**Chapter 1: Numbers**

**Section 5: The Real Numbers**

In the mid-1800’s, many European mathematicians were working on irrationals (see p64). There were two issues; we’ve discussed the first: are all irrationals infinite decimals? The second issue is the problem of limits. Answering these questions was part of a movement: the *arithmetisation* of geometry, eliminating intuitive ideas from geometry, and replacing them with better defined notions of arithmetic and algebra.

Limits, by this time, were important to all kinds of mathematics. For example, we can compute \( \sqrt{2} \). Take \( \{g_1, g_2, \ldots\} \) where \( g_2 = \frac{1}{2} \left( g_1 + \frac{2}{g_1} \right) \), etc, and if we believe that \( \{g_1, g_2, \ldots\} \) has a limit, \( g \), then the limit has to be \( g = \frac{1}{2} \left( g + \frac{2}{g} \right) \). Solving this gives \( g^2 = 2 \). We suspected the \( \{g_1, g_2, \ldots\} \) became progressively better approximations to \( \sqrt{2} \); with a theory of limits, we can say the limit of the approximations is \( \sqrt{2} \).

A bit after Newton, calculus was differentiation, integration, approximation by tangent lines, and infinite series. Especially infinite series. These could be treated as three separate techniques, until the work of Cauchy, Figure 55, who in 1821 had published the first calculus book: *Cours d’analyse*. Cauchy gave careful proofs of the main results of calculus (see p64), but he also showed how the three topics of traditional calculus could be done carefully by reducing them to problems about limits. No understanding of limits: no calculus. Cauchy was aware of techniques for approximating numbers like \( \pi, \sqrt{2} \), and also of how one could talk about the accuracy of approximations, using the same ideas we discussed for computing \( \frac{1}{7} \) on p47: the *error* in approximating the true value \( t \) by an approximation \( a \) is given by \( |t - a| \), and we can check how small the error is by writing inequalities like \( |t - a| < 2.35 \times 10^{-6} \), as we did with \( \frac{1}{7} \) (see p46).

But, leftover from Newton and Stifel, p52, there was still no idea of what \( \sqrt{2} \) is as a number rather than a ‘magnitude’. We might prove \( \{g_1, g_2, \ldots\} \) converges by showing \(|\sqrt{2} - g_n|\) is small – but to do this, we first need to know \( \sqrt{2} \) is a number. Standoff.

The mathematician Georg Cantor resolved these issues, giving a construction of the real numbers. Why *construction*? Because it wasn’t as though everyone suddenly hit on the one idea that was out there waiting to be discovered. Nor did everyone say ‘Why of course, that’s what I was thinking all along’. In reality, several other constructions competed for acceptance, and others besides Cantor’s are still used.
The real numbers were built, not handed down.

We’re going to do an overview of Cantor’s construction; a large part of modern mathematics depends on this. Cantor took important ideas from from Cauchy; what he needed was Cauchy’s idea of how to talk about convergence without mentioning the limit. That idea starts with a list of numbers (a sequence) \((a_1, a_2, a_3, \ldots)\) abbreviated as \((a_k)\); it’s understood that \(k\) goes through all the natural numbers, one after another (sequentially).

If a sequence \((a_k)\) converges, then after a while all the \(a_k\) have to be close to the limit, so they have to be close to each other. The phrase ‘after a while’ translates to the existence of a number \(N\) specifying exactly when that closeness begins to happen; the word ‘all’ translates to ‘all numbers bigger than \(N\’). In symbols, \(j \geq N, k \geq N\).

"Close to each other’ means that the distance between them is small; that translates into saying that \(|a_j - a_k|\) is small. How small? Well, to be a limit, it has to get smaller and smaller. Put another way, if you tell me how small you want it, I can make it that small. ‘How small you want it’ translates to ‘for any \(\epsilon > 0\’, and ‘I can make it that small’ now translates to \(|a_j - a_k| < \epsilon\). When you put it all together, you have a definition of convergence that doesn’t mention the actual limit:

**Cauchy Condition for Convergence:** If the sequence \((a_k)\) converges, then for every \(\epsilon > 0\), there is a number \(N\) such that, whenever both \(j \geq \ N, k \geq N\), then \(|a_j - a_k| < \epsilon\).

Cantor introduced a second idea: we believe that the sequence \((3, 3.1, 3.14, 3.145, \ldots)\) converges to \(\pi\), but we don’t know what \(\pi\) is. Why not define \(\pi\) to be the sequence \((3, 3.1, 3.14, 3.145, \ldots)\)? In general, a real number is defined to be any sequence? Well, not any sequence: the sequence \((1, 0, 4, 0, 9, 0, 16, 0, \ldots)\) doesn’t converge, so we don’t want that; we want only the convergent sequences. That is, Cauchy sequences! Let’s check this with \(\pi\) defined as \((3, 3.1, 3.14, 3.145, \ldots)\):

\[
\begin{align*}
|a_1 - a_2| &= |3 - 3.1| = .1 = \frac{1}{10} \\
|a_2 - a_3| &= |3.1 - 3.14| = .04 = \frac{4}{10^2} \\
|a_3 - a_4| &= |3.14 - 3.145| = .005 = \frac{5}{10^3}
\end{align*}
\]

These are getting smaller, so it’s a nice start on the road to showing the sequence is Cauchy, but it isn’t enough: we need all \(j \geq N, k \geq N\), and here we only have \(j \geq N, k = j + 1\). See p64 for a full proof.

The third idea is this: go back to \((3, 3.1, 3.14, 3.145, \ldots)\). All of these are finite decimals; they’re rational numbers. So: a real number is a Cauchy sequence of rational numbers.
There’s a fourth idea, and it’s that pesky $\epsilon$ being infinitely small issue, again. Cantor avoided it by restating the Cauchy condition: instead of saying ‘for every $\epsilon > 0$’, he says ‘for every rational number $\epsilon > 0$’. The Archimedean Principle for Rationals (p46) now eliminates any worries about infinitely small epsilons.

Cantor calls the collection of Cauchy sequences the real numbers, $\mathbb{R}$.

There are lots of questions:

**Question:** What are rational numbers in this new definition?
**Answer:** A rational number, say $\frac{1}{2}$, would be the sequence $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots\}$.

**Question:** How am I supposed to add, subtract, etc, these things?
**Answer:** This turns out to be amazingly easy: take two real numbers $R = (r_1, r_2, \ldots)$ and $S = (s_1, s_2, \ldots)$; $R + S = (r_1 + s_1, r_2 + s_2, \ldots)$. Same for multiplication, etc. These are sequences of rational numbers, and it’s easy to show they’re Cauchy.

**Question:** What happens to the whole theory of limits?
**Answer:** We’d like to have the usual definitions, but the tricky part is defining statements like $|R_k - R| < \epsilon$. That’s just $R - \epsilon < R_k < R + \epsilon$, so what we really need to understand is inequalities, like $R < S$. That’s the same as asking for the meaning of $0 < S - R$, so in the end, we just need to know what it means for a real number $R = (r_1, r_2, \ldots)$ to be positive.

Here’s where Cantor’s sneaky fourth idea comes in: using rational numbers to define inequalities. We’d like to just say that $R$ is positive if the rational numbers making up $R$ are positive, but that doesn’t work: take $R = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$. Intuitively, $R$ is the ‘limit’, and that ‘limit’ is zero, not positive. So Cantor defined $R > 0$ as follows: $R = (r_1, r_2, \ldots) > 0$ means there is a rational number $r > 0$ and an $N$ such that if $k \geq N$, then $r_k \geq r$: the rational $r > 0$ keeps the ‘limit’ away from zero.

We also put on the extra words ‘if $k \geq N$’. Again, it’s about limits: the sequence $(-1, -\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$ has limit $-1$, which is positive: what the first three terms do is irrelevant. That what $N$ does: if I take $N = 4$, then $n \geq N$ says first three are irrelevant. Now with $r = .5$, and $N = 4$ for all $k \geq N$, $r_k \geq r$.

**Question:** Many sequences of rationals converge to $\sqrt{2}$, so which am I supposed to use?
**Answer:** We can use any sequence we like. All you need to do is check: sequences $(a_k)$, $(b_k)$ define the same real number if, intuitively, they have the same limit; see p64 for the technical details.
Question: What if I take a limit of sequences of reals? Do I get some weird new kind of number?

Answer: Actually, no. If you took a sequence of real numbers \((R_1, R_2, \ldots)\) with limit \(R\), then you can also find rational numbers \((r_1, r_2, \ldots)\) with \(R = (r_1, r_2, \ldots)\). Very roughly, the idea is this: if
\[
R_1 = (a_1, a_2, a_3, \ldots)
\]
\[
R_2 = (b_1, b_2, b_3, \ldots)
\]
\[
R_3 = (c_1, c_2, c_3, \ldots)
\]
Then
\[
R = (a_1, b_2, c_3, \ldots),
\] which will work.

With this out of the way, Cantor proves four easy results:

One: Every Cauchy sequence of real numbers converges to a real number. This is called the completeness of the real numbers.

Two: Every real number is a limit of rational numbers; in fact, if
\[
R = (r_1, r_2, \ldots),
\]
then, with sloppy notation, \(\lim r_k = R\). This is called the density of the rational numbers.

Three: All the known irrationals were limits of rational numbers, so all the known irrationals are now official real numbers.

Finally, Cantor’s definition of \(R > 0\), using rational numbers, allows us to show the real numbers \(\mathbb{R}\) don’t contain infinitely small numbers. Here’s why:

Take a supposed infinitely small real number \(\gamma = \{r_1, r_2, \ldots\} > 0\).

\(\gamma > 0\) means there’s a rational number \(r > 0\) and an \(N\) such that for all \(k \geq N, r_k \geq r\).

Now we have our rational number \(r = \frac{p}{q} \leq \gamma < \frac{1}{10^k}\) for all \(k\). But this contradicts the Archimedean Principle for Rationals (p46), that for a rational \(r, 0 < r < 1\), there’s an \(m\) with \(\frac{1}{10^m} < r\). No infinitely small rational numbers, no infinitely small real numbers. And, BTW, .9999 . . . actually is equal to 1.

We used decimal expansions to get an idea of what is in the infinite collection \(\mathbb{N}\); we’d like something similar for \(\mathbb{R}\). We know each real number is a limit of rational numbers, and each rational number is a repeating decimal, but it might be that the limit is some very weird kind of object. For example, we’d like \(\pi = (3, 3.1, 3.14, \ldots)\), but it might instead be more like \((7, 2.1, 6.3, 2.4 \ldots)\). We’ll look at it in the next section, but for now, it’s worth noting, that \((3, 3.1, 3.14, \ldots)\) isn’t necessarily the best way to approximate and to compute \(\pi\); see p65.

What we have right now is that the real numbers are all limits of rationals. We also know that every infinite decimal gives rise to a Cauchy sequence like \((3, 3.1, 3.14, \ldots)\), whose limit is the real num-
ber representing that infinite decimal. What we haven’t yet shown is that every irrational is an infinite decimal; we’ll do exactly that in Section 6 Part 2.

Cantor’s theory raised troubling questions. For example: we can actually compute the decimal approximations for $\sqrt{2}$ and $\pi$ with the methods given by the Mesopotamians and by Lui Hui. We can even write computer programs to do the computations. Call numbers like $\pi$, $\sqrt{2}$ computable numbers. Are all real numbers computable? Cantor showed they are not: most real numbers are not computable (see p65).

If there are so many (most, actually) real numbers that aren’t even computable, how real is $\mathbb{R}$? Put another way, we have more numbers than we can understand – did we go too far? It certainly could seem so. After all, we have a very concrete understanding of what a number ought to be; recall the discussion of animal counting from p10. Numbers are likely built into the way our brain sees the world. Cantor’s construction is nothing like our built-in view of a number.

On the other hand, Eudoxus’ theory of magnitude, p52, views $\sqrt{2}$ as an undefined ‘magnitude,’ whatever that is. It may well be that the basic construct of number that evolution has given us for picking berries is not adequate for understanding a wider and wider universe.

Another answer is this: $\mathbb{R}$ is the smallest collection of numbers that contains all the limits of numbers from $\mathbb{Q}$ (and in which the ordinary rules of arithmetic hold). In technical terms, $\mathbb{R}$ is the smallest complete ordered field containing $\mathbb{Q}$.

And here’s a third answer: we talked about algebraic numbers being solutions to polynomial equations (with integer coefficients; see p52), but numbers like $\pi$, $e$ are solutions to equations with trig or logs. From this standpoint, $\mathbb{R}$ is, in a way we’ll make precise, just what we need to get solutions to all equations. In technical terms, if $f(x)$ is a continuous function, and the equation $f(x) = 0$ has a solution, then that solution is a real number. Of course, as with $x^2 + 1 = 0$, the solution could be a complex number. But the complex numbers $z$ are all of the form $z = x + iy$ where $x$ and $y$ are real numbers, and we’re back to real numbers.

So – like the story of the three bears – $\mathbb{R}$ is just right.
Notes for Chapter 1 Section 5: The Real numbers


For a list of contemporary, alternative constructions of the real numbers, see The Real Numbers – A Survey of Constructions by Ittaty Weiss, Rocky Mountain J. Math. Volume 45, Number 3 (2015), 737-762.

p59 Cauchy’s work on convergence wasn’t motivated by the status of the irrationals. He was trying to give the methods of calculus a logical justification by supplying rigorous proofs for all the results. This type of work is ‘working on the foundations’ of the calculus; the analogy being that without good foundations, buildings collapse. Would mathematics collapse without Cauchy?

After Newton and Leibnitz developed calculus, mathematicians worked to extend their ideas and give applications to physics, mechanics and engineering; meaning and proof were low priority (see Grabiner, Judith The Origins of Cauchy’s Rigorous Calculus, The MIT Press 1981). This began to change in the mid-eighteen hundreds when it became apparent that some of these applications were actually wrong. Cauchy was one of several mathematicians working to separate out the true results from the false; we’ll meet some of these later on. So, Yes: without foundations, mathematics was collapsing.

p61 Two Cauchy sequences in Cantor’s construction are equivalent if they have the same limit. Since Cantor was trying to develop a theory of limits, he used a Cauchy sequence idea: Cauchy sequences \((a_k), (b_k)\) are equivalent if for every \(\epsilon > 0\), there is a number \(N\) such that, whenever \(k \geq N\), then \(|a_k - b_k| < \epsilon\). The collection of all sequences equivalent to \((a_k)\) is denoted \([[(a_k)]]\), and the real number corresponding to \((a_k)\) is actually the equivalence class \([[(a_k)]]\).

p60 For Cauchy, it isn’t enough to show \(|a_3 - a_4| < \epsilon\) etc.; you also have to show \(|a_3 - a_5| < \epsilon\) and \(|a_3 - a_6| < \epsilon, etc. But for some sequences there’s a cheat: if \(|a_n - a_{n+1}| < \frac{C}{10^n}\) for all \(n\), then the sequence is Cauchy. We certainly have that with 3, 3.1, 3.14, 3.145, . . . in fact it’s easy to show any infinite decimal gives a Cauchy sequence. Say you have .8769539 . . . ; make a sequence out of it as we did with \(\pi\): (.8, .87, .876, . . .). Then you get \(|a_n - a_{n+1}| < \frac{9}{10^n}\), because decimal digits can’t be greater than 9.
What real numbers are ‘computable?’ To understand ‘computable’, look at our basic example, the computation of $\sqrt{2}$. We start with a guess $g_1$, then set $g_2 = \frac{1}{2} \left( g_1 + \frac{2}{g_1} \right)$. The side note shows how to make a computer program for this. We could even say that the computable numbers are the numbers that you can get from computer programs. It’s a practical definition of computing!

This means that the collection of computable numbers will be found in the collection of all computer programs (along with a lot of junk, like ‘hello world’). But the collection of all programs is contained in the collection of all finitely long collections of words. We’ll see the collection of all real numbers is the same as the collection of infinite decimals, which has the same size as the collection of all infinite lists. Cantor showed that the collection of all infinite lists is much larger than the collection of finite lists, so the computable numbers are a smaller collection than the collection of all real numbers.

Put another way, if you randomly picked a number from all the reals, the probability that the number is computable is zero. This argument is part of Cantor’s theory of transfinite numbers. For a popular, non-technical presentation, see Rudy Rucker’s *Infinity and the Mind*, Princeton University Press 2004. A basic technical introduction is in Michael J. Schramm’s *Introduction to Real Analysis*, Dover Publications 2008.

On a more hopeful side, we do know that all the algebraic irrationals are computable, and of course this includes all the irrationals known to Euclid.

We have a method to approximate numbers like $\sqrt{2}$ by decimals: the method gives the successive approximations 1, 1.4, 1.41, 1.414, … If you square these numbers, you get 1, 1.96, 1.9881, 1.999396, …, but there are other ways to approximate $\sqrt{2}$.

There’s an old technique (going back to the Greeks), called continued fractions, that gives a very general way to approximate a large class of irrationals. Here’s the idea: $\sqrt{2}$ satisfies the quadratic $x^2 - 2 = 0$. I can’t factor that, but I can factor $x^2 - 1 = 1$ as $(x - 1)(x + 1) = 1$, hence

$$x - 1 = \frac{1}{1 + x} \quad \text{or} \quad x = 1 + \frac{1}{1 + x}.$$

Oops, there are $x$’s on both sides. That’s okay, I already solved for $x$; let’s plug that into the right side:

$$x = 1 + \frac{1}{1 + x} = 1 + \frac{1}{1 + \left(1 + \frac{1}{1 + x}\right)} = 1 + \frac{1}{2 + \frac{1}{1 + x}}.$$
If we continue, we get

$$x = 1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}}$$

Now let’s compute $\sqrt{2}$. If we start with $x = 1$, we get $x = 3$, then $\frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \ldots$. The squares are 2.25, 1.96, 2.006944, ..., 1.99888109, ...

For a more serious overview, see http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/cfINTRO.html