Chapter 1: Numbers

Section 4: The Problem of Irrationals

We’ve been talking about mathematics as an extension of our mental perception of number, but number is just one quality we perceive. We also know qualities like size, position, length, area, angle, volume, weight . . . . Very early on, numbers were linked with these other kinds of qualities. That link created an association leading to entirely new kinds of number: those connected to geometry.

How exactly qualities like length and area came to be understood as number is a complex story, one which is still being worked out. We’ll discuss the neurophysiology in detail, in Section 9, Time and Space. See also Geometry and Decimals, p70, and Units & Standards, p75.

Geometric qualities: length, distance, area, volume ... Why? Because bureaucracies always tax, and how do you compute tax? If you just take part of the harvest from a farmer, the farmer can hide the harvest before the tax collector visits. Instead, compute the area of the land planted, then compute how much produce the land should yield, then take part of that (actual Egyptian practices were much more sophisticated than this; see p54). A simple scheme, but ancient inheritance involved subdividing land amongst many children, so taxable lands had complicated shapes. Figure 46 shows an Egyptian computation for the area of a complicated figure.

The area of a triangle is half the base times the height, so geometers had to understand the connection between numbers and lengths. This unites counting with measuring; quantity and geometry. We use yardsticks all the time, but the link was less obvious in ancient times: Figure 47 implies tools for surveying are gifts from the Gods. People took their lengths seriously – and state taxation depended on public trust in those tools (public standards, once more).

Figure 48 shows a problem in geometry from Mesopotamia, about 1700 BCE. It’s a right triangle, with two equal sides; each side has length one. The Pythagorean theorem tells us if $s$ is the length of the two equal sides, and $h$ that of the hypotenuse, $s^2 + s^2 = h^2$, and for $s = 1, 2 = h^2$. We’d write $h = \sqrt{2}$, but it’s very unlikely a scribe would write any of the above; see p54). On the tablet, in base 60, it’s written 1, 24, 51, 10; in decimal notation, 1.41421. We’d call it $\sqrt{2}$. The Mesopotamian answer in Figure 48 looks like the kind of approximations we used for $\frac{1}{7}$: it seems there’s nothing new or surprising here.

How did the Mesopotamian scribes get their approximations? No-
one knows, but here’s the best idea historians came up with (this method gives the same answers the Mesopotamian scribe wrote).

Start with a first guess for $\sqrt{2}$: say, $\frac{3}{2}$ or 1.5. That’s too big, but $2/\frac{3}{2} = \frac{4}{3} = 1.33\ldots$ is too small. Take the average; that’ll be in-between, so it will be closer to the true value than either guess. So, if $g_1$ is the first guess, then you get a second, better guess with

$$g_2 = \frac{1}{2} \left( g_1 + \frac{2}{g_1} \right)$$

With $g_1 = \frac{3}{2}$, then

$$g_2 = \frac{1}{2} \left( \frac{3}{2} + \frac{2}{\frac{3}{2}} \right) = \frac{1}{2} \left( \frac{3}{2} + \frac{4}{3} \right) = \frac{17}{12}$$

And again:

$$g_3 = \frac{1}{2} \left( \frac{17}{12} + \frac{2}{\frac{17}{12}} \right) = \frac{1}{2} \left( \frac{17}{12} + \frac{24}{17} \right) = \frac{577}{408}$$

How good are these approximations? Square them, and compare with 2:

$$g_1^2 = \frac{9}{4} = 2.25; \quad g_2^2 = \frac{289}{144} = 2.00694; \quad g_3^2 = \frac{332929}{166464} = 2.0000060073\ldots$$

We’re back in the Lui Hui situation: not just an approximation, but a way to get better and better approximations. And as before, you carry out as many decimal places as you need, get on with your job, report to the Chief Scribe, get your beer ration and go home.

But Greek mathematicians discovered $\sqrt{2}$ is not like $\frac{1}{7}$: it cannot be written as a quotient $p/q$, and it is not a repeating decimal (for the proof, see p 55). $\sqrt{2}$ isn’t rational; it’s ir-rational (‘ir-’ means ‘not’ as in ‘ir-relevant’). For some history of irrationals, see see p 55.

So: what is $\sqrt{2}$? We could still say it’s like $\frac{1}{7}$, because you can get more and more digits of $\sqrt{2}$. While you can’t get them by long division, at least you can get the numbers $g_1$, $g_2$, $g_3$, whose squares approximate 2.

But this is a kind of fraud. When you do the long division for $\frac{1}{7}$, you can see exactly where the decimal starts to repeat and why. As we saw, that means you can find the error in any one approximation. With the $g_1$, $g_2$, $g_3$, you don’t know what the $g_1$, $g_2$, $g_3$, are going to do, or why. Could $g_4$ be $\frac{17}{12}$ again? Then $g_5$ would be $\frac{577}{408}$ again, and you wouldn’t get better approximations. How can you rule that out? And how can you show it does get ‘better and better’?

The final answer came something like 2000 years later, so these are hard questions. The most significant answer (speaking historically)
was to avoid the question, which involved rethinking the relation between geometry and number.

The Greek mathematician Eudoxus of Cnidos (408 - 355 BCE) undid the link between quantity and geometry by developing a consistent theory of magnitude. Eudoxus used magnitude as an undefined term; one could think of magnitudes as being the lengths of lines, or areas and volumes of figures; he showed how to manipulate magnitudes as ratios, analogous to manipulation of ratios of numbers. For example, we can define \( \frac{m}{n} = \frac{p}{q} \) to mean \( mq = np \). For certain line segments, such as the diagonal of a square, you can no longer think of the magnitude as being the length (as it’s irrational); you have to think that the diagonal itself is the magnitude. You can then answer questions such as, if you double the magnitudes of the sides of a square, do you double the magnitude of the diagonal (yes).

Ratios were enough to do the geometry Eudoxus and most Greek mathematicians wanted, for example, the construction of figures using a ruler and a compass (see p56).

The Eudoxian theory was influential for centuries; even Newton, in his *Arithmetica Universalis* of 1707 defined numbers as ratios of line segments. The prevailing opinion was stated by the German mathematician Michael Stifel (1486-67), who was critical of using approximations to define an irrational:

\[ \ldots \text{considerations compel us to deny that irrational numbers are numbers at all. To wit, when we seek to subject them to [decimal representation] . . . we find they flee away perpetually, so that not one of them can be apprehended precisely . . . Now that cannot be called a true number which which is of such a nature that it lacks precision . . . so an irrational number . . . is hidden in a kind of cloud of infinity.} \]

Newton’s refusal to accept irrationals may seem inconsistent with his discovery of limits and calculus. However, Newton had a very classical training, beginning with Euclid, and tended to think geometrically. The idea of expressing an irrational number as a limit of rational numbers might have made no sense to him.

In Europe there were no alternatives to Eudoxus for over a thousand years, which didn’t prevent (some) mathematicians from using algebra to deal with "numbers" like \( \sqrt{2} \). The mathematician Leonardo of Pisa, who wrote under the name ‘Fibonacci’, was aware of Arabic work on algebra; in 1225 he published the solution to a problem mentioned by Omar Khayyam, in his book *Al-jabr*: solve the equation \( x^3 + 2x^2 + 10x = 20 \) (see Figure 49).

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**Figure 49: Solving The Cubic**

Five hundred years after Omar Khayyam, the Italian mathematician Girolamo Cardano found a formula for finding roots of cubics. Here, the formula is applied to Khayyam’s equation by the Wolfram Alpha computer program.

For the method, see David W. Henderson, *Geometric Solutions of Quadratic and Cubic Equations*, www.math.cornell.edu/
Fibonacci showed there were no integer solutions, no rational solutions, and that the solution could not be constructed by ruler and compass. So the number was irrational, but of some unknown kind.

The Mesopotamian scribe on p47 would probably shrug his shoulders: what did it matter, as long as he could compute three or four digits of these numbers, and keep the Chief Scribe happy?

For us it’s more difficult: am I going to run into new kinds of irrationals each time I solve a new equation?

In an attempt to create some kind of order, mathematicians began to rethink their irrationals. Fractions like \( \frac{1}{7} \), and irrationals like \( \sqrt{2} \), \( \frac{1+\sqrt{5}}{2} \ldots \) and even Fibonacci’s irrational, are all solutions of equations:

\[
7x - 1 = 0, \quad x^2 - 2 = 0, \quad x^2 - x - 1 = 0, \quad x^3 + 2x^2 + 10x - 20 = 0
\]

Solutions to these equations were called algebraic numbers, since they could be obtained by solving algebra equations with integer coefficients. Mathematicians were leaving behind Eudoxean geometric methods, moving to ideas that come from algebra.

The numbers \( \pi \) and \( e \) didn’t seem to be similar to algebraic numbers; the mathematician Euler remarked (1744) that these two seemed to go beyond the techniques of algebra. As the Latin for ‘go beyond’ is ‘transcend’, Euler suggested that these two were transcendental numbers. In 1878 the number \( e \), and in 1882, \( \pi \), were each shown to be transcendental: that is, they were not solutions of algebraic equations with integer coefficients.

Of course \( \pi \) is a solution to the equation \( \cos(\theta) = -1 \), and \( e \) to the equation \( \ln(x) = 1 \); the functions \( \cos(\theta) \) and \( \ln(x) \) are transcendental functions. Again, new equations, new irrationals. By this time, European mathematicians knew many new kinds of functions, for example the Bessel function \( J_1(x) \), which describes the pattern of rings when light is diffracted through a small hole (see Figure 50). All light through microscopes and telescopes gets diffracted, and the presence of the first dark ring determines how close two objects have to be to blur into one, through the lens. In short, \( J_1(x) \) determines the resolution of the lens. The dark rings occur when the intensity is zero, that is, \( J_1(x) = 0 \). Is this going to involve totally new kinds of numbers?

Just how many kinds of irrational are there?

There’s another issue: we know we can compute more and more decimal places of accuracy for \( \frac{1}{7} \), and we believe we can do that for \( \sqrt{2} \) and \( \pi \), but what about these new numbers? Do we even know they’re decimals?
Answering this question is an invitation to a new world: chains of numbers with no decimal expansion, ascending and descending into the infinitely large and infinitely small.

To enter this world, all you have to do is imagine .9999... doesn’t equal 1. Intuitively, the decimal ‘never gets there.’ So there’s a gap: then $\gamma = 1 - .9999... > 0$ measures the size of the gap: what kind of things are inside that gap? We’re going to discover many kinds of numbers inside there – and none of those numbers are decimals. This is the problem: the world of numbers might be strange, almost beyond imagining.

Let’s look a bit at $\gamma$. How big is it? What’s the decimal expansion?

If the decimal expansion of $\gamma$ starts with something like .276..., then that first decimal place makes $\gamma > .1$. But $\gamma = 1 - .9999... = 1 - .9 - (.09999...) < 1 - .9 = .1$. We can’t have $.1 < \gamma < .1$, so $\gamma$ has to start with something like $\gamma = .0276...$. The problem is that $\gamma = 1 - .9999... < 1 - .99 = .01$ and again, we can’t have $.02 < \gamma < .01$, so $\gamma$ has to start with something like $\gamma = .00276...$

The problem is, actually, that this never stops: $0 < \gamma < \frac{1}{10^k}$ for all $k$. $\gamma$ has the decimal expansion $\gamma = 0.00000...$ – but still $\gamma$ isn’t zero. So we have what we feared: a number that doesn’t have a decimal expansion at all. There goes the beer ration for our scribe.

We could say $\gamma$ is an infinitely small number, but not zero. Then $\frac{\gamma}{2} > \frac{\gamma}{3} > \frac{\gamma}{4} > \ldots$ are also infinitely small numbers with no decimal expansion. And so are $\gamma > \gamma^2 > \gamma^3 \ldots$.

So we don’t have just one infinitely small number, we have whole chains of them, getting smaller and smaller. We’ll see it gets worse – infinitely worse.

Since $\gamma < \frac{1}{10^n}$ for all $n$, then $\omega = \frac{1}{7} > 10^n$ for all $n$. Again, this means $\omega$ can’t start with $\omega = 1.374...$, and it can’t start with $\omega = 1384.732...$ because $1384.732... < 10,000 = 10^4$ but $\omega > 10^4$. So $\omega$ can’t start with any numbers before the decimal point: $\omega$ is infinitely large. And so are $\omega^2 < \omega^3 \ldots$.

And as if those aren’t large enough, $\mu = \omega^{\omega}$ is larger than all of them. Now we start again and we get $\nu = \mu^\mu \ldots$ and we get chains of larger and larger infinities.

Now let’s get more infinitely small numbers:

$$\gamma > \gamma^2 > \gamma^3 \ldots > \gamma^\gamma \ldots > \gamma^\mu \ldots$$

The gap contains a nightmare of infinities!
There’s supposed to be a way out, if you know limits. We’re supposed to know \( .999\ldots = 1 \) because the collection \{\( .9, .99, .999\ldots \)\} has 1 as a limit. Let’s try that. First, a little notation:

\[
.9 = \frac{9}{10}; \quad .99 = \frac{9}{10} + \frac{9}{10^2}; \quad .999 = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} \ldots
\]

Now we can talk about the limit: to say the limit of the \{\( .9, .99, .999\ldots \)\} is 1, is to say that for every \( \varepsilon > 0 \) there’s a point after which the sequence is at least \( \varepsilon \) close to 1:

\[
|1 - \left( \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} \ldots + \frac{9}{10^k} \right)| < \varepsilon
\]

But

\[
1 - \left( \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} \ldots + \frac{9}{10^k} \right) = \frac{1}{10^k}
\]

So we’re saying there’s a point after which \( \frac{1}{10^k} < \varepsilon \). But, this is the whole thing about numbers like \( \gamma \): if \( \varepsilon \) is one of our infinitely small numbers, it’s the other way around: \( \frac{1}{10^k} < \varepsilon < \frac{1}{10^k} \).

Not only do numbers like \( \gamma \) mess up ideas about decimals, they mess up the whole theory of limits.

Maybe our logic is bad? We can find a mistake in one of those computations? To tell us no such \( \gamma \) could ever exist?

No.

The British mathematician John Conway, Figure 52, invented a number system called the surreal numbers, S, which has infinitely small and infinitely large numbers (though his construction is nothing at all like our simple intuition for \( \gamma \)). Surreal numbers have a kind of not-really-a-decimal expansion; very very roughly, an infinitely large number might be something like

\[
\cdots + 3\gamma^2 - 5\gamma + \cdots + \frac{3}{10^2} + 7 + 2 \cdot 10^5 + \cdots + 3\omega - 5\omega^3 + \cdots + 3\mu + 2\mu^2 \cdots + 5\nu + \cdots
\]

But they’re not decimals, or any finite kind of thing; it even requires rethinking what it means to take infinite sums. See the references on p56.

There is no simple way to eliminate the surreals; you have to define real numbers in a way that will exclude them right from the start. We are making a choice; we are constructing real numbers to be the way we think they should be. The next section shows how it was done.
Notes for Chapter 1 Section 4: The Problem of Irrationals

p48 The amount of tax depends not just the area of a field, but also on the quality: fields were graded by their size and position as well as the presence of canals, trees and wells, and possible damage inflicted by floods. A second assessment was made before the harvest, followed by a final weighing of the harvested and threshed grain. See Corrina Rossi, Mixing, Building and Feeding: Mathematics and Technology in Ancient Egypt, in Eleanor Robson and Jacqueline Stedall, The Oxford Handbook of the History of Mathematics, Oxford University Press 2011.

p48 We found the hypotenuse of a right triangle as $\sqrt{2}$, using a bit of algebra. How did the Mesopotamian scribes think of it? It’s long been assumed that the scribes used a kind of algebra; the of problems they were able to solve were like this: I totaled the area and (the side of) my square: it is $0; 45$. We assume this means 'If $x$ denotes the side length, solve $x^2 + x = 0; 45$, and the scribe would solve the equation by completing the square.

This might not be anything like what the scribes were thinking. The historian Jens Hørup retranslated a group of tablets, trying to stay as close to the original as possible. He noted that the scribes don’t mention variables; they discuss lengths, areas and volumes. The word used for subtract, Akkadian $nas\text{"}hum$, is more literally rendered "to tear out". Hørup then suggests that the scribes had a kind of 'geometric algebra', a set of geometric techniques for solving problems. He further speculated that these techniques developed from the concrete problems of surveys. See Jens Hørup, Lengths, Widths, Surfaces: A Portrait of Old Babylonian Algebra and Its Kin Springer 2002.

See also Rahul Roy On Ancient Babylonian Algebra and Geometry at https://www.ias.ac.in/article/fulltext/reso/008/08/0027-0042

Along these lines, Greek philosophers and historians attributed the development of their own geometry to Egyptians and Mesopotamians: Herodotus (about 400-500 BCE) wrote: "They said also that this king [Sesostris] divided the land among all Egyptians so as to give each one a quadrangle of equal size and to draw from each his revenues, by imposing a tax to be levied yearly. But every one from whose part the river tore away anything, had to go to him and notify what had happened; he then sent the overseers, who had to measure out by how much the land had become smaller, in order that the owner might pay on what was left, in proportion to the entire tax imposed. In this way, it appears to me, geometry originated, which passed thence to Hellas."
A photograph of the Yale Babylonian Collection’s Tablet YBC 7289 (c. 1800 to 1600 BCE), showing a Babylonian approximation to the square root of 2 \((1, 24, 51, 10 \text{ base } 60)\). Babylonian mathematicians knew Pythagoras’ Theorem relating the sides of a right triangle. The photo is by Yale professor Bill Casselman; see http://www.math.ubc.ca/ cass/Euclid/ybc/ybc.html.

The result that \(\sqrt{2}\) is not a rational is contained in the works of the Greek mathematician Euclid. The original proof is ‘lost in the mists of time’, but there’s reason to believe it was more of a plausible picture than a proof. The rough sketch of a proof that we give is a modernized version of the one Euclid gave (and is an algebraic version of that more ancient geometric picture-proof).

The first result we need is that every integer is either even (has a factor of 2: 2, 4, 6, 8, \ldots) or odd (has no factor of 2: 1, 3, 5, 7, \ldots).

This is clear enough from the list, and is easy to prove using induction. A little algebra gives that the squares of even numbers are even (4, 16, 36, 64, \ldots), and the squares of odd numbers are odd (1, 9, 25, 49, \ldots). Easy, but important, because it allows us to do things where we’d want to use square-root: normally, if \(n\) had a factor of 2, all we could say that \(\sqrt{n^2}\) has a factor of \(\sqrt{2}\). But, using even/odd, we can do more: if \(n^2\) has a factor of 2, then \(n\) has a factor of 2, instead of just a factor of \(\sqrt{2}\).

The next result we need is that you can simplify fractions by canceling out common factors: \(60/36 = 30/18 = 15/9 = 5/3\) and now there’s no longer any common factors. This is a bit harder to prove; it needs our strong induction.

Now we can start: say you can find \(\sqrt{2}\) as a fraction \(p/q\). You might as well cancel out common factors. Then

\[
\frac{p}{q} = \sqrt{2} \quad \text{so} \quad \frac{p^2}{q^2} = 2 \quad \text{so} \quad p^2 = 2q^2
\]

This shows \(p^2\) has a factor of 2, so \(p\) also has a factor of 2; write it \(p = 2r\). Then \(p^2 = 2q^2\) becomes \((2r)^2 = 2q^2\) or \(4r^2 = 2q^2\). Divide by 2 to get \(2r^2 = q^2\); now \(q^2\) has a factor of 2, so \(q\) also has a factor of 2. But we already cancelled all the 2’s, and so, no such \(p\) and \(q\) can exist.

The discovery of irrational numbers is attributed to the philosopher Hippasus of Metapontum; the irrational in question was likely \(1 + \sqrt{2}/2\), derived from a pentagram. Hippasus lived in the late fifth century BC (that is, from 500 to 401 BCE, closer to 401), and was a member of the Pythagoreans, (see p18). Pythagoreans explained all the world by integers, so the discovery of irrationals was a serious
challenge to their beliefs.

A note on ‘rational’ and ‘irrational’. For much of Greek mathematics, fractions were thought of as ratios. Hence, our quotients \( \frac{p}{q} \) were, to the Greeks, the rational numbers, because they were ratios. Historically, ‘rational’ is from the word ‘ratio,’ from the Latin word ‘to compute’. This in turn is from Proto Indo-European, *reh-, to ‘put in order’. None of this relates to our uses like ‘rational thought’, which comes the French, meaning "right, just, fitting, fair ".

New irrationals appeared in Book X of Euclid’s Geometry; these came from ‘ruler and compass’ constructions. For example, a right triangle with side lengths one has hypotenuse \( \sqrt{2} \). Now start using \( \sqrt{2} \) as a base of a right triangle; continuing on, you can get \( \sqrt{1 + \sqrt{2}}, \sqrt{3 + \sqrt{5}}, \sqrt{\sqrt{5} + \sqrt{3}} \). Euclid showed that the theory of magnitudes could generate these; his irrationals all came from geometry.

Both Archimedes and Hero of Alexandria worked on finding the value of \( \pi \), the ratio of the circumference of a circle to its diameter. They approximated the circle by \( n \)-agons, dissected these into triangles, as in Figure 53, and computed areas or lengths of their sides; either led to irrationals. But they needed numbers; both used variations on the estimates

\[
p + \frac{r}{2p + 1} \leq \sqrt{p^2 + r} \leq p + \frac{r}{2p}
\]

While the estimate is reminiscent of the Mesopotamian computations for \( \frac{1}{7} \), the computation was motivated, once more, by geometry.

Figure 54 gives a geometric proof that \((p + q)^2 = p^2 + 2pq + q^2\): it shows that if you start with a square of area \( p^2 \), you can get a square of larger area \((p + q)^2\) by adding two rectangles of area \( pq \), and a square of area \( q^2 \).

Here’s how this led to an approximation for \( \sqrt{2} \): \( \sqrt{(p + q)^2} = p + q \), so \( \sqrt{p^2 + 2pq + q^2} = p + q \). If \( q \) is a small number (think \( q = .01 \)) then \( q^2 \) is even smaller (think \( q^2 = .0001 \)). In our approximation, we can ignore it; then \( \sqrt{p^2 + 2pq} \) is about the same as \( \sqrt{p^2 + 2pq + q^2} = p + q \). Now let \( r = 2pq \), then \( q = \frac{r}{2p} \) and we get Archimedes’ approximation, \( \sqrt{p^2 + r} \) is about \( p + \frac{r}{2p} \). We’ll use tangent lines, to get the same result.