like the Babylonian mathematicians (p15), he wanted to predict the position of the planets; the difference was that he believed the planets moved in circular orbits. The mathematics available at that time was Euclidean geometry, and he used trigonometry to relate lengths along circular orbits to angles. Which involved computing the sines of all the angles, or at least a large number of them. He was also aware of trig identities, like \( \sin(2\theta) = 2 \sin(\theta) \cos(\theta) \). If he could compute the sine of half a degree, this would give him the sine of a degree, and with other identities, the sine of all integer angles.

So – how to get the sine of a half-degree from the sine of known angles? He used a result of Archimedes:

\[
\begin{align*}
\text{If } \theta > \psi & \text{ Then } \frac{\theta}{\psi} > \frac{\sin(\theta)}{\sin(\psi)} \\
\end{align*}
\]

Using \( \theta = \frac{3}{4} \) and \( \psi = \frac{1}{2} \), he got

\[
\frac{\sin(\frac{3}{4})}{\sin(\frac{1}{2})} < \frac{3}{2}
\]

Rearranging, \( \frac{3}{2} \sin(\frac{3}{4}) < \sin(\frac{1}{2}) \). Numerically, using modern values (and degrees, not radians) we get 0.008726397 < 0.0087265355, and a similar inequality on the other side, which is remarkably accurate.
Notes for Chapter 1 Section 3: The Problem of Fractions


p45 The Archimedean Principle for rational numbers begins with the division algorithm (not surprising; \( \frac{p}{q} \) is a divisor!). Roughly, a number like \( \frac{80}{9} \) can be written as \( 8 + \frac{8}{9} \); the second term is a fraction less than one. In Archimedean terms, this shows that if we have \( 10^0 \leq 8 < 10^1 \), it's still true that \( 10^0 \leq 8 + \frac{8}{9} < 10^1 \).

The second half, dealing with \( 0 < r < 1 \), follows by applying the the Archimedean Theorem to \( \frac{1}{r} > 1 \) and then inverting the inequalities.

p45 The work of the mathematician Lui Hui appeared in commentaries and solutions to the Chinese text *The Nine Chapters on the Mathematical Art*, written in 263 CE.

p46 Technically, a sequence is more than just a collection of numbers; there's also a sense of one number following another (think of the cognate word *sequel* for the movie that follows the original; similarly the word *second* is also a cognate: it’s the number following the first). To say that approximations get better and better, we need a sense of the direction to go so we can get better; the sequence provides that direction. So, a sequence comes with a first number, a second number, etc. For 'first number' we write \( a_1 \), the second would be \( a_2 \), and so on. The sequence is then \( (a_1, a_2, \ldots) \).