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Contents

1. Complex Numbers

1.1. The Algebra of Complex Numbers.

1.1.5. *Inequalities.*
1.1.5.1. Prove that

$$\left|\frac{a-b}{1-\overline{a}b}\right| < 1$$

if $|a| < 1$ and $|b| < 1$.

Solution: The inequality is equivalent to showing that

$$|a-b| < |1-\overline{a}b|.$$

By definition of the norm,

$$\begin{aligned}
|a-b|^2 &= (a-b)\overline{(a-b)} = (a-b)(\overline{a}-\overline{b}) \\
&= |a|^2 - (a\overline{b} + \overline{a}b) + |b|^2 \\
|1-\overline{a}b|^2 &= (1-\overline{a}b)\overline{(1-\overline{a}b)} = (1-\overline{a}b)(1-a\overline{b}) \\
&= 1 - (a\overline{b} + \overline{a}b) + |a|^2|b|^2.
\end{aligned}$$

Thus it suffices to show

$$|a|^2 + |b|^2 < 1 + |a|^2|b|^2.$$

Starting with $|a| < 1$, we have that $|a|^2 < 1$ and thus $|a|^2 < (1-|b|^2)/(1-|b|^2)$. since $0 < |b|^2 < 1$, $1-|b|^2$ is positive and therefore $|a|^2(1-|b|^2) < 1-|b|^2$. Rearranging this gives us the above inequality.

Note that equality holds when either $|a| = 1$ or $|b| = 1$.

1.1.5.2. Prove Cauchy's inequality by induction.

Solution: Cauchy's inequality states that

$$\left|\sum_{j=1}^{n} a_j b_j\right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2.$$

The case $n = 1$ is always an equality since

$$\left|\sum_{j=1}^{1} a_j b_j\right|^2 = |a_1 b_1|^2 = |a_1|^2|b_1|^2 = \sum_{j=1}^{1} |a_j|^2 \sum_{j=1}^{1} |b_j|^2.$$

For the case $n = 2$, first note that

$$(|a_1||b_2| - |a_2||b_1|)^2 = |a_1|^2|b_2|^2 + |a_2|^2|b_1|^2 - 2|a_1|^2|a_2|^2|b_1|^2|b_2|^2 \geq 0.$$

Adding $|a_1|^2|b_1|^2 + |a_2|^2|b_2|^2$ to each side and rearranging gives

$$\begin{aligned}
(|a_1 b_1| + |a_2 b_2|)^2 &= |a_1|^2|b_1|^2 + 2|a_1|^2|b_1|^2|a_2|^2|b_2|^2 + |a_2|^2|b_2|^2 \\
&\leq |a_1|^2|b_1|^2 + |a_1|^2|b_2|^2 + |a_2|^2|b_1|^2 + |a_2|^2|b_2|^2 \\
&= (|a_1|^2 + |a_2|^2)(|b_1|^2 + |b_2|^2)
\end{aligned}$$

Then, by the triangle inequality

$$\begin{aligned}
\left|\sum_{j=1}^{2} a_j b_j\right|^2 &= |a_1 b_1 + a_2 b_2|^2 \leq (|a_1 b_1| + |a_2 b_2|)^2 \\
&\leq (|a_1|^2 + |a_2|^2)(|b_1|^2 + |b_2|^2) = \sum_{j=1}^{2} |a_j|^2 \sum_{j=1}^{2} |b_j|^2.
\end{aligned}$$

Now suppose it holds for $n = k$; we wish to show it holds for $n = k + 1$. By the triangle inequality and direct application of the inductive hypothesis,

$$\begin{aligned}\left|\sum_{j=1}^{k+1} a_j b_j\right|^2 &= \left|\sum_{j=1}^{k} a_j b_j + a_{k+1}b_{k+1}\right|^2 \\ &\leq \left|\left|\sum_{j=1}^{k} a_j b_j\right| + |a_{k+1}b_{k+1}|\right|^2 \leq \left|\sqrt{\sum_{j=1}^{k}|a_j|^2}\sqrt{\sum_{j=1}^{k}|b_j|^2} + |a_{k+1}b_{k+1}|\right|^2 \\ &\leq \left(\sum_{j=1}^{k}|a_j|^2 + |a_{k+1}|^2\right)\left(\sum_{j=1}^{k}|b_j|^2 + |b_{k+1}|^2\right) = \sum_{j=1}^{k+1}|a_j|^2 \sum_{j=1}^{k+1}|b_j|^2\end{aligned}$$

where we used the same inequality trick to pass from the second to third line as in the $n = 2$ case.

1.1.5.3. If $|a_i| < 1$, $\lambda_i \geq 0$ for $i = 1, ..., n$ and $\lambda_1 + ... + \lambda_n = 1$, show that

$$|\lambda_1 a_1 + \lambda_2 a_2 + ... + \lambda_n a_n| < 1.$$

Solution: By the general triangle inequality,

$$|\lambda_1 a_1 + ... + \lambda_n a_n| \leq \sum_{j=1}^{n} |\lambda_j||a_j| < \sum_{j=1}^{n} \lambda_j = 1.$$

1.1.5.4. Show that there are complex numbers z and a satisfying

$$|z - a| + |z + a| = 2|c|$$

if and only if $|a| \leq |c|$. If this condition is fulfilled, what are the smallest and largest possible values of $|z|$?

Solution: By the reverse triangle inequality, if such a and z exist then

$$2|c| = |z - a| + |z + a| \geq |z - a - (z + a)| = 2|a|$$

so that $|a| \leq |c|$. For the reverse implication, consider the two circles

$$\begin{aligned}|z - a| &= |c| \\ |z + a| &= |c|.\end{aligned}$$

Clearly if z lies on both circles, it will satisfy $|z - a| + |z + a| = 2|c|$. Since the circles have the same radius, finding a solution is easy. It will lie on the line perpendicular to the line connecting a and $-a$ and passing through 0. One such solution is $\sqrt{|c|^2 - |a|^2}/|a|ia$.

As a remark, the solution set to this is an ellipse (which may degenerate to a line if $|a| = |c|$). The focii are at $\pm a$. This gives insight into determining the maximum and minimum values of $|z|$, they are just the major and minor radii.

1.2. **The Geometric Representation of Complex Numbers.**

1.2.1. *Geometric Addition and Multiplication.*

1.2.1.1. Find the symmetric points of a with respect to the lines which bisect the angles between the coordinate axes.

Solution: Let L_1 and L_2 denote the real axis rotated counterclockwise by $\pi/4$ and $3\pi/4$ respectively. The symmetric point of a about L_1 can be computed by rotating the entire frame by $-\pi/4$, reflecting a across the real axis (i.e., taking the complex conjugate of this new point), then rotating back by $\pi/4$. Notice that the magnitude does not change, so we just need to find the argument. An important property about the argument is that $\arg(\overline{z}) = -\arg(z)$. So, after the first rotation by $-\pi/4$ the argument is $\arg(a) - \pi/4$. Taking the complex conjugate gives $-(\arg(a) - \pi/4) = \pi/4 - \arg(a)$. Finally, rotating back by $\pi/4$ gives a final argument of $\pi/2 - \arg(a)$. The same computation works for L_2, just changing $\pi/4$ with $3\pi/4$ everywhere. The final argument in this case is $3\pi/2 - \arg(a)$.

Call these symmetric points a_1 and a_2. Then we have

$$\arg(a_1) = \arg(\pi/2) - \arg(a) = \arg(i) + \arg(\overline{a}) = \arg(i\overline{a})$$
$$\arg(a_2) = \arg(3\pi/2) - \arg(a) = \arg(-i) + \arg(\overline{a}) = \arg(-i\overline{a})$$

Since $|i\overline{a}| = |a| = |a_1|$ and similarly for a_2, it follows that $a_1 = i\overline{a}$ and $a_2 = -i\overline{a}$.

1.2.1.2. Prove that the points a_1, a_2, a_3 are vertices of an equilateral triangle if and only if $a_1^2 + a_2^2 + a_3^2 = a_1a_2 + a_2a_3 + a_3a_1$.

Solution: At a first glance, this problem has too much generality if we do not fix the triangle ins some particular location. Let us show that the equality given is actually translation invariant – this is a helpful trick for many problems, for example one often uses scale invariance to deduce information about geometric inequalities.

Suppose that all the vertices a_1, a_2, a_3 are translated by a single complex number z. Then,

$$\begin{aligned}(a_1 - z)^2 + (a_2 - z)^2 + (a_3 - z)^2 &= a_1^2 + a_2^2 + a_3^2 - a_1z - z(a_1 - z) - a_2z - z(a_2 - z) - a_3z - z(a_3 - z)\\ &= a_1a_2 + a_2a_3 + a_3a_1 - a_1z - z(a_1 - z) - a_2z - z(a_2 - z) - a_3z - z(a_3 - z)\\ &= [a_1a_2 - a_1z - z(a_1 - z)] + [a_2a_3 - a_2z - z(a_2 - z)] + [a_3a_1 - a_3z - z(a_3 - z)]\\ &= (a_1 - z)(a_2 - z) + (a_2 - z)(a_3 - z) + (a_3 - z)(a_1 - z).\end{aligned}$$

So, the equality is translation invariant. Now we assume that $a_1 = 0$, which will kill two of the terms in the right hand side! Thus, we want to show that $0, a_2, a_3$ are vertices of an equilateral triangle if and only if $a_2^2 + a_3^2 = a_2a_3$. We can reduce one more level of generality by rotating and dilating the frame. That is, multiply each point by $\overline{a_2}/|a_2|^2$ (which we can assume is nonzero, since otherwise $a_3 = 0$ automatically). The resulting triangle has vertices at 0, 1, and $\tilde{a}_3 = a_3\overline{a}_2/|a_2|^2$. We finally wish to show that this forms an equilateral triangle if and only if $1 + \tilde{a}_3^2 = \tilde{a}_3$. Note that we have an equilateral triangle if and only if $\tilde{a}_3 = 1/2 \pm \sqrt{3}/2i$. These are exactly the two roots of the quadratic equation we have above.

1.2.1.3. Suppose that a and b are two vertices of a square. Find the two other vertices in all possible cases.

Solution: Note that a square has four-fold symmetry, so that multiplication by i permutes the vertices. Hence, all the vertices are $a, ia, -a, -ia$. One of these will be b.

1.2.1.4. Find the center and the radius of the circle which circumscribes the triangle with vertices a_1, a_2, a_3. Express the result in symmetric form.

Solution: Let c denote the center of the circumscribed circle and r its radius. Since the circle passes through all the vertices a_i,

$$|c|^2 + |a_i|^2 - c\overline{a_i} - a_i\overline{c} = \overline{(c - a_i)}(c - a_i) = |c - a_i|^2 = r^2.$$

Rewriting each of these gives

$$\begin{aligned}\overline{a_1}c + a_1\overline{c} + [r^2 - |c|^2] &= |a_1|^2\\ \overline{a_2}c + a_2\overline{c} + [r^2 - |c|^2] &= |a_2|^2\\ \overline{a_3}c + a_3\overline{c} + [r^2 - |c|^2] &= |a_2|^2\end{aligned}$$

Now translate a_1 and a_2 by a_3 so that we get a new triangle with vertices $0, a_1 - a_3, a_2 - a_3$. The circumcenter will now be $c - a_3$ but the radius XXX

1.2.2. *The Binomial Equation.*

1.2.2.1. Express $\cos 3\varphi$, $\cos 4\varphi$, and $\sin 5\varphi$ in terms of $\cos\varphi$ and $\sin\varphi$.

Solution: By de Moivre's formula,

$$\begin{aligned}
\cos 3\varphi + i\sin 3\varphi &= (\cos\varphi + i\sin\varphi)^3 = ([\cos^2\varphi - \sin^2\varphi] + i[2\cos\varphi\sin\varphi])(\cos\varphi + i\sin\varphi) \\
&= [\cos^3\varphi - 3\cos\varphi\sin^2\varphi] + i[3\cos^2\varphi\sin\varphi - \sin^3\varphi] \\
\cos 4\varphi + i\sin 4\varphi &= (\cos\varphi + i\sin\varphi)^4 = (\cos\varphi + i\sin\varphi)^3(\cos\varphi + i\sin\varphi) \\
&= [\cos^4\varphi - 6\cos^2\varphi\sin^2\varphi + \sin^4\varphi] + i[4\cos^3\varphi\sin\varphi - 4\cos\varphi\sin^3\varphi] \\
\cos 5\varphi + i\sin 5\varphi &= (\cos\varphi + i\sin\varphi)^5 = (\cos\varphi + i\sin\varphi)^4(\cos\varphi + i\sin\varphi) \\
&= [\cos^5\varphi - 10\cos^3\varphi\sin^2\varphi + 5\cos\varphi\sin^4\varphi] \\
&\quad i[5\cos^4\varphi\sin\varphi - 10\cos^2\varphi\sin^3\varphi + \sin^5\varphi].
\end{aligned}$$

Equating real and imaginary parts gives

$$\begin{aligned}
\cos 3\varphi &= \cos^3\varphi - 3\cos\varphi\sin^2\varphi \\
\cos 4\varphi &= \cos^4\varphi - 6\cos^2\varphi\sin^2\varphi + \sin^4\varphi \\
\sin 5\varphi &= 5\cos^4\varphi\sin\varphi - 10\cos^2\varphi\sin^3\varphi + \sin^5\varphi.
\end{aligned}$$

There is an easy way to compute these in general. Consider the binomial expansion of $(x+y)^n$. Starting with x^n, select every other term and add them together in order. Next, collect the remaining terms and add them together. In each sum x with $\cos\varphi$ and y with $\sin\varphi$, then change the signs so they alternate (starting with $+$). The first sum is $\cos n\varphi$ while the latter is $\sin n\varphi$.

1.2.2.2. Simplify $1 + \cos\varphi + \cos 2\varphi + ... + \cos n\varphi$ and $1 + \sin\varphi + \sin 2\varphi + ... + \sin n\varphi$.

Solution: Let $z = \cos\varphi + i\sin\varphi$ and consider the sum

$$S = 1 + z + z^2 + ... + z^n.$$

By the standard geometric series formula, this is

$$\begin{aligned}
S &= \frac{1 - z^{n+1}}{1 - z} = \frac{(1 - z^{n+1})(1 - \overline{z})}{1 - 2\operatorname{Re}(z) + |z|^2} = \frac{(1 - \cos(n+1)\varphi - i\sin(n+1)\varphi)(1 - \cos\varphi + i\sin\varphi)}{2 - 2\cos\varphi} \\
&= \frac{(1 - \cos\varphi)(1 - \cos(n+1)\varphi) + \sin\varphi\sin(n+1)\varphi}{2(1 - \cos\varphi)} + \frac{\sin\varphi(1 - \cos(n+1)\varphi) - (1 - \cos\varphi)\sin(n+1)\varphi}{2(1 - \cos\varphi)} i \\
&= \left[\frac{1 - \cos(n+1)\varphi}{2} + \frac{\sin\varphi\sin(n+1)\varphi}{2(1 - \cos\varphi)}\right] + \left[\frac{\sin\varphi(1 - \cos(n+1)\varphi)}{2(1 - \cos\varphi)} - \frac{\sin(n+1)\varphi}{2}\right] i
\end{aligned}$$

The real and imaginary parts of these are the desired sums via de Moivre's theorem.

1.2.2.3. Express the fifth and tenth roots of unity in algebraic form.

Solution: Observe that

$$1 + \omega + ... + \omega^{n-1} = 0$$

when ω is an n-th root of unity. Hence, XXX

1.2.2.4. If ω is given by

$$\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n},$$

prove that

$$1 + \omega^h + \omega^{2h} + ... + \omega^{(n-1)h} = 0$$

for any integer h which is not a multiple of n.

Solution: It suffices to show that ω^h is an n-th root of unity. We have that

$$\arg(\omega^h) \text{ “}=\text{” } h\arg(\omega) = \frac{2\pi h}{n}$$

Here we need the assumption that h is not a multiple of n to proceed. Because of this, we can take $2\pi h/n$ and reduce it modulo 2π. This will just be one of $2\pi/n, ..., 2(n-1)\pi/n$, which are the arguments of the n-th roots of unity not equal to 1. Since the modulus is multiplicative, $|\omega^h| = |\omega| = 1$, and ω^h is an n-th root of unity.

1.2.2.5. What is the value of

$$1 - \omega^h + \omega^{2h} - ... + (-1)^{n-1}\omega^{(n-1)h}?$$

Solution: Notice the above sum is

$$S = 1 + (-\omega)^h + (-\omega)^{2h} + ... + (-\omega)^{(n-1)h} = \sum_{k=0}^{n-1}(-\omega^h)^k.$$

If $\omega^h = -1$, then the above sum is clearly just n. Note that this can only occur when n is even, since the n-th roots of unity are vertices of a regular n-gon with one vertex at 1. In particular, $\omega^h = -1$ only when $h = n/2 + nm = n(2m+1)/2$ for some integer m. In all other cases, we apply the geometric series formula to obtain

$$S = \frac{1-(-\omega^h)^n}{1-(-\omega^h)} = \frac{1-(-1)^n(\omega^n)^h}{1+\omega^h} = \frac{1-(-1)^n}{1+\omega^h}$$

since $\omega^n = 1$. Observe that if n is even, then we have $S = 0$. This suggests a nice geometric proof. Indeed, in this case the regular polygon generated is symmetric with respect to reflection about the origin. In particular, if ω is a root of unity then so too is $-\omega$. Hence, we can apply the standard result we know.

1.2.3. *Analytic Geometry.*
1.2.3.1. When does $az + b\overline{z} + c = 0$ represent a line?

Solution: Suppose this does represent the equation of a line. Then for solutions u, v it should be that $u - v$ is always a multiple of a particular complex number, say λw. Then, $\overline{u} - \overline{v} = \lambda\overline{z}$. By assumption, $au + b\overline{u} + c = 0$ and similarly for v, so subtracting them gives

$$\lambda[aw + b\overline{w}] = 0$$

for any λ. Thus, $aw + b\overline{w} = 0$ for some w. This implies that $|a| = |b|$. Moreover, since u satisfies the equation we have $\overline{a}\overline{u} + \overline{b}u + \overline{c} = 0$. Multiply this by b and the original equation by $\overline{a}$, then take the difference. This yields

$$\overline{a}b\overline{u} + \overline{b}bu + b\overline{c} - [a\overline{a}u + \overline{a}b\overline{u} + \overline{a}c] = 0$$

Two terms obviously cancel, and two more cancel since $|a| = |b|$. Hence, we are left with $\overline{a}c + b\overline{c} = 0$. Clearly, we cannot have $a = b = 0$ since this implies $c = 0$, which may not be the case a priori. One can also check that this is sufficient. XXX

1.2.3.2.

2. Complex Functions

2.1. **Introduction to the Concept of Analytic Function.**

2.1.2. *Analytic Functions.*

2.1.2.1. If $g(w)$ and $f(z)$ are analytic functions, show that $g(f(z))$ is also analytic.

Solution: Write $g(w) = u_g(x,y) + iv_g(x,y)$ and $f(z) = u_f(x,y) + iv_f(x,y)$. Then u_g, v_g and u_f, v_f all have continuous first partial derivatives and pairwise satisfy the Cauchy-Riemann equations. Let $h(z) = g(f(z))$ so that

$$h(z) = u_h(x,y) + iv_h(x,y) = u_g(u_f(x,y), v_f(x,y)) + iv_g(u_f(x,y), v_f(x,y)).$$

By the classical chain rule for multivariable real functions,

$$\begin{aligned}
\frac{\partial u_h}{\partial x} &= \frac{\partial u_g}{\partial x}\frac{\partial u_f}{\partial x} + \frac{\partial u_g}{\partial y}\frac{\partial v_f}{\partial x} & \qquad \frac{\partial v_h}{\partial x} &= \frac{\partial v_g}{\partial x}\frac{\partial u_f}{\partial x} + \frac{\partial v_g}{\partial y}\frac{\partial v_f}{\partial x} \\
\frac{\partial u_h}{\partial y} &= \frac{\partial u_g}{\partial x}\frac{\partial u_f}{\partial y} + \frac{\partial u_g}{\partial y}\frac{\partial v_f}{\partial y} & \qquad \frac{\partial v_h}{\partial y} &= \frac{\partial v_g}{\partial x}\frac{\partial u_f}{\partial y} + \frac{\partial v_g}{\partial y}\frac{\partial v_f}{\partial y}
\end{aligned}$$

which all exist and are continuous. Next we check they satisfy the Cauchy-Riemann equations

$$\begin{aligned}
\frac{\partial u_h}{\partial x} &= \frac{\partial u_g}{\partial x}\frac{\partial u_f}{\partial x} + \frac{\partial u_g}{\partial y}\frac{\partial v_f}{\partial x} & \qquad \frac{\partial u_h}{\partial y} &= \frac{\partial u_g}{\partial x}\frac{\partial u_f}{\partial y} + \frac{\partial u_g}{\partial y}\frac{\partial v_f}{\partial y} \\
&= \frac{\partial u_g}{\partial x}\frac{\partial v_f}{\partial y} - \frac{\partial u_g}{\partial y}\frac{\partial u_f}{\partial y} & &= -\frac{\partial u_g}{\partial x}\frac{\partial v_f}{\partial x} + \frac{\partial u_g}{\partial y}\frac{\partial u_f}{\partial x} \\
&= \frac{\partial v_g}{\partial y}\frac{\partial v_f}{\partial y} + \frac{\partial v_g}{\partial x}\frac{\partial u_f}{\partial y} = \frac{\partial v_h}{\partial y} & &= -\frac{\partial v_g}{\partial y}\frac{\partial v_f}{\partial x} - \frac{\partial v_g}{\partial x}\frac{\partial u_f}{\partial x} = -\frac{\partial v_h}{\partial x}
\end{aligned}$$

where, the move from the first to second line in each we use the Cauchy-Riemann equations for f, then to move from the second to third line we use the Cauchy-Riemann equations for g. It follows that $h(z) = g(f(z))$ is harmonic.

2.1.2.2. Verify Cauchy-Riemann's equations for the functions z^2 and z^3

Solution: We must first compute the real and imaginary parts of these functions:

$$\begin{aligned}
z^2 &= (x+iy)^2 = x^2 - y^2 + 2xyi \\
z^3 &= (x+iy)^3 = (x^2 - y^2 + 2xyi)(x+yi) = [x^3 - 3xy^2] + [3x^2y - y^3]i
\end{aligned}$$

First, for z^2, let $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$ so that $z^2 = u + iv$. Then,

$$\begin{aligned}
\frac{\partial u}{\partial x} &= 2x = \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -2y = -\frac{\partial v}{\partial x}
\end{aligned}$$

so that the Cauchy-Riemann equations hold. Next let $u(x,y) = x^3 - 3xy^2$ and $v(x,y) = 3x^2y - y^3$ so that $z^3 = u + iv$. Then,

$$\begin{aligned}
\frac{\partial u}{\partial x} &= 3x^2 - 3y^2 = \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -6xy = -\frac{\partial v}{\partial x}
\end{aligned}$$

so that the Cauchy-Riemann equations also hold.

2.1.2.3. Find the most general harmonic polynomial of the form $ax^3 + bx^2y + cxy^2 + dy^3$. Determine the conjugate harmonic function and the corresponding analytic function by integration and by the formal method.

Solution: Let $u(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$. Then,

$$\frac{\partial u}{\partial x} = 3ax^2 + 2bxy + cy^2 \qquad \frac{\partial u}{\partial y} = bx^2 + 2cxy + 3dy^2$$
$$\frac{\partial^2 u}{\partial x^2} = 6ax + 2by \qquad \frac{\partial^2 u}{\partial y^2} = 2cx + 6dy$$

So, for u to be harmonic, it must be that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2(3a+c)x + 2(b+3d)y = 0$$

evidently, this occurs if and only if $c = -3a$ and $b = -3d$. Now, to find the conjugate harmonic polynomial v it must be that $\partial v/\partial x = -\partial u/\partial y$ and $\partial v/\partial y = \partial u/\partial x$. Using the first relation, we find that

$$v = -\int bx^2 + 2cxy + 3dy^2 \; dx = \int 3dx^2 + 6axy - 3dy^2 \; dx = dx^3 + 3ax^2y - 3dxy^2 + \varphi(y)$$

where we have substituted the values of c and b found, and $\varphi(y)$ is some function of y. Differentiating this with respect to y gives

$$3ax^2 - 6dxy + \varphi'(y) = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3ax^2 + 2bxy + cy^2 = 3ax^2 - 6xy - 3ay^2.$$

It follows that $\varphi'(y) = -3ay^2$ so that $\varphi(y) = -ay^3 + C$. In total,

$$v = dx^3 + 3ax^2y - 3dxy^2 - ay^3 + C$$

for a real constant C. The corresponding analytic function is

$$f(z) = [ax^3 - 3dx^2y - 3axy^2 + dy^2] + [dx^3 + 3ax^2y - 3dxy^2 - ay^3 + C]i.$$

The formal method states that we can construct $f(z)$ by

$$f(z) = 2u(z/2, z/(2i)) - u(0,0) + Ci$$

where C is a real constant. Reducing this gives

$$\begin{aligned} f(z) &= 2\left(a\left(\frac{z}{2}\right)^3 - 3d\left(\frac{z}{2}\right)^2\left(\frac{z}{2i}\right) - 3a\left(\frac{z}{2}\right)\left(\frac{z}{2i}\right)^2 + d\left(\frac{z}{2i}\right)^3\right) + Ci \\ &= \frac{1}{4}\left(az^3 - 3dz^2\left(\frac{z}{i}\right) - 3az\left(\frac{z}{i}\right)^2 + d\left(\frac{z}{i}\right)^3\right) + Ci \\ &= (a+di)\,z^3 + Ci. \end{aligned}$$

Expanding out the last expression recovers our formula for $f(z)$ deduced from integration.

2.1.2.4. Show that an analytic function cannot have a constant absolute value without reducing to a constant.

Solution: First, if $|f(z)| = 0$ then $f(z) = 0$, so there is nothing to prove. Otherwise, let $|f(z)| = c > 0$. We have that

$$\overline{f(z)} = \frac{|f(z)|^2}{f(z)} = \frac{c^2}{f(z)}$$

so that $\overline{f(z)}$ is analytic whenever $f(z)$ is nonzero. But, by assumption $f(z)$ is never zero so that $\overline{f(z)}$ is analytic. Writing $f(z) = u + iv$, the Cauchy-Riemann equations applied to f and $\overline{f}$ imply that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

Note the sign change that occurs in the Cauchy-Riemann equations for $\overline{f}$. Together, these imply that all the partial derivatives are zero, and thus u and v are constant functions.

2.1.2.5. Prove rigorously that the functions $f(z)$ and $\overline{f(\overline{z})}$ are simultaneously analytic.

Solution: Suppose first that $f(z)$ is analytic. Let $g(z) = \overline{f(\overline{z})}$. Let u_f, v_f and u_g, v_g denote the real and imaginary parts of f and g respectively. Then, these are related by

$$u_g(x,y) = u_f(x,-y) \qquad v_g(x,y) = -v_f(x,-y).$$

Since f is analytic, u_f and v_f have continuous partial derivatives – so too do u_g, v_g then. Next we verify the Cauchy-Riemann equations:

$$\begin{aligned} \frac{\partial u_g}{\partial x} &= \frac{\partial u_f}{\partial x} & \qquad \frac{\partial v_g}{\partial x} &= -\frac{\partial v_f}{\partial x} \\ \frac{\partial u_g}{\partial y} &= -\frac{\partial u_f}{\partial y} & \qquad \frac{\partial v_g}{\partial y} &= \frac{\partial v_f}{\partial y} \end{aligned}$$

Since f is analytic, u_f and v_f satisfy the Cauchy-Riemann equations. Thus,

$$\begin{aligned} \frac{\partial u_g}{\partial x} &= \frac{\partial u_f}{\partial x} = \frac{\partial v_f}{\partial y} = \frac{\partial v_g}{\partial y} \\ \frac{\partial u_g}{\partial y} &= -\frac{\partial u_f}{\partial y} = \frac{\partial v_f}{\partial x} = -\frac{\partial v_g}{\partial y} \end{aligned}$$

Thus, u_g and v_g satisfy the Cauchy-Riemann equations. It follows that $g(z) = \overline{f(\overline{z})}$ is analytic. Finally, observe that $\overline{g(\overline{z})} = f(z)$ so that the above proof works to show the converse, with the roles of f and g reversed.

2.1.2.6. Prove that the functions $u(z)$ and $u(\overline{z})$ are simultaneously harmonic.

Solution: Let $f(z) = u(z)$ and $g(z) = \overline{f(\overline{z})} = u(\overline{z})$, where $u : \mathbb{C} \to \mathbb{R}$. Then apply Exercise 2.1.2.5, we see that f and g are simultaneously analytic. This implies that their real parts – $u(z)$ and $u(\overline{z})$ – are harmonic.

2.1.2.7. Show that a harmonic function satisfies the formal differential equation

$$\frac{\partial^2 u}{\partial z \partial \overline{z}} = 0.$$

Solution: Recall if f is analytic with $f = u + iv$, then we formally have

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \qquad \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Now consider the function $f = u$, which has no imaginary part. Then,

$$\begin{aligned} \frac{\partial^2 u}{\partial z \partial \overline{z}} &= \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial \overline{z}}\right) = \frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) = \frac{1}{4}\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) - i\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)\right) \\ &= \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + i\frac{\partial^2 u}{\partial x \partial y} - i\frac{\partial^2 u}{\partial y \partial x}\right) = 0 \end{aligned}$$

We use the fact that u is harmonic and symmetry of the mixed partials to conclude.

2.1.4. *Rational Functions.*

2.1.4.1. Use the method of the text to develop

$$\frac{z^4}{z^3 - 1} \quad \text{and} \quad \frac{1}{z(z+1)^2(z+2)^3}$$

in partial fractions.

Solution: XXX

2.1.4.2. If Q is a polynomial with distinct roots $\alpha_1, ..., \alpha_n$, and if P is a polynomial of degree $< n$, show that

$$\frac{P(z)}{Q(z)} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z-\alpha_k)}.$$

Solution: I'm fairly certain we should assume that Q has simple roots $\alpha_1, ..., \alpha_n$, otherwise the right hand side makes little sense. For example, $Q'(\alpha_k) = 0$ at a root with order greater than 1, so the right hand-side is ill defined in this case. We thus assume that $Q(z)$ has degree n. Also, we may assume the leading coefficient is 1, by just multiplying both sides of the above equation by the leading coefficient (were there one). Hence, Q takes the form

$$Q(z) = \prod_{j=1}^{n} (z - \alpha_j)$$

and its derivative

$$Q'(z) = \sum_{l=1}^{n} \prod_{j \neq l} (z - \alpha_j).$$

So, at some zero α_k we have

$$Q'(\alpha_k) = \prod_{j \neq k} (\alpha_k - \alpha_j).$$

Now consider the quantity

$$\tilde{Q}_k(z) := \frac{Q(z)}{Q'(\alpha_k)(z-\alpha_k)} = \frac{\prod_{j=1}^{n}(z-\alpha_j)}{(z-\alpha_k)\prod_{j\neq k}(\alpha_k - \alpha_j)} = \frac{\prod_{j\neq k}(z-\alpha_j)}{\prod_{j\neq k}(\alpha_k-\alpha_j)}$$

for any $k = 1, ..., n$. It is clear that $\tilde{Q}_k(\alpha_l) = \delta_{kl}$.

Next, observe that we can prove the desired equality by instead showing

$$P(z) = \sum_{k=1}^{n} \frac{P(\alpha_k)Q(z)}{Q'(\alpha_k)(z-\alpha_k)} = \sum_{k=1}^{n} P(\alpha_k)\tilde{Q}_k(z).$$

Moreover, a polynomial of degree n is determined by its values at n distinct points. Since $\deg(P) < n$, we just need to show the above equality at less than n points. In particular, we can show it at $z = \alpha_k$ for each k. For fixed α_l we have

$$\sum_{k=1}^{n} P(\alpha_k)\tilde{Q}_k(\alpha_l) = \sum_{k=1}^{n} P(\alpha_k)\delta_{kl} = P(\alpha_l).$$

2.1.4.3. What is the general form of a rational function which has absolute value 1 on the circle $|z| = 1$? In particular, how are the zeros and poles related to each other?

Solution: Consider the quantity

$$R(z)\overline{R(1/\overline{z})}.$$

Observe that if $|z| = 1$, then $z\overline{z} = 1$ and so $z = 1/\overline{z}$. Hence, on $|z| = 1$

$$R(z)\overline{R(1/\overline{z})} = R(z)\overline{R(z)} = |R(z)|^2 = 1.$$

It follows that

$$R(z) = 1/\overline{R(1/\overline{z})}$$

on $|z| = 1$. But, both sides are rational functions. If two rational functions agree on an infinite set, then they must be the same everywhere (take the difference, which has infinitely many zeros – but, as the book shows, every rational function has finitely many zeros). So the above identity actually

holds everywhere. This tells us that if w is a zero (resp. zero) of order k of R, then it has a pole (resp. zero) of order k at $1/\overline{w} = w/|w|^2$. We thus write $R(z)$ as

$$R(z) = z_0 z^{k_0} \prod_{j=1}^{n} \left(\frac{z - \alpha_j}{z - 1/\overline{\alpha_j}} \right)^{k_j}$$

where the α_j are the distinct zeros and poles of $R(z)$ inside the unit disc, with order k_j. The leading coefficient z_0 is such that $|z_0| = 1$ since $R(z) = 1/\overline{R(1/\overline{z})}$.

2.1.4.4. XXX

Solution: XXX

2.1.4.5. If $R(z)$ is a rational function of order n, how large and how small can the order of $R'(z)$ be?

Solution: We need only count the poles. Note that if $R(z)$ has a pole at $w \in \mathbb{C}$ of order k, then $R'(z)$ has a pole at $w \in \mathbb{C}$ of order $k+1$. We must also account for poles at infinity. Note that the order of a pole at infinity is determined by computing the order the pole at $z = 0$ of $R_1(z) := R(1/z)$. By the chain rule,

$$\frac{d}{dz} R_1(z) = -\frac{1}{z^2} R'(1/z)$$

so that we have

$$R'(1/z) = -z^2 \frac{d}{dz} R_1(z).$$

To determine the order of a pole at infinity of $R'(z)$, we look at the order of the pole at $z = 0$ of $R'(1/z)$. If $n = \deg(P) > \deg(Q) = m$ then $R_1(z)$ has a pole at 0 of degree $k = n - m$. Applying the above analysis for finite poles, $d/dz R_1(z)$ has a pole at $z = 0$ of order $k+1$. The additional factor of z^2 in front decreases the order of a pole by 2, so that $R'(1/z)$ has a pole of order $k-1$ at $z = 0$.

More explicitly,

$$\begin{aligned}
R(z) &= \frac{p_0 + p_1 z + ... + p_n z^n}{q_0 + q_1 z + ... + q_m z^m} \\
R'(z) &= \frac{(p_1 + 2p_2 z + ... + np_n z^{n-1})(q_0 + ... + q_m z^m) - (q_1 + 2q_2 z + ... + mq_m z^{m-1})(p_0 + ... + p_n z^n)}{(q_0 + q_1 z + ... + q_m z^m)^2} \\
&= \frac{[p_1 q_0 - q_1 p_0] + 2[p_2 q_0 - p_0 q_2] z + ... + (n-m)[p_n q_m] z^{n+m-1}}{(q_0 + q_1 z + ... + q_m z^m)^2} \\
R'(1/z) &= \frac{z^{2m}}{z^{n+m-1}} \frac{[p_1 q_0 - q_1 p_0] z^{n+m-1} + 2[p_2 q_0 - p_0 q_2] z^{n+m} + ... + (n-m)[p_n q_m]}{(q_0 q^m + q_1 z^{m-1} + ... + q_m)^2}
\end{aligned}$$

so that $R'(1/z)$ has a pole at $z = 0$ of order $n - m - 1 = k - 1$.

To determine the order of R', we simply need to add up the orders of all the poles. Thus,

$$\deg(R') = \sum_j [k_j + 1] + [k_\infty - 1]$$

where k_j is the order of pole w_j of $R(z)$ and $k_\infty \geq 1$ is the order of the infinite pole of $R(z)$. Since $\deg(R) = \sum_j k_j + k_\infty$, we have

$$\deg(R') = \deg(R) + \sum_j 1 - 1 = \deg(R) + d - 1$$

where d is the number of distinct finite poles of $R(z)$. If all the poles are distinct, then $\deg(R') = 2\deg(R)$ and if all the poles are infinite, then $\deg(R') = \deg(R) - 1$. XXX

2.2. **Elementary Theory of Power Series.**

2.2.3. *Uniform Convergence.*

2.2.3.1. Prove that a convergent sequence is bounded.

Solution: Let $\{z_n\}_{n=1}^{\infty}$ be a convergent sequence of complex numbers such that $z_n \to z \in \mathbb{C}$. Let $\epsilon > 0$, then there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $|z_n - z| < \epsilon$. Consequently,

$$|z_n| = |z_n - z + z| \leq |z_n - z| + |z| < |z| + \epsilon$$

for $n \geq N$. For concreteness, choose $\epsilon = 1$. Now set $M = \max\{|z_1|, ..., |z_N|, |z| + 1\}$. By the above, we see that $|z_n| < M$ for all n.

2.2.3.2. If $\lim_{n\to\infty} z_n = A$, prove that

$$\lim_{n\to\infty} \frac{1}{n}(z_1 + z_2 + ... + z_n) = A.$$

Solution: It suffices to prove the real case, since $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ are continuous functions. So, consider a convergent sequence of real numbers $a_n \to a$. For $\epsilon > 0$ there exists an N such that if $n \geq N$, $|a_n - a| < \epsilon$. Expanding this, we have $a - \epsilon < a_n < a + \epsilon$. Define $s_n := 1/n(a_1 + ... + a_n)$. Then, if $n = N + k$ for $k \geq 0$,

$$s_n = \frac{1}{n}(a_1 + ... + a_{N-1} + a_N + ... + a_{N+k}) < \frac{1}{n}(a_1 + ... + a_{N-1} + (n - N + 1)(a + \epsilon)),$$

and we get a similar lower bound. Since

$$\frac{a_1 + ... + a_{N-1}}{n} \to 0$$
$$\frac{(n - N + 1)(a \pm \epsilon)}{n} \to a \pm \epsilon$$

as $n \to \infty$, we have that

$$a - \epsilon < s_n < a + \epsilon$$

for $n \geq N$. Of course, this says that for any $\epsilon > 0$ there exists an N such that if $n \geq N$ then $|s_n - a| < \epsilon$. So, $s_n \to a$.

2.2.3.3. Show that the sum of an absolutely convergent series does not change if the terms are rearranged.

Solution: Let $\sum_{n=1}^{\infty} z_n$ be an absolutely convergent series and $\sum_{n=1}^{\infty} w_n$ any rearrangement (that is, there exists a bijective map $\sigma : \mathbb{N} \to \mathbb{N}$ such that $w_n = z_{\sigma(n)}$). Since $\sum_n z_n$ is absolutely convergent, the series itself converges to some complex number z. Let s_n and t_n be the n-th partial sums of $\sum z_n$ and $\sum w_n$ respectively. Then,

$$|t_n - z| \leq |t_n - s_n| + |s_n - z|.$$

The second term evidently tends to zero. We now wish to estimate $|t_n - s_n|$. Since t_n is a partial sum of the rearrangement, some of the terms will cancel with those in s_n. Let

$$S(n) = \{1, ..., n\}\Delta\{\sigma(1), ..., \sigma(n)\}$$

and set $K = \max S(n)$ and $k = \min S(n)$. It follows that the remaining terms in $t_n - s_n$ have indices between k and K. Thus,

$$|t_n - s_n| \leq \sum_{j=k}^{K} |z_j| \leq \sum_{j=k}^{\infty} |z_j|.$$

As n increases, so too does k. By absolute convergence of $\sum z_j$, we see that the remainders tend to zero. Hence also $|t_n - s_n|$ does.

2.2.3.4. Discuss completely the convergence and uniform convergence of the sequence $\{nz^n\}_{n=1}^{\infty}$.

Solution: First if $|z| < 1$, then we have $|nz^n| = n|z|^n \to 0$, and so $nz^n \to 0$. If $|z| \geq 1$, then we have $|nz^n| = n|z|^n \geq n \to \infty$, and thus $nz^n \to \infty$. Defining $f_n(z) = nz^n$, if the convergence were uniform then the limit would be continuous. But the limit function (in the extended complex numbers) is 0 inside the unit disc and ∞ outside it. This is discontinuous, and therefore the limit is not uniform. We can ask, however, if the limit is uniform in $|z| < 1$. First, consider convergence on the set $|z| < r$ where $|r| < 1$. Then, $|nz^n| < nr^n$, where by continuity of xr^x and since $|r| < 1$, it is clear that the convergence is uniform. On the other hand, if we try to obtain uniform convergence for $|z| < 1$, we see it is impossible. For example, along the sequence $z_n = n^{-1/n}$ we actually have $f_n(z_n) = 1$. One can show (for example, via L'hopitals rule) that $z_n \to 1$. Yet, uniform convergence implies that $f_n(z_n) \to f(z)$.

The intuition for this problem is that multiplication of complex numbers amounts to multiplying their distance to the origin and adding their arguments. So, z^n is rotates around the origin as n increases, and either shrinks ($|z| < 1$) or grows ($|z| > 1$). This causes a spiraling effect.

2.2.3.5. Discuss the uniform convergence of the series

$$\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$$

for real values of x.

Solution: Define $f_n(x) = x/(n(1+nx^2))$, so that the series above is $\sum_n f_n(x)$. Note that $\lim_{x\to\infty} f_n(x) = 0$ for any n. Next,

$$f_n'(x) = \frac{n(1+nx^2) - nx(2nx)}{n^2(1+nx^2)^2} = \frac{1-nx^2}{n(1+nx^2)^2}.$$

So, $f_n'(x) = 0$ when $1 - nx^2 = 0$, or when $x = \pm 1/\sqrt{n}$. Substituting these into $f_n(x)$ gives

$$-f_n(-1/\sqrt{n}) = f_n(1/\sqrt{n}) = \frac{1}{2n\sqrt{n}}.$$

Combining this with the fact that each f_n tends to zero, we have $|f_n| \leq 1/(2n\sqrt{n})$. If $a_n = 1/2n^{-3/2}$, then the series $\sum_{n=1}^{\infty} a_n$ converges. Hence by the Weierstrass M-test, the above series converges uniformly.

2.2.3.6. XXX

Solution:

2.2.4. *Power Series.*
2.2.4.1. Expand $(1-z)^{-m}$, m a positive integer, in powers of z.

Solution: We know that

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + ...$$

so we just need to compute the derivatives of $f(z) = (1-z)^{-m}$ and evaluate them at zero. Observe that

$$\begin{aligned} f(z) &= (1-z)^{-m} \\ f'(z) &= m(1-z)^{-m-1} \\ f''(z) &= m(m+1)(1-z)^{-m-2}... \end{aligned}$$

By repeated differentiation, we see that in general

$$f^{(n)}(z) = \frac{(m+n-1)!}{(m-1)!}(1-z)^{-m-n}.$$

Thus,

$$f^{(n)}(0) = \frac{(m+n-1)!}{(m-1)!}.$$

The power series expansion is then

$$f(z) = \sum_{n=0}^{\infty} \frac{(m+n-1)!}{(m-1)!n!} z^n = \sum_{n=0}^{\infty} \binom{m+n-1}{n} z^n.$$

Note the similarity to the binomial theorem.

2.2.4.2. Expand $(2z+3)/(z+1)$ in powers of $z-1$. What is the radius of convergence?

Solution: We can write

$$\begin{aligned}\frac{2z+3}{z+1} &= 2 + \frac{1}{z+1} = 2 + \frac{1}{2-(-(z-1))} = 2 + \frac{1/2}{1-(-(z-1)/2)} \\ &= 2 + \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{(z-1)}{2}\right)^n = 2 + \sum_{n=0}^{\infty} \frac{(-1)^n (z-1)^n}{2^{n+1}}\end{aligned}$$

By the geometric series test, it follows that this series converges when

$$\left|\frac{-(x-1)}{2}\right| < 1$$

so that the radius of convergence is $R = 2$.

2.2.4.3. Find the radius of convergence of the following power series:

$$\sum n^p z^n, \quad \sum \frac{z^n}{n!}, \quad \sum n! z^n, \quad \sum q^{n^2} z^n, \quad \sum z^{n!}$$

where $|q| < 1$.

Solution: Recall Hadamard's formula, which states that

$$1/R = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

defines the radius of convergence for the power series $\sum a_n z^n$. Note that every power series converges for at least one z, namely $z = 0$; This occurs if $R = 0$ and $1/R = \infty$. On the other hand, a power series can converge everywhere, in which case $R = \infty$ and $1/R = 0$. For the latter, since $\sqrt[n]{|\cdot|}$ is a continuous function, if $a_n \to 0$ then $1/R = 0$. We now treat each power series separately.

i) $(\sum n^p z^n)$. Since $a_n = n^p$, we have

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \left(\limsup_{n\to\infty} \sqrt[n]{n}\right)^p = 1^p = 1.$$

So, $R = 1$.

ii) $(\sum z^n/n!)$. Since $a_n = 1/n!$, which converges to 0, we have $R = \infty$.

iii) $(\sum n! z^n)$. Here $a_n = n!$, and by Stirling's approximation $n! \approx \sqrt{2\pi n}(n/e)^n$. So, by L'hopitals rule one can show $\sqrt[n]{n!} \approx (2\pi n)^{1/(2n)} n/e \to \infty$. It follows that $R = 0$, so the power series converges only at $z = 0$.

iv) $(\sum q^{n^2} z^n)$. Now $a_n = q^{n^2}$, where $|q| < 1$. It is clear to see that $|a_n|^{1/n} = |q|^n \to 0$, so that $R = \infty$.

v) $(\sum z^{n!})$. This one is slightly odd, since $a_k = 1$ when $k = n!$ for some n, and is zero otherwise. Let $b_k = \sqrt[k]{|a_k|}$, then $b_k = 1$ when $k = n!$ and zero else. Since $n!$ grows, for any tail $\{k, k+1, ...\}$ one of the indices will be a factorial. Hence, the limsup is always 1, and $R = 1$.

2.2.4.4. If $\sum a_n z^n$ has radius of convergence R, what is the radius of convergence of $\sum a_n z^{2n}$? of $\sum a_n^2 z^n$?

Solution: Let $b_n = a_{n/2}$ if n is even and zero otherwise. Then $\sum a_n z^{2n} = \sum b_n z^n$. So, to find the radius of convergence we can apply Hadamard's formula. Note that if $n = 2k$ then,

$$\sqrt[n]{|b_n|} = \sqrt[2k]{|a_k|} = \left(\sqrt[k]{|a_k|}\right)^{1/2}.$$

By definition,

$$1/R_1 = \limsup_{n\to\infty} \sqrt[n]{|b_n|} = \limsup_{k\to\infty} \left(\sqrt[k]{|a_k|}\right)^{1/2} = \left(\limsup_{k\to\infty} \sqrt[k]{|a_k|}\right)^{1/2} = 1/R^{1/2}$$

since $\sqrt{x}$ is a monotone continuous function and we can discard the the remaining terms of $\sqrt[n]{|b_n|}$ since they are just zero.

Now let $c_n = a_n^2$ so that $\sum c_n z^n = \sum a_n^2 z^n$. By Hadamard's formula,

$$1/R_2 = \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \left(\limsup_{n\to\infty} \sqrt[n]{|a_n|}\right)^2 = 1/R^2$$

using the same logic as above.

2.2.4.5. If $f(z) = \sum a_n z^n$, what is $\sum n^3 a_n z^n$?

Solution: It is clear to see that

$$zf'(z) = z\sum_{n=0}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} n a_n z^n$$

by applying Abel's theorem for the differentiation of power series. We inductively see that

$$\sum_{n=0}^{\infty} n^k a_n z^n = z\frac{d}{dz}\left(z\frac{d}{dz}\ldots\left(z\frac{df}{dz}\right)\right)$$

where there are k many d/dz. Hence,

$$\begin{aligned}\sum_{n=0}^{\infty} n^3 a_n z^n &= z\frac{d}{dz}\left(z\frac{d}{dz}\left(zf'(z)\right)\right) = z\frac{d}{dz}\left(zf'(z) + z^2 f''(z)\right)\\ &= zf'(z) + 3z^2 f''(z) + z^3 f'''(z)\end{aligned}$$

2.2.4.6. If $\sum a_n z^n$ and $\sum b_n z^n$ have radii of convergence R_1 and R_2, show that the radius of convergence of $\sum a_n b_n z^n$ is at least $R_1 R_2$.

Solution: We first show that if x_n and y_n are sequences of nonnegative real numbers then

$$\limsup_{n\to\infty} x_n y_n \le \limsup_{n\to\infty} x_n \limsup_{n\to\infty} y_n$$

Observe that

$$\sup_{k\ge n}\{x_k y_k\} \le \sup_{j\ge n}\sup_{k\ge n}\{x_j y_k\} \le \sup_{j\ge n}\{x_j \sup_{k\ge n}\{y_k\}\} = \sup_{j\ge n}\{x_j\}\sup_{k\ge n}\{y_k\}$$

The first inequality comes from taking the sup over a larger set, while the second comes from considering $x_j y_k$ for fixed j and taking the sup over $k \ge n$. Taking the limit as $n \to \infty$ on both sides shows the desired inequality. To solve the problem, apply it with $x_n = \sqrt[n]{|a_n|}$ and $y_n = \sqrt[n]{|b_n|}$, noting that $\sqrt[n]{|a_n b_n|} = \sqrt[n]{|a_n|}\sqrt[n]{|b_n|} = x_n y_n$.

2.2.4.7. If $\lim_{n\to\infty} |a_n|/|a_{n+1}| = R$, prove that $\sum a_n z^n$ has radius of convergence R.

Solution: We show that if b_n is a sequence of positive real numbers then

$$\liminf_{n\to\infty} \frac{b_{n+1}}{b_n} \le \liminf_{n\to\infty} \sqrt[n]{b_n} \le \limsup_{n\to\infty} \sqrt[n]{b_n} \le \limsup_{n\to\infty} \frac{b_{n+1}}{b_n}.$$

Set l, m, M, L as the above limits respectively, so we wish to show $l \le m \le M \le L$. Let $\epsilon > 0$, then there exists an N such that if $n \ge N$ then $\sup_{n\ge N} b_{n+1}/b_n < L + \epsilon$. In particular, for all $n \ge N$ we have $b_{n+1} < (L+\epsilon)b_n$. If $n = N + k$ with $k \ge 1$ then

$$b_n < (L+\epsilon)b_{n-1} < (L+\epsilon)^2 b_{n-2} < ... < (L+\epsilon)^k b_{n-k} = (L+\epsilon)^{n-N} b_N.$$

Recall that $c^{1/n} \to 1$ for $c > 0$. Hence, taking n-th roots and the limsup of both sides, we get

$$\limsup_{n\to\infty} \sqrt[n]{b_n} \le (L+\epsilon) \limsup_{n\to\infty} \left(\frac{b_N}{(L+\epsilon)^N} \right)^{1/n} = L + \epsilon.$$

This holds for all $\epsilon > 0$, so that $M \le L$. Showing $l \le m$ is similar, except using $l - \epsilon$. The inner inequality is trivial.

Now, if $\lim_{n\to\infty} |a_n|/|a_{n+1}| = R$ then $\lim_{n\to\infty} |a_{n+1}|/|a_n| = 1/R$. Setting $b_n = |a_n|$ and using the fact that, for convergent sequences, the limsup and liminf agree, we have

$$\frac{1}{R} \le \liminf_{n\to\infty} \sqrt[n]{|a_n|} \le \limsup_{n\to\infty} \sqrt[n]{|a_n|} \le \frac{1}{R}.$$

We conclude by applying Hadamard's formula.

2.2.4.8. For what values of z is

$$\sum_{n=0}^{\infty} \left(\frac{z}{1+z} \right)^n$$

convergent?

Solution: As a geometric series, it will converge when $|z/(1+z)| < 1$, or equivalently $|z| < |1+z|$. Squaring both sides does not affect the solution set to this inequality, so we get

$$z\overline{z} = |z|^2 < |1+z|^2 = (1+z)(1+\overline{z}) = 1 + z + \overline{z} + z\overline{z}.$$

Recalling that $z + \overline{z} = 2\,\mathrm{Re}(z)$, this is equivalent to

$$0 < 1 + 2\,\mathrm{Re}(z) \quad \Leftrightarrow \quad \mathrm{Re}(z) > -1/2.$$

2.2.4.9. Same question for

$$\sum_{n=0}^{\infty} \frac{z^n}{1+z^{2n}}.$$

Solution: Suppose that $|z| = 1$. Then, $|1 + z^{2n}| \le 1 + |z|^{2n} = 2$. Consequently,

$$\left| \frac{z^n}{1+z^{2n}} \right| = \frac{1}{|1+z^{2n}|} \ge \frac{1}{2}.$$

So, the terms never go to zero, and thus the series diverges. Also note that

$$\sum_{n=0}^{\infty} \frac{z^n}{1+z^{2n}} = \sum_{n=0}^{\infty} \frac{1}{z^{-n} + z^n}$$

so that the behavior for $|z| > 1$ is the same as that of $|z| < 1$. Now, if $|z| > 1$ then for any $r > 1$ there exists an N such that if $n \ge N$,

$$|1 + z^{2n}| \ge |1 - |z|^{2n}| \ge \frac{|z|^{2n}}{r}.$$

The first inequality is just the reverse triangle inequality. The latter comes from the limit

$$\limsup_{n\to\infty} \frac{|1-|z|^{2n}|}{|z|^{2n}} = 1$$

and the fact that $r > 1$. Next, applying the root test

$$\limsup_{n\to\infty} \sqrt[n]{\left|\frac{z^n}{1+z^{2n}}\right|} \leq \limsup_{n\to\infty} \sqrt[n]{\frac{|z|^n r}{|z|^{2n}}} = \limsup_{n\to\infty} \frac{\sqrt[n]{r}}{|z|} = \frac{1}{|z|} < 1.$$

Since the limit is less than one, the series converges.

2.3. **The Exponential and Trigonometric Functions.**

2.3.2. *The Trigonometric Functions.*

2.3.2.1. Find the values of $\sin i$, $\cos i$, $\tan(1+i)$.

Solution: We recall that

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

So that direct substitution of $z = i$ gives

$$\begin{aligned}
\sin i &= \frac{e^{i^2} - e^{-i^2}}{2i} = \frac{1/e - e}{2i} = \frac{(e^2-1)i}{2e} \\
\cos i &= \frac{e^{i^2} + e^{-i^2}}{2} = \frac{1/e + e}{2} = \frac{1+e^2}{2e}
\end{aligned}$$

By the addition formulas, we have

$$\begin{aligned}
\sin(1+i) &= \cos(1)\sin(i) + \sin(1)\cos(i) = \frac{(1+e^2)\sin(1) + (e^2-1)\cos(1)i}{2e} \\
\cos(1+i) &= \cos(1)\cos(i) - \sin(1)\sin(i) = \frac{(1+e^2)\cos(1) + (1-e^2)\sin(1)i}{2e}
\end{aligned}$$

Hence,

$$\begin{aligned}
\tan(1+i) &= \frac{\sin(1+i)}{\cos(1+i)} = \frac{(1+e^2)\sin(1) + (e^2-1)\cos(1)i}{(1+e^2)\cos(1) + (1-e^2)\sin(1)i} \\
&= \frac{((1+e^2)\sin(1) + (e^2-1)\cos(1)i)((1+e^2)\cos(1) + (e^2-1)\sin(1)i)}{(1+2e^2+e^4)\cos(1)^2 + (1-2e^2+e^4)\sin(1)^2} \\
&= \frac{4e^2\sin(1)\cos(1) + (e^4-1)i}{1+e^4+2e^2\cos(2)}
\end{aligned}$$

If we adopt the definitions of $\cosh(z)$ and $\sinh(z)$ in the next problem, then observe that

$$\begin{aligned}
\cosh(2) &= \frac{e^2 + 1/e^2}{2} = \frac{e^4+1}{2e^2} \\
\sinh(2) &= \frac{e^2 - 1/e^2}{2} = \frac{e^4-1}{2e^2}
\end{aligned}$$

Thus, we can write $\tan(1+i)$ as

$$\tan(1+i) = \frac{2\sin(1)\cos(1) + (e^4-1)/(2e^2)i}{(1+e^4)/(2e^2) + \cos(2)} = \frac{\sin(2) + \sinh(2)i}{\cos(2) + \cosh(2)}.$$

2.3.2.2. The hyperbolic cosine and sine are defined by $\cosh z = (e^z + e^{-z})/2$, $\sinh z = (e^z - e^{-z})/2$. Express them through $\cos iz$, $\sin iz$. Derive the addition formulas, and formulas for $\cosh 2z$, $\sinh 2z$.

Solution: It is easy to see that

$$\cos(iz) = \frac{e^{i^2 z} + e^{-i^2 z}}{2} = \frac{e^z + e^{-z}}{2} = \cosh(z)$$
$$\sin(iz) = \frac{e^{i^2 z} - e^{-i^2 z}}{2i} = \frac{(e^z - e^{-z})i}{2} = \sinh(z)i.$$

The addition formulas are thus

$$\begin{aligned}\cosh(a+b) &= \cos(ai+bi) = \cos(ai)\cos(bi) - \sin(ai)\sin(bi)\\ &= \cosh(a)\cosh(b) - \sinh(a)\sinh(b)i^2 = \cosh(a)\cosh(b) + \sinh(a)\sinh(b)\\ \sinh(a+b) &= -i\sin(ai+bi) = -i[\sin(ai)\cos(bi) + \sin(bi)\cos(ai)]\\ &= [-i\sin(ai)]\cos(bi) + [-i\sin(bi)\cos(ai)] = \sinh(a)\cosh(b) + \sinh(b)\cosh(a)\end{aligned}$$

and from these we derive the double angle formulae

$$\cosh(2z) = \cosh^2(z) + \sinh^2(z)$$
$$\sinh(2z) = 2\sinh(z)\cosh(z)$$

2.3.2.3. Use the addition formulas to separate $\cos(x+iy)$, $\sin(x+iy)$ in real and imaginary parts.

Solution: We essentially did this in Exercise 2.3.2.1, but without the aid of the hyperbolic trig functions. Direct application of the addition formulae give

$$\begin{aligned}\cos(x+iy) &= \cos(x)\cos(iy) - \sin(x)\sin(iy) = \cos(x)\cos(iy) - i[-i\sin(iy)]\sin(x)\\ &= \cos(x)\cosh(y) - \sin(x)\sinh(y)i\\ \sin(x+iy) &= \sin(x)\cos(iy) + \sin(iy)\cos(x) = \sin(x)\cos(iy) + i[-i\sin(iy)]\cos(x)\\ &= \sin(x)\cosh(y) + \cos(x)\sinh(y)i\end{aligned}$$

2.3.2.4. Show that

$$|\cos z|^2 = \sinh^2 y + \cos^2 x = \cosh^2 y - \sin^2 x = \frac{1}{2}(\cosh 2y + \cos 2x)$$

and

$$|\sin z|^2 = \sinh^2 y + \sin^2 x = \cosh^2 y - \cos^2 x = \frac{1}{2}(\cosh 2y - \cos 2x).$$

Solution: If $z = x + iy$ then by Exercise 2.3.2.3 we have

$$\begin{aligned}|\cos z|^2 &= \mathrm{Re}(\cos(z))^2 + \mathrm{Im}(\cos(z))^2 = \cos^2(x)\cosh^2(y) + \sin^2(x)\sinh^2(y)\\ &= \cos^2(x)(1+\sinh^2(y)) + (1-\cos^2(x))\sinh^2(y)\\ &= \cos^2(x) + \sinh^2(y)\end{aligned}$$

where we have made use of the fact that $\cosh^2(y) - \sinh^2(y) = 1$ and $\cos^2(x) + \sin^2(x) = 1$. To convert to the second formulation, we simply apply these identities again

$$|\cos z|^2 = [1 - \sin^2(x)] + [\cosh^2(y) - 1] = \cosh^2(y) - \sin^2(x).$$

To get the third formulation add both of these identities and apply the double angle formulae.

For $\sin z$ we have

$$\begin{aligned}|\sin z|^2 &= \sin^2(x)\cosh^2(y) + \cos^2(x)\sinh^2(y)\\ &= \sin^2(x)(1+\sinh^2(y)) + (1-\sin^2(x))\sinh^2(y) = \sin^2(x) + \sinh^2(y).\end{aligned}$$

Deriving the other identities proceeds in virtually the same way as for $\cos z$. Note that this aligns with Euler's identity $e^{iz} = \cos z + i\sin z$ and the modulus of the exponential, $|e^{x+iy}| = e^x$.

2.3.4. *The Logarithm.*

2.3.4.1. XXX

Solution: XXX

2.3.4.2. XXX

Solution:

2.3.4.3. Find the value of e^z for $z = -\pi i/2, 3\pi i/4, 2\pi i/3$.

Solution: We apply Euler's identity $e^{iz} = \cos z + i \sin z$.

$$\begin{aligned} e^{-\pi/2i} &= \cos\frac{-\pi}{2} + i\sin\frac{-\pi}{2} = -i \\ e^{3\pi/4i} &= \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \\ e^{2\pi/3i} &= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \end{aligned}$$

2.3.4.4. For what values of z is e^z equal to $2, -1, i, -i/2, -1-i, 1+2i$?

Solution: For each w in the list above, we want to find

$$\log w = \log|w| + i\arg w$$

where $\arg w \in (-\pi, \pi]$. Then, every value can be obtained from adding integer multiples of $2\pi i$. One can easily find that

$$\arg 2 = 0, \quad \arg -1 = \pi, \quad \arg i = \pi/2, \quad \arg -i/2 = \arg -i = -\pi/2,$$
$$\arg(-1-i) = \arg\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = -3\pi/4, \quad \arg(1+2i) = \arctan(2)$$

since arguments are not affected by dilations. Hence, all the values of z such that e^z is equal to the above listed numbers are

$$\begin{aligned} z(2) &= \log 2 + 2\pi n i = \log 2 + 2\pi n i \\ z(-1) &= \log(-1) + 2\pi n i = (2n+1)\pi i \\ z(i) &= \log(i) + 2\pi n i = (4n+1)\pi/2i \\ z(-i/2) &= \log(-i/2) + 2\pi n i = -\log 2 + (4n-1)\pi/2i \\ z(-1-i) &= \log(-1-i) + 2\pi n i = 1/2\log 2 + (8n-3)\pi/4i \\ z(1+2i) &= \log(1+2i) + 2\pi n i = 1/2\log 5 + (2\pi n + \arctan(2))i \end{aligned}$$

2.3.4.5. Find the real and imaginary parts of $\exp(e^z)$.

Solution: Write $z = a + bi$ so that

$$e^z = e^a e^{bi} = e^a(\cos b + i\sin b).$$

Then, applying the exponential once more gives

$$\exp(e^z) = e^{e^a(\cos b + i\sin b)} = e^{e^a\cos b} e^{ie^a\sin b} = e^{e^a\cos b}(\cos(e^a\sin b) + i\sin(e^a\sin b)).$$

Hence,

$$\begin{aligned} \mathrm{Re}(\exp(e^z)) &= e^{e^a\cos b}\cos(e^a\sin b) \\ \mathrm{Im}(\exp(e^z)) &= e^{e^a\cos b}\sin(e^a\sin b) \end{aligned}$$

2.3.4.6. Determine all values of $2^i, i^i, (-1)^2 i$.

Solution: We recall that $a^b := e^{b \log a}$. We simply apply this definition with Euler's identity several times. Thus,

$$\begin{aligned} 2^i &= e^{i \log 2} = e^{i(\log 2 + 2\pi n i)} = e^{-2\pi b} e^{i \log 2} = e^{-2\pi n}(\cos \log 2 + i \sin \log 2) \\ i^i &= e^{i \log i} = e^{i(2\pi n i + \pi/2 i)} = e^{-\pi/2} e^{-2\pi n} \\ (-1)^{2i} &= e^{2i \log(-1)} = e^{2i(2n+1)\pi i} = e^{-2(2n+1)\pi} \end{aligned}$$

for $n \in \mathbb{Z}$.

2.3.4.7. Determine the real and imaginary parts of z^z.

Solution: If $z = a + bi$ we have that

$$\begin{aligned} z^z &= e^{z \log z} = e^{z(\log|z| + i \arg z)} = e^{a \log|z| - b \arg z + i(\arg z + b \log|z|)} \\ &= e^{a \log|z| - b \arg z} e^{i(\arg z + b \log|z|)} = e^{a \log|z| - b \arg z}(\cos(\arg z + b \log|z|) + i \sin(\arg z + b \log|z|)) \end{aligned}$$

Thus, the real and imaginary parts are

$$\begin{aligned} \mathrm{Re}(z^z) &= e^{a \log|z| - b \arg z} \cos(\arg z + b \log|z|) \\ \mathrm{Im}(z^z) &= e^{a \log|z| - b \arg z} \sin(\arg z + b \log|z|) \end{aligned}$$

Of course, this assumes we use the principal values of $\arg z$.

2.3.4.8. Express $\arctan w$ in terms of the logarithm.

Solution: We have that

$$w = \tan z = -i \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = -i \frac{e^{2iz} - 1}{e^{2iz} + 1}.$$

Rearranging this gives

$$e^{2iz} = \frac{i - w}{w + i}$$

so that taking the logarithm yields

$$\arctan w = z = \frac{1}{2i} \log\left(\frac{i - w}{i + w}\right).$$

3. Analytic Functions as Mappings

3.1. **Elementary Point Set Topology.**

3.1.2. *Metric Spaces.*

3.1.2.1. If S is a metric space with distance function $d(x, y)$, show that S with the distance function $\delta(x, y) = d(x, y)/[1 + d(x, y)]$ is also a metric space. The latter space is bounded in the sense that all distances lie under a fixed bound.

Solution: That $\delta(x, y) \geq 0$, $\delta(x, y) = 0$ if and only if $x = y$, and $\delta(x, y) = \delta(y, x)$ follow immediately from the fact that $d(x, y)$ is a metric. Next, observe that

$$\delta(x, y) = 1 - \frac{1}{1 + d(x, y)}.$$

Since $d(x, y) \leq d(x, z) + d(z, y)$, it follows that

$$\delta(x, y) = 1 - \frac{1}{1 + d(x, y)} \leq 1 - \frac{1}{1 + d(x, z) + d(z, y)}.$$

Next, one can show that if $x, y \geq 0$ then

$$\frac{1}{1 + x} + \frac{1}{1 + y} \leq 1 + \frac{1}{1 + x + y}$$

Applying this with $d(x,z)$ and $d(y,z)$ gives

$$1-\frac{1}{1+d(x,z)+d(z,y)} \leq 1-\frac{1}{1+d(x,y)}+1-\frac{1}{1+d(z,y)} = \delta(x,z)+\delta(z,y).$$

One can also appeal to monotonicity of the function $x \mapsto x/(1+x)$.

3.1.2.2. Suppose that there are given two distance functions $d(x,y)$ and $d_1(x,y)$ on the same space S. They are said to be equivalent if they determine the same open sets. Show that d and d_1 are equivalent if to every $\epsilon > 0$ there exists a $\delta > 0$ such that $d(x,y) < \delta$ implies $d_1(x,y) < \epsilon$, and vice versa. Verify that this condition is fulfilled in the preceding exercise.

Solution: Recall that a set U is open in a topology τ induced by a metric d if for each $x \in U$ there exists an $r_x > 0$ such that $B_d(x, r_x) \subset U$. We wish to show that d_1-balls are open in the d-topology. That is, we wish to verify the above with $U = B_{d_1}(x,r)$ for any $r > 0$ and $x \in S$. Assume for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d(x,y) < \delta$ implies $d_1(x,y) < \epsilon$. Let $y \in B_{d_1}(x,r)$ and set $\epsilon = \min\{d_1(x,y), r - d_1(x,y)\}$ and extract $\delta > 0$ according to the above. Now if $z \in B_d(y,\delta)$ then $d(y,z) < \delta$ and $d_1(y,z) < \epsilon \leq r - d_1(x,y)$. By the triangle inequality,

$$d_1(x,z) \leq d_1(y,z) + d_1(x,y) < r - d_1(x,y) + d_1(x,y) < r$$

so that $z \in B_{d_1}(x,r)$. The converse is analogous.

Now, it is clear if d_1 is the bounded metric from Exercise 3.1.2.1, $d(x,y) \geq d_1(x,y)$. So, we need only show for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_1(x,y) < \delta$ implies $d(x,y) < \epsilon$. Observe that $d_1(x,y) < \delta$ implies

$$d(x,y) < \delta[d(x,y)+1] \quad \Leftrightarrow \quad d(x,y) < \frac{\delta}{1-\delta}.$$

So, choose δ such that $\delta/(1-\delta) < \epsilon$ and $\delta < 1$.

3.1.2.3. Show by strict application of the definition that the closure of $|z - z_0| < \delta$ is $|z - z_0| \leq \delta$.

Solution: We show that $|z - z_0| \leq \delta$ is closed by showing $|z - z_0| > \delta$ is open. Let z be such that $|z - z_0| > \delta$ and set $r = |z - z_0| - \delta$. The minimum distance between a point w with $|w - z| \leq r$ and z_0 is attained at $w = z + r(z_0 - z)/|z_0 - z|$. Then,

$$|w - z_0| = |z - z_0 + (1 - \delta/|z - z_0|)(z_0 - z)| = \delta.$$

But, note that $|w - z| = r$ so that in fact if $w \in B(z,r)$ then $|w - z_0| > \delta$.

Since $|z - z_0| \leq \delta$ (denoted S) is closed, it follows that $\mathrm{Cl}(B(z_0,\delta)) \subset S$. Now let U be open and disjoint from $B(z_0,\delta)$. We show that U is disjoint from S. So, let $z \in S$ be such that $|z - z_0| = \delta$. Suppose that $z \in U$. Since U is open, there exists an $\delta > r > 0$ such that $B(z,r) \subset U$. Consider the point $w = z + r/2(z_0 - z)/|z - z_0|$. We see that

$$|w - z| = |z - z + r/2(z_0 - z)/|z - z_0|| = r/2$$

$$|w - z_0| = |z - z_0 + r/2(z_0 - z)/\delta| = \frac{1}{\delta}|\delta(z - z_0) + r/2(z_0 - z)| = |r/2 - \delta| < \delta$$

Hence, $w \in B(z_0,\delta) \cap B(z,r)$, a contradiction since $B(z_0,\delta) \cap U = \emptyset$. This shows that every closed set containing $B(z_0,\delta)$ must contain S. Hence, $S = \mathrm{Cl}(B(z_0,\delta))$.

3.1.2.4. If X is the set of complex numbers whose real and imaginary parts are rational, what is $\mathrm{Int}\, X$, $\mathrm{Cl}\, X$, ∂X?

Solution: Let $z \in X$ and consider $B(z,\delta)$ for some $\delta > 0$. Let w be such that $\mathrm{Im}\, w = \mathrm{Im}\, z$ but $|\mathrm{Re}\, w - \mathrm{Re}\, z| < \delta$. There are obviously irrational real numbers satisfying this, so we can always find a $w \in B(z,\delta)$ whose real part is irrational. Hence, $B(z,\delta)$ is not a subset of X for any δ, and X is not a neighborhood of any of its points. Thus, $\mathrm{Int}\, X = \emptyset$. Now, $\mathrm{Cl}\, X = (\mathrm{Int}\, X^c)^c$. By a

similar reasoning to the above, for any $z \in X^c$ and $B(z,\delta)$, we can find a $w \in B(z,\delta)$ whose real and imaginary parts are rational. So, $\operatorname{Int} X^c = \emptyset$ and $\operatorname{Cl} X = \mathbb{C}$. By definition, $\partial X = \operatorname{Cl} X \setminus \operatorname{Int} X = \mathbb{C}$.

3.1.2.5. It is sometimes typographically simpler to write X' for X^c. With this notation, how is $(\operatorname{Cl} X^c)^c$ related to X? Show that $(\operatorname{Cl}(\operatorname{Cl}(\operatorname{Cl}(\operatorname{Cl} X)^c)^c)^c)^c = (\operatorname{Cl}(\operatorname{Cl} X)^c)^c$

Solution: By definition,

$$\operatorname{Cl} X^c = \bigcap_{X^c \subset C} C$$

where C is closed. Note that if $X^c \subset C$ then $C^c \subset X$, and C^c is open. So,

$$(\operatorname{Cl} X^c)^c = \left(\bigcap_{X^c \subset C} C \right)^c = \bigcup_{X^c \subset C} C^c = \operatorname{Int} X.$$

Next, we show that if X is open

$$(\operatorname{Cl}(\operatorname{Cl} X)^c)^c = X.$$

It is equivalent to show that

$$\operatorname{Cl}(\operatorname{Cl} X)^c = \operatorname{Cl} X^c = X^c$$

since X is open. If $Y = X^c$ then the above shows that

$$(\operatorname{Cl} X)^c = (\operatorname{Cl} Y^c)^c = \operatorname{Int} Y.$$

By taking closures and noting that Y is closed, we see $\operatorname{Cl} \operatorname{Int} Y = Y = X^c$. The statement we are asked to show reduces to the original one above.

3.1.2.6. A set is said to be discrete if all its points are isolated. Show that a discrete set in $\mathbb{R}$ or $\mathbb{C}$ is countable.

Solution: We showed in Exercise 3.1.2.4 that the set A of complex numbers with rational real and imaginary parts is dense in $\mathbb{C}$. Let S be a discrete set in $\mathbb{C}$. Then for any $z \in S$ there exists an $r > 0$ such that $B(z,r) \cap S = \{z\}$. But by density this ball will contain a point w with rational real and imaginary parts. We may choose w so that if $r' > |w - z|$ then $B(w,r') \subset B(z,r)$ and $z \in B(w,r')$. It follows that $B(w,r') \cap S = \{z\}$, and thus we can define a map $\varphi : S \to A$ such that $\varphi(z) = w$. Note that our choice of w is not unique, but we have chosen it so that φ is injective. Since A is countable we are done. The case for $\mathbb{R}$ follows from this by way of the subspace topology.

3.1.2.7. Show that the accumulation points of any set form a closed set.

Solution: Let $X \subset \mathbb{C}$ and denote the set of accumulation points of X by A. Then,

$$A = \{x \in \operatorname{Cl} X \mid x \text{ is not an isolated point of } \operatorname{Cl} X\}.$$

Consequently, A^c is defined by

$$A^c = (\operatorname{Cl} X)^c \cup \{x \in \operatorname{Cl} X \mid x \text{ is an isolated point of } \operatorname{Cl} X\}.$$

The set $(\operatorname{Cl} X)^c$ is obviously open, so it suffices to show that the isolated points of $\operatorname{Cl} X$ is open. XXX

3.2. **Conformality.**

3.2.2. *Analytic Functions in Regions.*

3.2.2.1. Give a precise definition of a single-valued branch of $\sqrt{1+z}+\sqrt{1-z}$ in a suitable region, and prove that it is analytic.

Solution: We recall that for $\sqrt{z}$ we choose the region Ω which is the complement of the negative real axis $x \leq 0$, $y = 0$. Hence, to define $\sqrt{1+z}$ we should choose Ω_1 as the complement of $x \leq -1$, $y = 0$ and for $\sqrt{1-z}$ we should choose Ω_2 as the complement of $1 \leq x$, $y = 0$. In total, to define $\sqrt{1+z}+\sqrt{1-z}$ we choose the region $\Omega = \Omega_1 \cap \Omega_2$. This is actually the same region used in defining $\arccos z$. The branch chosen is that which has positive real part. As simple transformations of $\sqrt{z}$ in an analytic region, it follows that $\sqrt{1+z}+\sqrt{1-z}$ is analytic.

3.2.2.2. Same problem for $\log\log z$.

Solution: We look at how the principal branch of $\log z$ transforms the region Ω defined by $x \leq 0$, $y = 0$. The ray $x > 0$, $y = 0$ has $\arg(z) = 0$ so that $\log z = \log|z| = \log x$. So, this gets mapped to the real axis. All other points get mapped to the strips $-\pi < y < 0$ and $0 < y < \pi$. To define the logarithm again, we must remove those points with $x \leq 0$ and $y = 0$. As above, this corresponds to the points in Ω with $x \leq 1$ and $y = 0$. We once more choose the branch $|\operatorname{Im}\log\log z| < \pi$ and $|\operatorname{Im}\log z| < \pi$. It is analytic by way of a composition of analytic functions.

3.2.2.3. Suppose that $f(z)$ is analytic and satisfies the condition $|f(z)^2 - 1| < 1$ in a region Ω. Show that either $\operatorname{Re} f(z) > 0$ or $\operatorname{Re} f(z) < 0$ throughout Ω.

Solution: Suppose $\operatorname{Re} f(z) = 0$ at a point $z \in \Omega$. Then $f(z) = iy^2$ for some $y \in \mathbb{R}$, and thus $f(z)^2 = -y^2$. By the modulus condition $|f(z)^2 - 1| < 1$ we have $|-y^2 - 1| < 1$ and thus $|y^2 + 1| < 1$. This is clearly impossible, so that $\operatorname{Re} f(z) \neq 0$ throughout Ω. But, $\operatorname{Re} f(z)$ is continuous and Ω is connected, so either $\operatorname{Re} f(z) > 0$ or $\operatorname{Re} f(z) < 0$.

3.3. **Linear Transformations.**

3.3.1. *The Linear Group.*

3.3.1.1. Prove that the reflection $z \mapsto \overline{z}$ is not a linear transformation.

Solution: Suppose there exist $a, b, c, d \in \mathbb{C}$ such that

$$\overline{z} = \frac{az+b}{cz+d}.$$

First, if $z = 0$ then $\overline{z} = 0$ and so $b/d = 0$. It follows that $b = 0$. Next, multiply through by $cz + d$. Then

$$c|z|^2 + d\overline{z} = \overline{z}(cz+d) = az.$$

Consider $z = r$ for real $r > 0$, then $cr^2 = (a-d)r$, or $cr = a - d$. But this must hold for all r, which is impossible since a, c, d are constants.

A more intuitive approach may be to prove that $\overline{z}$ is generally not analytic, but linear transformations are.

3.3.1.2. If

$$T_1 z = \frac{z+2}{z+3} \qquad T_2 z = \frac{z}{z+1}$$

find T_1T_2z, T_2T_1z, and $T_1^{-1}T_2z$.

Solution: Generally consider two linear transformations

$$T_1 z = \frac{a_1 z + b_1}{c_1 z + d_1} \qquad T_2 z = \frac{a_2 z + b_2}{c_2 z + d_2}.$$

Then, the composition $T_1T_2z = T_1(T_2z)$ is given by

$$\begin{aligned} T_1(T_2z) &= \frac{a_1(a_2z+b_2)/(c_2z+d_2)+b_1}{c_1(a_2z+b_2)/(c_2z+d_2)+d_1} = \frac{a_1(a_2z+b_2)+b_1(c_2z+d_2)}{c_1(a_2z+b_2)+d_1(c_2z+d_2)} \\ &= \frac{(a_1a_2+b_1c_2)z+(a_1b_2+b_1d_2)}{(c_1a_2+d_1c_2)z+(c_1b_2+d_1d_2)} \end{aligned}$$

Observe that if we write T_1 and T_2 as matrices

$$T_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \qquad T_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

then T_1T_2 is just given by the matrix product

$$T_1T_2 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2+b_1c_2 & a_1b_2+b_1d_2 \\ c_1a_2+d_1c_2 & c_1b_2+d_1d_2 \end{pmatrix}.$$

With our particular T_1, T_2, we have that

$$T_1 = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad T_1^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}.$$

Thus,

$$\begin{aligned} T_1T_2 &= \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \\ T_2T_1 &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \\ T_1^{-1}T_2 &= \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

So, the compositions are given by

$$T_1T_2z = \frac{3z+2}{4z+3} \qquad T_2T_1z = \frac{z+2}{2z+5} \qquad T_1^{-1}T_2 = z-2$$

3.3.1.3. Prove that the most general transformation which leaves the origin fixed and preserves all distances is either a rotation or a rotation followed by reflection in the real axis.

Solution: Consider a general linear transformation

$$Tz = \frac{az+b}{cz+d}.$$

If the origin is fixed, then $T(0) = 0$ and so $b/d = 0$. Thus, $b = 0$. Next, if distances are preserved then

$$|z-w| = |Tz - Tw| = \left|\frac{az}{cz+d} - \frac{aw}{cw+d}\right| = \left|\frac{aczw + adz - aczw - adw}{(cz+d)(cw+d)}\right| = \frac{|ad||z-w|}{|cz+d||cw+d|}.$$

It follows that $|cz+d||cw+d| = |ad|$ for $z \neq w$. In particular, this shows that $d \neq 0$, and if $w = 0$ then $|cz+d| = |a|$. Since the right hand side is constant, it must be that $c = 0$. Substituting this into $|cz+d||cw+d| = |ad|$, we see that $|d| = |a|$. So, also, $a \neq 0$. Finally we are left with $Tz = az/d$, where $|a/d| = 1$. If we set $k = a/d$ then $Tz = kz$ where $|k| = 1$, and it is clear that T is a rotation.

3.3.1.4. Show that any linear transformation which transforms the real axis into itself can be written with real coefficients.

Solution: Suppose that $Tz = (az+b)/(cz+d)$. We can assume that a is real, since it is either 0 or we can multiply the numerator and denominator by $\overline{a}$.

3.4. **Elementary Conformal Mappings.**

3.4.2. *A Survey of Elementary Mappings.* Throughout this section, I ignore the symmetry condition. I skipped that section.

3.4.2.1. Map the common part of the disks $|z| < 1$ and $|z-1| < 1$ on the inside of the unit circle. Choose the mapping so that the two symmetries are preserved.

Solution: The region is bounded by two circular arcs with common endpoints, so we take the approach given in the book. The two common endpoints are $a = 1/2 + \sqrt{3}/2i$ and $b = 1/2 - \sqrt{3}/2i$. So, consider first the map

$$z_1 = \frac{z-a}{z-b} = \frac{z - 1/2 - \sqrt{3}/2i}{z - 1/2 + \sqrt{3}/2i} = \frac{2z - 1 - \sqrt{3}i}{2z - 1 + \sqrt{3}i}.$$

We know this maps the region Ω to a sector. To determine the boundary of this sector we make smart choices of z. First consider $z = 1$, where we have

$$z_1(1) = \frac{1-\sqrt{3}i}{1+\sqrt{3}i} = e^{-i\pi/3 - i\pi/3} = e^{-2\pi/3i} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

Similarly, by choosing $z = 0$ we get

$$z_1(0) = \frac{a}{b} = e^{i\pi/3 - (-i\pi/3)} = e^{2i\pi/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

To determine the interior of the sector we make one more choice of z. Consider $z = 1/2$,

$$z_1(1/2) = \frac{-\sqrt{3}i}{\sqrt{3}i} = -1.$$

It follows that z_1 maps Ω to the sector $2\pi/3 \le \theta \le 4\pi/3$.

Consider now the map $z^{3/2} = e^{3/2 \log z}$. Exactly as we define $z^{1/2}$, we may consider the same branch cut by removing the negative real axis. But if we do that, we will remove a portion of $z_1(\Omega)$. To avoid this, we first rotate $z_1(\Omega)$. Let $z_2 = e^{i\pi/3} z_1$. Then z_2 rotates the sector $2\pi/3 \le \theta \le 4\pi/3$ to the sector $\pi \le \theta \le 5\pi/3$. Now we can apply $z_3 = z_2^{3/2}$, which expands this to the region $3\pi/2 \le \theta \le 5\pi/2 \Leftrightarrow -\pi/2 \le \theta \le \pi/2$. This is the right half-plane. We know that the final transformation $w = (z_3 - 1)/(z_3 + 1)$ maps this half plane to the unit disc $|w| < 1$. The full composition is

$$w = \frac{z_3 - 1}{z_3 + 1} = \frac{z_2^{3/2} - 1}{z_2^{3/2} + 1} = \frac{(e^{i\pi/3} z_1)^{3/2} - 1}{(e^{i\pi/3} z_1)^{3/2} + 1} = \frac{i z_1^{3/2} - 1}{i z_1^{3/2} + 1} = \frac{i(z - e^{i\pi/3})^{3/2}/(z - e^{-i\pi/3})^{3/2} - 1}{i(z - e^{i\pi/3})^{3/2}/(z - e^{-i\pi/3})^{3/2} + 1}.$$

I see no good way to simplify this from here.

3.4.2.2. Map the region between $|z| = 1$ and $|z - 1/2| = 1/2$ on a half plane.

Solution: The region is bounded by two circles which are tangent at $a = 1$. The map $z_1 = 1/(z-a) = 1/(z-1)$ sends this region to a parallel strip. We test z_1 at different points to determine the boundary curves. First consider $z = -1$ and $z = i$ which lie on $|z| = 1$. Then, $z_1(-1) = -1/2$ and $z_1(i) = 1/(i-1) = (-1-i)/2$. This shows that one boundary curve is the line $x = -1/2$ in the z_1-plane. Since the other boundary curve is parallel to this, we only need to test one point. Consider $z = 0$, which lies on the other circle. Then $z_1(0) = -1$, and so the other boundary curve is $x = -1$. We can map this to the upper half plane if the strip was bounded by $y = 0$ and $y = \pi$. But this is easily done by a rotation, translation, and dilation. First applying the rotation $z_2 = -iz_1$ converts the strip between $y = 1/2$ and $y = 1$. Next, the translation $z_3 = z_2 - i/2$ converts this to the strip between $y = 0$ and $y = 1/2$. Now, the dilation $z_4 = 2\pi z_3$ converts this to the strip between $y = 0$ and $y = \pi$ as desired. The final map $w = e^{z_4}$ transforms this to the upper half plane. In total, the map is

$$w = e^{z_4} = e^{2\pi z_3} = e^{2\pi(z_2 - i/2)} = e^{2\pi z_2 - i\pi} = e^{-2\pi i z_1 - i\pi} = e^{-i\pi} e^{-2\pi i/(z-1)} = -e^{-2\pi i/(z-1)}.$$

3.4.2.3. Map the complement of the arc $|z| = 1$, $y \geq 0$ on the outside of the unit circle so that the points at ∞ correspond to each other.

Solution: Consider the map $z_1 = (z+1)/(z-1)$. This maps the arc $|z| = 1, y \geq 0$ to a straight line. Choose $z = i$ so that

$$z_1(i) = \frac{1+i}{-1+i} = \frac{(1+i)(-1-i)}{2} = -i.$$

Since $z = i$ belongs to the specified arc, it follows that z_1 maps the arc to the ray $x = 0$, $y \leq 0$. This map sends ∞ to 1. Rotating the domain using $z_2 = -iz_1$ takes the ray to the negative real axis, and the composition maps ∞ to $-i$. Using the standard branch cut for the square root, we use $z_3 = z_2^{1/2}$ to map the region to the right half plane. Moreover, ∞ gets sent to $\sqrt{-i} = (e^{-\pi/2i})^{1/2} = e^{-\pi/4i}$. Now translate and dilate so that $e^{-\pi/4i}$ is sent to 1 – this is achieved via $z_4 = \sqrt{2}(z_3 + \sqrt{2}/2i) = \sqrt{2}z_3 + i$. Note that this keeps the region fixed to the right half plane, and we have sent ∞ to 1 in the composition. Finally, we use the map $w = (z_4+1)/(z_4-1)$ to map this to $|w| > 1$. It is clear that $\infty \mapsto \infty$. The full composition is

$$\begin{aligned} w &= \frac{z_4+1}{z_4-1} = \frac{\sqrt{2}z_3+i+1}{\sqrt{2}z_3+i-1} = \frac{\sqrt{2}z_2^{1/2}+i+1}{\sqrt{2}z_2^{1/2}+i-1} = \frac{\sqrt{2}(-iz_1)^{1/2}+i+1}{\sqrt{2}(-iz_1)^{1/2}+i-1} \\ &= \frac{\sqrt{2}e^{-\pi/4i}z_1^{1/2}+i+1}{\sqrt{2}e^{-\pi/4i}z_1^{1/2}+i-1} = \frac{(1-i)z_1^{1/2}+i+1}{(1-i)z_1^{1/2}+i-1} = \frac{(1-i)\sqrt{(z+1)/(z-1)}+i+1}{(1-i)\sqrt{(z+1)/(z-1)}+i-1} \\ &= \frac{\sqrt{(z+1)/(z-1)}+i}{\sqrt{(z+1)/(z-1)}-1} \end{aligned}$$

To verify $w(\infty) = \infty$, define $\tilde{w}$ by $\tilde{w} = w(1/z)$. Then,

$$\tilde{w}(z) = \frac{\sqrt{(1+z)/(1-z)}+i}{\sqrt{(1+z)/(1-z)}-1}.$$

Clearly $\tilde{w}(0) = \infty$ so that $w(\infty) = \infty$. Interestingly, our simplified version of w suggests that we only need three steps – two linear transformations and a square root in between.

3.4.2.4. Map the outside of the parabola $y^2 = 2px$ on the disk $|w| < 1$ so that $z = 0$ and $z = -p/2$ correspond to $w = 1$ and $w = 0$. (Lindelöf.)

Solution: We work backwards. XXX

3.4.2.5. Map the inside of the right-hand branch of the hyperbola $x^2 - y^2 = a^2$ on the disk $|w| < 1$ so that the focus corresponds to $w = 0$ and the vertex to $w = -1$. (Lindelöf.)

Solution: We are immediately guided towards using z^2 in some way, since $\mathrm{Re}(z^2) = x^2 - y^2$. So, the specified region is mapped to $x > 0$ by $z_1 = z^2 - a^2$. The focus is at $\sqrt{2}a$ while the vertex is at a. Under $z_1 = z^2 - a^2$ these get mapped to a^2 and 0, respectively. Recall that the map $w = (z-1)/(z+1)$ sends $x > 0$ to $|w| < 1$. Furthermore, $w(1) = 0$ and $w(0) = -1$. So, if we squish $x > 0$ so that the boundary stays fixed but a^2 is sent to 1, then composing it with the above finishes the problem. Hence we set $z_2 = z_1/a^2$ and $w = (z_2-1)/(z_2+1)$. The full composition is

$$w = \frac{z_2-1}{z_2+1} = \frac{z_1/a^2-1}{z_1/a^2+1} = \frac{z_1-a^2}{z_1+a^2} = \frac{z^2-2a^2}{z^2} = 1 - 2a^2z^{-2}.$$

3.4.2.6. Map the inside of the lemniscate $|z^2 - a^2| = \rho^2$ ($\rho > a$) on the disk $|w| < 1$ so that symmetries are preserved. (Lindelöf.)

Solution: If $w = z^2$ then the lemniscate is transformed into $|w - a^2| = \rho^2$, a circle with center a^2 and radius ρ^2. The interior is mapped to the interior. Composing with a translation and a

dilation gives the required map. In total

$$w = \frac{1}{\rho^2}(z^2 - a^2) = \frac{z^2}{\rho^2} - \frac{a^2}{\rho^2}.$$

3.4.2.7. Map the outside of the ellipse $(x/a)^2 + (y/b)^2 = 1$ onto $|w| < 1$ with preservation of symmetries.

4. Complex Integration

4.1. **The Fundamental Theorems.**

4.1.3. *Line Integrals as Functions of Arcs.*

4.1.3.1. Compute

$$\int_\gamma x\ dz$$

where γ is the directed line segment from 0 to $1+i$.

Solution: Let $z(t) = (1+i)t = t + it$. Then $z(t)$ parameterizes γ where $z(0) = 0$ and $z(1) = 1+i$. Thus,

$$\int_\gamma x dz = \int_0^1 x(t)z'(t)\ dt = \int_0^1 t(1+i)\ dt = (1+i)\frac{t}{2}\bigg|_0^1 = \frac{1}{2} + \frac{1}{2}i.$$

As an aside, recall that if $F(z)$ is analytic and $F(z) = U(x,y) + iV(x,y)$ then

$$F'(z) = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial x}i = \frac{\partial V}{\partial y} - i\frac{\partial U}{\partial y}.$$

Since F is analytic, U and V are harmonic functions. Thus,

$$0 = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \frac{\partial}{\partial x}\operatorname{Re}(F'(z)) - \frac{\partial}{\partial y}\operatorname{Im}(F'(z))$$
$$0 = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial}{\partial x}\operatorname{Im}(F'(z)) + \frac{\partial}{\partial y}\operatorname{Re}(F'(z))$$

Let $f(z) = u(x,y) + iv(x,y)$ where $f(z) = F'(z)$. Then, the above can be written as

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \qquad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0.$$

In this problem, we have $f(z) = x$ so that $u(x,y) = x$ and $v(x,y) = 0$. The first equation fails, so that f is not the derivative of an analytic function. Indeed, considering the circle $z(t) = 1/2 + 1/\sqrt{2}e^{it}$ gives a different value of the integral, so it is path dependent.

4.1.3.2. Compute

$$\int_{|z|=r} x\ dz$$

for the positive sense of the circle, in two ways: first, by use of a parameter, and second, by observing that $x = 1/2(z + \overline{z}) = 1/2(z + r^2/z)$ on the circle.

Solution: As seen in the previous problem, $f(z) = x$ is not the derivative of an analytic function so that this contour integral is not necessarily zero. Letting $z(t) = re^{it}$ we have $z'(t) = ire^{it}$ and thus

$$\int_{|z|=r} x\ dz = \int_0^{2\pi} x(t)z'(t)\ dt = \int_0^{2\pi} (r\cos t)(-r\sin t + ir\cos t)\ dt = \int_0^{2\pi} -r^2\cos t\sin t + ir^2\cos^2 t\ dt$$
$$= \int_0^{2\pi} -\frac{r^2}{2}\sin(2t) + \frac{ir^2}{2} + \frac{ir^2}{2}\cos(2t)\ dt = \int_0^{2\pi} \frac{ir^2}{2} = \pi r^2 i$$

Now recall that if C is the circle with center a and radius ρ then

$$\int_C \frac{1}{z-a}\ dz = 2\pi i.$$

Consequently,

$$\int_{|z|=r} x\ dz = \frac{1}{2}\int_{|z|=r} z + \frac{r^2}{z}\ dz = \frac{1}{2}\int_{|z|=r} z\ dz + \frac{1}{2}\int_{|z|=r} \frac{r^2}{z}\ dz = \frac{r^2}{2}\int_{|z|=r} \frac{1}{z}\ dz = \pi r^2 i$$

where we emphasize that the first integral vanishes since $f(z) = z$ is analytic.

4.1.3.3. Compute

$$\int_{|z|=2} \frac{dz}{z^2-1}$$

for the positive sense of the circle.

Solution: Here's one way to do the problem, which leads to some insight. We first split the integral as

$$\int_{|z|=2} \frac{dz}{z^2-1} = \int_{|z|=2} \left(\frac{1}{2(z-1)} - \frac{1}{2(z+1)}\right) dz = \frac{1}{2}\int_{|z|=2} \frac{dz}{z-1} - \frac{1}{2}\int_{|z|=2} \frac{dz}{z+1}.$$

Let $a \in \mathbb{C}$ and suppose that γ is any simple closed curve which bounds a compact region containing a. We wish to compute

$$\int_\gamma \frac{dz}{z-a}.$$

We can assume WLOG that $a = 0$ by a change of variables and that γ is positively oriented. By this, note that γ is diffeomorphic to S^1, and so induces an orientation on S^1 by requiring this diffeomorphism be orientation preserving. If the induced orientation on S^1 is positive (counterclockwise), we say that γ is positively oriented. Recall that $F(z) = \log z$ is analytic except along $x \leq 0$, and that $F(z) = 1/z$ (due to our choice of branch cut). Since γ is a closed loop around $z = 0$, it intersects the positive real axis somewhere. Denote this point by b. Let K denote the compact region γ bounds. Since $\operatorname{Int} K$ is open and $z \in \operatorname{Int} K$, there exists a ball of some radius r about z.

Let γ_1 be a straight line from b to r, γ_2 the negatively oriented circle $|z| = r$, and $\gamma_3 = -\gamma_1$. Let $\tilde{\gamma} = \gamma + \gamma_1 + \gamma_2 + \gamma_3$. Then γ is a simple closed curve which does not bound 0. It follows, since $1/z$ is the derivative of an analytic function in this region, that the integral vanishes. Also, since $\gamma_3 = -\gamma_1$ we have

$$0 = \int_{\tilde{\gamma}} \frac{dz}{z} = \int_\gamma \frac{dz}{z} + \int_{\gamma_1} \frac{dz}{z} + \int_{\gamma_2} \frac{dz}{z} - \int_{\gamma_1} \frac{dz}{z} = \int_\gamma \frac{dz}{z} + \int_{\gamma_2} \frac{dz}{z}.$$

Consequently,

$$\int_\gamma \frac{dz}{z} = -\int_{\gamma_2} \frac{dz}{z} = 2\pi i$$

since γ_2 is a negatively oriented circle centered at 0.

Returning to our problem, since ± 1 are in the region bounded by $|z| = 2$, we may apply the above and conclude

$$\int_{|z|=2} \frac{dz}{z^2-1} = \pi i - \pi i = 0.$$

This suggests that maybe $f(z) = 1/(z^2-1)$ is the derivative of an analytic function. Indeed, we know that $f(z) = F'(z)$ where $F(z) = -\operatorname{arctanh} z$. However, I think the branch points of $F(z)$ are ± 1...

One can also deduce that the real and imaginary parts are $u(x,y) = (x^2-y^2-1)/((x^2-y^2-1)^2 + (2xy)^2)$ and $v(x,y) = -2xy/((x^2-y^2-1)^2 + (2xy)^2)$. Letting $z(t) = 2e^{it} = 2\cos t + 2i \sin t$, one can painfully compute the integral.

4.1.3.4. Compute

$$\int_{|z|=1} |z-1|\ |dz|.$$

Solution: Parametrize $|z|=1$ by $z(t)=e^{it}=\cos t+i\sin t$. Then if $f(z)=|z-1|$, we have

$$f(z(t))=\sqrt{(\cos t-1)^2+\sin^2 t}=\sqrt{\cos^2 t-2\cos t+1+\sin^2 t}=\sqrt{2-2\cos t}=2|\sin(t/2)|.$$

Moreover,

$$|z'(t)|=\sqrt{(-\sin t)^2+(\cos t)^2}=1$$

so that

$$\int_{|z|=1} |z-1|\ |dz|=\int_0^{2\pi} 2|\sin(t/2)|\ dt=2\int_0^{2\pi}\sin(t/2)\ dt=-4\cos(t/2)\Big|_0^{2\pi}=-4(-1-1)=8.$$

4.1.3.5. Suppose that $f(z)$ is analytic on a closed curve γ (i.e., f is analytic in a region that contains γ). Show that

$$\int_\gamma \overline{f(z)}f'(z)\ dz$$

is purely imaginary. (The continuity of $f'(z)$ is taken for granted.)

Solution: Let $f(z)=u(x,y)+iv(x,y)$. Note that

$$\begin{aligned}\int_\gamma f(z)\ dz &= \int_a^b (u(x(t),y(t))+iv(x(t),y(t)))(x'(t)+iy'(t))\ dt\\ &= \int_a^b [u(x(t),y(t))x'(t)-v(x(t),y(t))y'(t)]+[u(x(t),y(t))y'(t)+v(x(t),y(t))x'(t)]i\ dt\end{aligned}$$

where $z(t)$, $a\le t\le b$ is a suitable parametrization of γ. If we write $dz=dx+idy$, then formally

$$f(z)dz=(u+iv)(dx+idy)=[udx-vdy]+[udy+vdx]i$$

which agrees with the above rigorous computation. We therefore have that $\mathrm{Re}(\int_\gamma f(z)dz)=\int_\gamma \mathrm{Re}(f(z)dz)$. Then,

$$\begin{aligned}\mathrm{Re}(\overline{f(z)}f'(z)dz) &= \mathrm{Re}((u-iv)(u_x+iv_x)(dx+idy))=\mathrm{Re}(([uu_x+vv_x]-[uu_y+vv_y]i)(dx+idy))\\ &= \mathrm{Re}([[uu_x+vv_x]dx+[uu_y+vv_y]dy]+[[uu_x+vv_x]dy-[uu_y+vv_y]dx]i)\\ &= [uu_x+vv_x]dx+[uu_y+vv_y]dy=\frac{1}{2}d(u^2+v^2)\end{aligned}$$

where we have used the Cauchy-Riemann equations throughout. It follows that we have an exact differential, and therefore the integral is zero. This shows that the overall integral is purely imaginary, since its real part vanishes.

4.1.3.6. Assume that $f(z)$ is analytic and satisfies the inequality $|f(z)-1|<1$ in a region Ω. Show that

$$\int_\gamma \frac{f'(z)}{f(z)}\ dz=0$$

for every closed curve in Ω. (The continuity of $f'(z)$ is taken for granted.)

Solution: If $w=f(z)$ then in the w-plane, the image of f lies in the disc $|w-1|<1$. This is a circle of radius 1 centered at 1. Importantly, $\mathrm{Re}\, f(z)>0$, and $-\pi<\arg f(z)<\pi$. Note that $\log z$ is well-defined and analytic on the complex plane minus $x\le 0$. Hence, the composition $\log f(z)$ is well-defined and analytic on Ω. Since $d/dz\log f(z)=f'(z)/f(z)$, we see that the integrand is the derivative of an analytic function. Hence the integral over any closed curve in that region vanishes.

4.1.3.7. If $P(z)$ is a polynomial and C denotes the circle $|z-a| = R$, what is the value of $\int_C P(z)d\bar{z}$? *Answer*: $-2\pi i R^2 P'(a)$.

Solution: By definition,

$$\int_C P(z)\ d\bar{z} := \overline{\int_C \overline{P(z)}\ dz}.$$

We can always write $P(z)$ as a power series

$$P(z) = \sum_{k=0}^{n} a_k (z-a)^k.$$

Parametrize C by $z(t) = a + Re^{it}$ with $0 \le t \le 2\pi$. Then,

$$\int_C \overline{P(z)}\ dz = \int_0^{2\pi} iRe^{it} \overline{\left(\sum_{k=0}^{n} a_k R^k e^{ikt}\right)}\ dt = i\sum_{k=0}^{n} a_k R^{k+1} \int_0^{2\pi} \cos((1-k)t) - i\sin((1-k)t)\ dt.$$

All of these evaluate to zero, except for the $k = 1$ term where we get

$$\int_C \overline{P(z)}\ dz = 2\pi i a_1 R^2.$$

It is easy to find that $a_1 = P'(a)/1! = P'(a)$. Hence,

$$\int_C P(z)\ d\bar{z} = -2\pi i R^2 P'(a).$$

4.1.3.8. Describe a set of circumstances under which the formula

$$\int_\gamma \log z\ dz = 0$$

is meaningful and true.

Solution: Consider $F(z) = z(\log z - 1)$. This map is single-valued and analytic in on the complex plane minus the negative real axis $x \le 0$. Since the choice of branch cut affects the values in the codomain, it does not put a restriction on γ. Hence any closed curve γ not intersecting the negative real axis is such that $\int_\gamma \log z\ dz = 0$.

4.2. **Cauchy's Integral Formula.**

4.2.2. *The Integral Formula.*
4.2.2.1. Compute

$$\int_{|z|=1} \frac{e^z}{z}\ dz.$$

Solution: The map $f(z) = e^z$ is analytic everywhere, and in particular on $|z| = 1$. By the representation formula,

$$f(0) = \frac{1}{2\pi i}\int_{|z|=1} \frac{f(z)}{z-0}\ dz = \frac{1}{2\pi i}\int_{|z|=1} \frac{e^z}{z}\ dz.$$

Hence, the integral is $2\pi i e^0 = 2\pi i$.

4.2.2.2. Compute

$$\int_{|z|=2} \frac{dz}{z^2+1}$$

by decomposition of the integrand in partial fractions.

Solution: We have that

$$\int_{|z|=2} \frac{dz}{z^2+1} = \int_{|z|=2} \left(\frac{1}{2i(z-i)} - \frac{1}{2i(z+i)}\right) dz = \frac{1}{2i}\int_{|z|=2} \frac{dz}{z-i} - \frac{1}{2i}\int_{|z|=2} \frac{dz}{z+i}.$$

By an application of Cauchy's representation formula with the constant function $f(z) = 1$ at $z = \pm i$, we arrive at

$$\int_{|z|=2} \frac{dz}{z^2+1} = \frac{1}{2i}\,(2\pi i - 2\pi i) = 0.$$

It is important that $\pm i$ lie in the bounded region determined by $|z| = 2$.

4.2.2.3. Compute

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2}$$

under the condition $|a| \neq \rho$. *Hint*: make use of the equations $z\overline{z} = \rho^2$ and $|dz| = -i\rho dz/z$.

Solution: Let's first show the second hint in the equation. Consider an arbitrary integrable function $f(z)$ and parametrize $|z| = \rho$ by $z(t) = \rho e^{it} = \rho\cos t + i\rho\sin t$. Then,

$$\int_{|z|=\rho} f(z)\ |dz| = \int_0^{2\pi} f(z(t))|z'(t)|\ dt = \int_0^{2\pi} \rho f(e^{it})\ dt.$$

On the other hand,

$$\int_{|z|=\rho} -i\rho\frac{f(z)}{z}\ dz = \int_0^{2\pi} -i\rho\frac{f(z(t))}{z(t)}z'(t)\ dt = \int_0^{2\pi} -i^2\rho^2\frac{f(e^{it})}{\rho e^{it}}e^{it}\ dt = \int_0^{2\pi} \rho f(e^{it})\ dt.$$

Since these integrals are equal, we are justified in formally writing $|dz| = -i\rho dz/z$.

Assume first that $a = 0$. Then,

$$\int_{|z|=\rho} \frac{|dz|}{|z|^2} = \int_{|z|=\rho} \frac{-i\rho dz/z}{\rho^2} = -\frac{i}{\rho}\int_{|z|=\rho} \frac{dz}{z} = -\frac{i}{\rho}(2\pi i) = \frac{2\pi}{\rho}.$$

Consider the change of variables $z \mapsto e^{i\theta}z$, $\theta \in [-\pi, \pi)$. The differential element transforms by $|e^{i\theta}dz| = |e^{i\theta}||dz| = |dz|$. Next,

$$|z-a|^2 = |e^{i\theta}z - a|^2 = |e^{i\theta}|^2|z - ae^{-i\theta}|^2.$$

By choosing $\theta = \arg a$, we may assume a is a nonnegative real number. Or,

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = \int_{|z|=\rho} \frac{|dz|}{|z-|a||^2}.$$

If z lies on $|z| = \rho$, then we have $\overline{z} = \rho^2/z$. Together with the above,

$$\begin{aligned}
\int_{|z|=\rho} \frac{|dz|}{|z-|a||^2} &= \int_{|z|=\rho} \frac{|dz|}{(z-|a|)(\overline{z}-|a|)} = \int_{|z|=\rho} \frac{-i\rho dz/z}{(z-|a|)(\rho^2/z-|a|)} = -i\rho\int_{|z|=\rho} \frac{dz}{(z-|a|)(\rho^2-|a|z)} \\
&= \frac{i\rho}{|a|^2-\rho^2}\left(\int_{|z|=\rho} \frac{dz}{z-|a|} + |a|\int_{|z|=\rho} \frac{dz}{\rho^2-|a|z}\right) \\
&= \frac{i\rho}{|a|^2-\rho^2}\left(\int_{|z|=\rho} \frac{dz}{z-|a|} - \int_{|z|=\rho} \frac{dz}{z-\rho^2/|a|}\right).
\end{aligned}$$

To evaluate the two integrals, we must determine if $|a|$ and $\rho^2/|a|$ lie the bounded region determined by $|z| = \rho$ or not. That is, if $|a| < \rho$ and/or $\rho^2/|a| < \rho$. The latter inequality reduces to $\rho < |a|$. So, there are two cases: $|a| < \rho$ and $|a| > \rho$. In the former, the first integral is $2\pi i$ while the second integral is zero (since $\rho^2/|a|$ is in the unbounded region). In the latter case, the roles are reversed. So,

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = \begin{cases} 2\pi\rho/(\rho^2-|a|^2) & |a| < \rho \\ 2\pi\rho/(|a|^2-\rho^2) & |a| > \rho \end{cases}.$$

Note that we can include the case $|a| = 0$ (in fact, we could have included this throughout if we consider the extend complex numbers).

4.2.3. *Higher Derivatives.*

4.2.3.1. Compute

$$\int_{|z|=1} e^z z^{-n}\ dz, \qquad \int_{|z|=2} z^n(1-z)^m\ dz, \qquad \int_{|z|=\rho} |z-a|^{-4}\ |dz|$$

where $|a| \neq \rho$.

Solution: Let the integrals be denoted by I_1, I_2, and I_3 respectively.

i) If $n \leq 0$ then $e^z z^{-n}$ is analytic, and hence $I_1 = 0$. If $n > 0$, then by Cauchy's integral formula:

$$I_1 = \int_{|z|=1} \frac{e^z}{z^n}\ dz = \frac{2\pi i}{(n-1)!} e^z \bigg|_{z=0} = \frac{2\pi i}{(n-1)!}.$$

ii) If $n, m \geq 0$ then $z^n(1-z)^m$ is analytic, and thus $I_2 = 0$. Suppose $n < 0$ but $m \geq 0$. Then,

$$\begin{aligned} I_2 &= \int_{|z|=2} \frac{(1-z)^m}{z^{|n|}}\ dz = \frac{2\pi i}{(|n|-1)!} \frac{d^{|n|-1}}{dz^{|n|-1}} (1-z)^m \bigg|_{z=0} \\ &= \frac{2\pi i m!(-1)^{|n|-1}}{(|n|-1)!(m-(|n|-1))!} = 2\pi i(-1)^{|n|-1} \binom{m}{|n|-1}. \end{aligned}$$

If $|n|-1 > m$, we interpret the above to be zero. Now suppose $m < 0$ but $n \geq 0$. Then,

$$\begin{aligned} I_2 &= \int_{|z|=2} \frac{z^n}{(1-z)^{|m|}}\ dz = (-1)^m \int_{|z|=2} \frac{z^n}{(z-1)^{|m|}}\ dz = \frac{2\pi i(-1)^m}{(|m|-1)!} \frac{d^{|m|-1}}{dz^{|m|-1}} z^n \bigg|_{z=1} \\ &= \frac{2\pi i n!(-1)^m}{(|m|-1)!(n-(|m|-1))!} = 2\pi i(-1)^m \binom{n}{|m|-1} \end{aligned}$$

which is also zero if $|m|-1 > n$. Assume now that both $n \leq 0$ and $m \leq 0$. Then,

$$I_2 = (-1)^m \int_{|z|=2} \frac{dz}{z^{|n|}(z-1)^{|m|}}.$$

This is probably going to be hellish to do... I think we're probably supposed to assume $n, m \geq 0$.

iii) This one also seems a little challenging, but is very similar to 4.2.2.3. We still have $|dz| = -i\rho dz/z$ and if $\overline{z} = \rho/z$. We can also go ahead and assume that a is a nonnegative real number, as before. So,

$$\begin{aligned} I_3 &= \int_{|z|=\rho} \frac{|dz|}{|z-a|^4} = -i\rho \int_{|z|=\rho} \frac{dz}{(z-a)^2(\rho^2/z-a)^2 z} \\ &= -i\rho \int_{|z|=\rho} \frac{z\ dz}{(z-a)^2(\rho^2-az)^2} = -\frac{i\rho}{a^2} \int_{|z|=\rho} \frac{z\ dz}{(z-a)^2(z-\rho^2/a)^2}. \end{aligned}$$

Note that if $|a| < \rho$, then $1/(z-\rho^2/a)^2$ is analytic inside $|z| = \rho$, and if $|a| > \rho$ then $1/(z-a)^2$ is analytic inside $|z| = \rho$. We assume $|a| < \rho$ first. Then,

$$I_3 = -\frac{i\rho}{a^2} \int_{|z|=\rho} \frac{z/(z-\rho^2/a)^2\ dz}{(z-a)^2} = -\frac{i\rho}{a^2} \frac{2\pi i}{1!} \frac{d}{dz} \left(\frac{z}{(z-\rho^2/a)^2} \right) \bigg|_{z=a}.$$

The derivative is

$$\frac{d}{dz} \left(\frac{z}{(z-\rho^2/a)^2} \right) = \frac{(z-\rho^2/a)^2 - 2z(z-\rho^2/a)}{(z-\rho^2/a)^4} = \frac{-z^2+\rho^4/a^2}{(z-\rho^2/a)^4} = \frac{\rho^4 a^2 - a^4 z^2}{(az-\rho^2)^4}.$$

Evaluating this at $z = a$ gives

$$\frac{d}{dz} \left(\frac{z}{(z-\rho^2/a)^2} \right) \bigg|_{z=a} = \frac{a^2(\rho^4-a^4)}{(a^2-\rho^2)^4}$$

so that I_3 is

$$I_3 = \frac{2\pi\rho(\rho^2+a^2)}{(\rho^2-a^2)^3}.$$

We now assume $|a| > \rho$. Then,

$$I_3 = -\frac{i\rho}{a^2}\int_{|z|=\rho} \frac{z/(z-a)^2\ dz}{(z-\rho^2/a)^2} = \frac{2\pi\rho}{a^2}\frac{d}{dz}\left(\frac{z}{(z-a)^2}\right)\bigg|_{z=\rho^2/a}.$$

The derivative is

$$\frac{d}{dz}\left(\frac{z}{(z-a)^2}\right) = \frac{a^2-z^2}{(z-a)^4} \quad\Rightarrow\quad \frac{d}{dz}\left(\frac{z}{(z-a)^2}\right)\bigg|_{z=\rho^2/a} = \frac{a^2(a^4-\rho^4)}{(\rho^2-a^2)^4} = \frac{a^2(a^2+\rho^2)}{(a^2-\rho^2)^3}.$$

Then, I_3 is simply

$$I_3 = \frac{2\pi\rho(a^2+\rho^2)}{(a^2-\rho^2)^3}.$$

Note that in the two cases, all that changes is a sign.

4.2.3.2. Prove that a function which is analytic in the whole plane and satisfies an inequality $|f(z)| \le |z|^n$ for some n and all sufficiently large $|z|$ reduces to a polynomial.

Solution: We have existence of an $R > 0$ such that $|f(z)| \le |z|^n$ for all $|z| > R$. Let $r > R$, then by Cauchy's integral formula,

$$|f^{(n+1)}(a)| \le \frac{(n+1)!}{2\pi}\int_{|z|=r}\frac{|f(z)|\ |dz|}{|z-a|^{n+2}} \le \frac{(n+1)!}{2\pi}\int_{|z|=r}\frac{r^n\ |dz|}{(r-|a|)^{n+2}} = \frac{(n+1)!r^{n+1}}{(r-|a|)^{n+2}}.$$

for any a with $|a| < r$. Of course, for any a we can find an r such that $|a| < r$ and $R < r$, so as to arrive at the above estimate. By taking $r \to \infty$ it follows that $|f^{(n+1)}(a)| \le 0$, and so $f^{(n+1)} \equiv 0$. Hence f is a polynomial of degree at most n.

4.2.3.3. If $f(z)$ is analytic and $|f(z)| \le M$ for $|z| \le R$, find an upper bound for $|f^{(n)}(z)|$ in $|z| \le \rho < R$.

Solution: By Cauchy's integral formula,

$$|f^{(n)}(a)| \le \frac{n!}{2\pi}\int_{|z|=R}\frac{|f(z)|\ |dz|}{|z-a|^{n+1}} \le \frac{n!MR}{(R-|a|)^{n+1}}.$$

The right hand side increases with increasing $|a|$. Hence, it is maximized when $|a| = \rho$, and for any a with $|z| \le \rho$,

$$|f^{(n)}(z)| \le \frac{n!MR}{(R-\rho)^{n+1}}.$$

4.2.3.4. If $f(z)$ is analytic for $|z| < 1$ and $|f(z)| \le 1/(1-|z|)$, find the best estimate of $|f^{(n)}(0)|$ that Cauchy's inequality will yield.

Solution: Fix $n \ge 0$ and let $R = n/(n+1)$ so that $f(z)$ is analytic on $|z| = R$. Then by Cauchy's inequality,

$$|f^{(n)}(0)| \le \frac{n!}{R^n}\max_{|z|=R}|f(z)| = \frac{n!}{R^n(1-R)} = \frac{n!(n+1)^{n+1}}{n^n} = n!\left(1+\frac{1}{n}\right)^n.$$

The optimal radius R was obtained by minimizing the function $g(r) = 1/(r^n(1-r))$ on $(0,1)$. Evaluation of $g(r)$ at R gives the quantity used in the estimate, $(n+1)^{n+1}/n^n$. Interestingly, this is approximately $e([n+1]+n)/2 = (n+1/2)e$.

4.2.3.5. Show that the successive derivatives of an analytic function at a point can never satisfy $|f^{(n)}(z)| > n!n^n$. Formulate a sharper theorem of the same kind.

Solution: Recall that analytic functions admit a power series

$$f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)(z-a)^n}{n!}.$$

Let $a_n = f^{(n)}(a)/n!$, so that for any a we have $|a_n| > n^n$. By Hadamard's formula, the radius of convergence is given by

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} n = \infty$$

so that the power series defined in the right hand side is valid only at a. Yet, as an analytic function it should be analytic in some radius of convergence for $R > 0$. Of course, if $g(n)$ is any increasing function (instead of n^n) such that $g(n) \to \infty$, then we arrive at the same conclusion.

Here's another way to go about the problem, which seems to be more of what Ahlfors intended by putting this problem here. Cauchy's estimate tells us that if $R > 0$,

$$|f^{(n)}(z)| \leq \frac{n!M}{R^n}$$

where $M = \max_{|z|=R} |f(z)|$. Note that M is finite since $f(z)$ is continuous and $|z| = R$ is compact. If also $|f^{(n)}(z)| > n!n^n$, then $n^n \leq M/R^n$ for all $n \geq 0$. For n large enough this inequality fails, since we can rearrange it to $(nR)^n \leq M$, and the left hand side is unbounded for every $R > 0$.

4.2.3.6. A more general form of Lemma 3 reads as follows:
Let the function $\varphi(z,t)$ be continuous as a function of both variables when z lies in a region Ω and $a \leq t \leq b$. Suppose further that $\varphi(z,t)$ is analytic as a function of $z \in \Omega$ for any fixed t. Then

$$F(z) = \int_\alpha^\beta \varphi(z,t)\ dt$$

is analytic in z and

$$F'(z) = \int_\alpha^\beta \frac{\partial \varphi(z,t)}{\partial z}\ dt \qquad (\star).$$

To prove this represent $\varphi(z,t)$ as a Cauchy integral

$$\varphi(z,t) = \frac{1}{2\pi i}\int_C \frac{\varphi(\zeta,t)}{\zeta - z}\ d\zeta.$$

Fill in the necessary details to obtain

$$F(z) = \int_C \left(\frac{1}{2\pi i}\int_\alpha^\beta \varphi(\zeta,t)\ dt\right)\frac{d\zeta}{\zeta - z}.$$

and use Lemma 3 to prove $(\star)$.

Solution: Recall that Cauchy's integral formula tells us that

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)\ d\zeta}{\zeta - z}$$

for z in the bounded region determined by C. Applying this with $F(z)$ gives

$$F(z) = \frac{1}{2\pi i}\int_C \frac{F(\zeta)\ d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_C \left(\int_\alpha^\beta \varphi(\zeta,t)\ dt\right)\frac{d\zeta}{\zeta - z} = \int_C \left(\frac{1}{2\pi i}\int_\alpha^\beta \varphi(\zeta,t)\ dt\right)\frac{d\zeta}{\zeta - z}$$

(one may also view it as a consequence of Fubini's theorem). Defining $\Phi(\zeta) = 1/2\pi i \int_\alpha^\beta \varphi(\zeta,t)\ dt$, Lemma 3 tells us that

$$F'(z) = \int_C \frac{\Phi(\zeta)\ d\zeta}{(\zeta - z)^2} = \int_C \left(\frac{1}{2\pi i}\int_\alpha^\beta \varphi(\zeta,t)\ dt\right)\frac{d\zeta}{(\zeta - z)^2}.$$

Now apply Lemma 3 once more to $\varphi(z,t)$ directly:

$$\frac{\partial \varphi(z,t)}{\partial z} = \frac{1}{2\pi i}\int_C \frac{\varphi(\zeta,t)\ d\zeta}{(\zeta - z)^2}.$$

Integrating this over $\alpha \leq t \leq \beta$ and using Fubini's theorem to switch integrals gives the result.

4.3. **Local Properties of Analytical Functions.**

4.3.2. *Zeros and Poles.*
4.3.2.1. If $f(z)$ and $g(z)$ have the algebraic orders h and k at $z = a$, show that fg has the order $h + k$, f/g the order $h - k$, and $f + g$ an order which does not exceed $\max(h, k)$.

Solution: By definition, $f(z)$ has an algebraic order h at $z = a$ if

a) $\lim_{z\to a} |z - a|^\alpha |f(z)| = 0$ for all real $\alpha > h$, and
b) $\lim_{z\to a} |z - a|^\alpha |f(z)| = \infty$ for all real $\alpha < h$.

Consider now the function fg. Suppose $\alpha > h + k$. Then we can write α as $\alpha_1 + \alpha_2$ with $\alpha_1 > h$ and $\alpha_2 > k$. Indeed, if $\epsilon = \alpha - h - k > 0$, then we can consider $\alpha_1 := h + \epsilon/2$ and $\alpha_2 := k + \epsilon/2$. Consequently,

$$\lim_{z\to a} |z - a|^\alpha |f(z)g(z)| = \left(\lim_{z\to a} |z - a|^{\alpha_1} |f(z)|\right) \left(\lim_{z\to a} |z - a|^{\alpha_2} |g(z)|\right) = 0.$$

Similarly, if $\alpha < h + k$ we can write $\alpha = \alpha_1 + \alpha_2$ with $\alpha_1 < h$ and $\alpha_2 < k$. Then, by the same logic we can split the limit to a product of limits, each evaluating to ∞.

Next, observe that constant, nonzero functions have algebraic order 0. By applying the above with g and $1/g$, we see that the orders must satisfy $k + \tilde{k} = 0$, or that the order of $1/g$ is $\tilde{k} = -k$. So, the order of f/g is simply the order of $f(1/g)$, which is $h + (-k) = h - k$.

Finally, suppose that $f + g$ has order greater than $\max(h, k)$. Then there is a nondegenerate interval between $\max(h, k)$ and the order of $f + g$ – let α be in the interior of this interval. Then, by the triangle inequality and applying the definitions of the orders:

$$\infty = \lim_{z\to a} |z - a|^\alpha |f(z) + g(z)| \le \lim_{z\to a} |z - a|^\alpha |f(z)| + \lim_{z\to a} |z - a|^\alpha |g(z)| = 0.$$

Clearly this is a contradiction, so that the order of $f + g$ is majorized by $\max(f, g)$.

4.3.2.2. Show that a function which is analytic in the whole plane and has a nonessential singularity at ∞ reduces to a polynomial.

Solution: If $\lim_{z\to\infty} f(z) \in \mathbb{C}$, then $f(z)$ is bounded in a region containing ∞. The complement of this region is compact, and hence $f(z)$ is bounded there too. Consequently, $f(z)$ is a bounded holomorphic function, and by Louisville's theorem $f(z)$ reduces to a constant (thus, is a polynomial). Suppose instead that $\lim_{z\to\infty} f(z) = \infty$. We recall that in Exercise 4.2.3.2, we show that if $|f(z)|$ grows like $|z|^n$ for $|z|$ large, then f is a polynomial. So, we aim to prove an estimate like this. Consider the function $g(z) = 1/f(1/z)$. Since $f(z) \to \infty$ as $z \to \infty$, we see that $g(z) \to 0$ as $z \to 0$. Hence, g has a removable singularity at 0. Let m be the algebraic order of this zero, so that $h(z)$ is a holomorphic function satisfying $g(z) = z^m h(z)$ and $h(0) \neq 0$. Because of this, $1/h$ is continuous in some ball $|z| < R$, and is therefore bounded by $M < \infty$ in this ball. Consequently,

$$\left| f\left(\frac{1}{z}\right) \right| = \frac{1}{|g(z)|} = \frac{1}{|z|^m} \left| \frac{1}{h(z)} \right| \le \frac{M}{|z|^m}.$$

Considering $z \mapsto 1/z$, we see that $|f(z)| \le M|z|^m$ for $|z| \ge 1/R$. This is exactly the estimate we want.

4.3.2.3. Show that the functions e^z, $\sin z$ and $\cos z$ have essential singularities at ∞.

Solution: All these functions are analytic, so that if they had a nonessential singularity at ∞, then by Exercise 4.3.2.2 they would be polynomials. But, this is not the case (the latter two have infinitely many zeros, and e^z is not zero and not constant).

One can also prove this directly. Consider first $f(z) = e^z$. To describe the nature of $f(z)$ at infinity, we instead analyze the nature of $g(z) = f(1/z) = e^{1/z}$ at $z = 0$. For any real α, $|z|^\alpha |e^{1/z}| = |z|^\alpha$. XXX. It is easily seen that this limit blows up for $\alpha < 0$ and goes to zero for $\alpha > 0$. But, $g(z)$ tends to zero as $z \to 0$, so that g has no algebraic order. Hence, e^z has no algebraic order; ∞ is thus an

essential singularity.

4.3.2.4. Show that any function which is meromorphic in the extended plane is rational.

Solution: Note that rational functions have finitely many poles, so we aim to show this first. Suppose that f is meromorphic in the extended plane and has poles a_1, a_2, a_3... Then, since the extended complex plane is compact, there exists a convergent subsequence (still denoted $a_1, a_2, ...$). This implies that the poles are not isolated, which contradicts f being meromoprhic. So, there are finitely many poles – say N. Let h_n be the order of the pole at a_n. Consider the function

$$h(z) = f(z) \prod_{n=1}^{N} (z - a_n)^{h_n}.$$

We wish to show that h is a polynomial so that $f(z) = h(z)/\prod_{n=1}^{N}(z-a_n)^{h_n}$ is a rational function. By construction, h is an analytic function and therefore only has a pole possibly at ∞. It follows that h is a polynomial, and we are done.

4.3.2.5. Prove that an isolated singularity of $f(z)$ is removable as soon as either $\operatorname{Re} f(z)$ or $\operatorname{Im} f(z)$ is bounded above or below. *Hint*: Apply a fractional linear transformation.

Solution: Note that the fractional linear transformation $T(z) = (z-1)/(z+1)$ maps $\{\mathrm{Re} > 0\}$ to the unit disc. Consequently, the map $T_c(z) = (z-1-c)/(z+1-c)$ maps $\{\mathrm{Re} > c\}$ onto the unit disc. If $\mathrm{Im} > c$, $\mathrm{Re} < c$, $\mathrm{Im} < c$, we can always rotate these domains onto $\mathrm{Re} > c$. In particular, we can always map these to the unit disc via

$$T_{c,n}(z) = \frac{i^n z - 1 - c}{i^n z + 1 - c}$$

for $n = 0, 1, 2, 3$ (corresponding to the four cases, in order). Hence, without loss of generality we may assume $\mathrm{Re} > 0$. Consider now the composite map

$$g(z) = \frac{f(z) - 1}{f(z) + 1}.$$

If g has a removable singularity at zero, then

$$\lim_{z \to 0} z g(z) = 0$$

and consequently,

$$\lim_{z \to 0} z[f(z) - 1] = \lim_{z \to 0} z f(z) = 0$$

so that

4.3.3. *The Local Mapping.*
4.3.3.1. Determine explicitly the largest disk about the origin whose image under the mapping $w = z^2 + z$ is one to one.

Solution: If $f(z) = z^2 + z$ then $f(0) = 0$ and $f'(z) = 2z + 1$ so that $f'(0) \neq 0$. So, $z = 0$ is a simple root and we can apply Theorem 11 to deduce existence of a disc. Note that if $f'(z) = 0$, then f cannot be injective. The only place where $f'(z) = 0$ is at $z = -1/2$, so that f is injective in a disc of radius at most $1/2$. Suppose now that $z_1 \neq z_2$ are such that $|z_1|, |z_2| < 1/2$. If $f(z_1) = f(z_2)$, then

$$(z_1 - z_2)(z_1 + z_2) = z_1^2 - z_2^2 = z_2 - z_1.$$

Consequently, $z_1 + z_2 = -1$. By the triangle inequality,

$$1 = |z_1 + z_2| \leq |z_1| + |z_2| < \frac{1}{2} + \frac{1}{2} = 1,$$

a contradiction. Hence f is injective on $|z| < 1/2$.

4.3.3.2. Same problem for $w = e^z$.

Solution: Note that $f(z) = e^z$ has nonzero derivative everywhere, so we cannot use the same logic as above to bound the injectivity radius. Suppose that $e^{z_1} = e^{z_2}$. It follows immediately (by taking absolute values) that $\operatorname{Re} z_1 = \operatorname{Re} z_2$. Dividing through by this, we have

$$\cos(\operatorname{Im} z_1) + i\sin(\operatorname{Im} z_1) = \cos(\operatorname{Im} z_2) + i\sin(\operatorname{Im} z_2).$$

By equating real and imaginary parts, one discovers that $\operatorname{Im} z_1 - \operatorname{Im} z_2 = 2\pi n$ for some integer n. Hence, the largest disc is one whose imaginary parts differ by less than 2π. In other words, a disc of radius π.

4.3.3.3. Apply the representation $f(z) = w_0 + \zeta(z)^n$ to $\cos z$ with $z_0 = 0$. Determine $\zeta(z)$ explicitly.

Solution: If $z_0 = 0$ then $w_0 = 1$. Now, $f'(z) = -\sin z$ so that $f'(0) = 0$, and $f''(z) = -\cos z$ so that $f''(0) = 1$. Hence, $n = 2$.

$$f(z) - 1 = \cos z - 1 = -2\sin\left(\frac{z}{2}\right)^2.$$

It follows that $\zeta(z) = \sqrt{2}i\sin(z/2)$.

4.3.3.4. If $f(z)$ is analytic at the origin and $f'(0) \neq 0$, prove the existence of an analytic $g(z)$ such that $f(z^n) = f(0) + g(z)^n$ in a neighborhood of 0.

Solution: Let $h(z) = f(z^n)$, which is analytic as the composition of analytic functions. Now observe that $h'(z) = nz^{n-1}f'(z^n)$. As we differentiate this, the product rule generates more terms but there is only one which contains $f'(z^n)$; the one obtained by differentiating the polynomial coefficient of $f'(z^n)$. All other terms contain z to some nonzero power multiplied by a higher order derivative of f. All of these evaluate to zero, so we write $h^{(k)}(z) = n!/(n-k)!f'(z^n) + h_k(z)$, where $h_k(z)$ is analytic and $h_k(0) = 0$. Accordingly, $h^{(n)}(0) = n!f'(0) + h_n(0) = n!f'(0) \neq 0$. So, $h(z) - f(0)$ has a zero of order n, and we may apply Theorem 11 to locally write

$$f(z^n) - f(0) = h(z) - f(0) = (z - z_0)^n\tilde{g}(z).$$

Then, if ϵ is such that $|\tilde{g}(z_0) - \tilde{g}(z)| < |\tilde{g}(z_0)|$ for $|z - z_0| < \epsilon$ we can choose a single-valued branch of $\sqrt[n]{\tilde{g}(z)}$. By defining $g(z) = (z - z_0)\sqrt[n]{\tilde{g}(z)}$, which is analytic, we have

$$f(z^n) - f(0) = h(z) - f(0) = (z - z_0)^n\tilde{g}(z) = g(z)^n$$

as desired.

4.3.4. *The Maximum Principle.*
4.3.4.1. Show by use of (36), or directly, that $|f(z)| \leq 1$ for $|z| \leq 1$ implies

$$\frac{|f'(z)|}{(1 - |f(z)|^2)} \leq \frac{1}{1 - |z|^2}.$$

Solution: First, (36) tells us that

$$\left|\frac{M(f(z) - w_0)}{M^2 - \overline{w_0}f(z)}\right| \leq \left|\frac{R(z - z_0)}{R^2 - \overline{z_0}z}\right|$$

when $|z_0| < R$ and $|w_0| < M$, $w_0 = f(z_0)$. In our particular problem, $R = M = 1$, and we can rewrite it as

$$\left|\frac{[f(z) - w_0]/[z - z_0]}{1 - \overline{w_0}f(z)}\right| \leq \left|\frac{1}{1 - \overline{z_0}z}\right|.$$

Now, taking $z \to z_0$ within $|z| < 1$ gives

$$\frac{|f'(z_0)|}{1 - |f(z_0)|^2} = \frac{|f'(z_0)|}{|1 - \overline{w_0}f(z_0)|} \leq \frac{1}{|1 - \overline{z_0}z_0|} = \frac{1}{1 - |z_0|^2}.$$

But, the choice of z_0 in $|z| < 1$ was arbitrary, so long as $|f(z_0)| < 1$. Consequently,

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}$$

on $|z| < 1$.

4.3.4.2. If $f(z)$ is analytic and $\operatorname{Im} f(z) \geq 0$ for $\operatorname{Im} z > 0$, show that

$$\frac{|f(z) - f(z_0)|}{|f(z) - \overline{f(z_0)}|} \leq \frac{|z - z_0|}{|z - \overline{z_0}|}$$

and

$$\frac{|f'(z)|}{\operatorname{Im} f(z)} \leq \frac{1}{y}$$

where $z = x + iy$.

Solution: Consider the linear transformations

$$Tz = \frac{z - z_0}{z - \overline{z_0}} \qquad Sw = \frac{w - f(z_0)}{w - \overline{f(z_0)}}.$$

As done in the book, if we can show that $Sf(T^{-1}\zeta)$ satisfies the hypotheses of the Schwarz lemma, then we have $|Sf(T^{-1}\zeta)| \leq |\zeta|$, or by setting $z = T^{-1}\zeta$, $|Sf(z)| \leq |Tz|$. Define $F(\zeta) := Sf(T^{-1}\zeta)$. We first want to show that $F(0) = 0$. Since $Tz_0 = 0$, we have $T^{-1}(0) = z_0$. Moreover, it is easily seen that $Sf(z_0) = 0$, so that $F(0) = 0$. Now, Tz maps $\operatorname{Im} z > 0$ to $|\zeta| < 1$ and Sw maps $\operatorname{Im} w \geq 0$ to $|Sw| \leq 1$. Since $f(z)$ maps $\operatorname{Im} z > 0$ to $\operatorname{Im} f(z) \geq 0$, it follows that $|F(\zeta)| \leq 1$ on $|\zeta| < 1$. As a composition of analytic functions, F is analytic. Thus the Schwarz lemma applies.

The second inequality follows from the first. If we rearrange it to

$$\left|\frac{f(z) - f(z_0)}{z - z_0}\right| \leq \left|\frac{f(z) - \overline{f(z_0)}}{z - \overline{z_0}}\right|$$

then taking $z \to z_0$ (from above?) yields

$$|f'(z_0)| \leq \left|\frac{f(z_0) - \overline{f(z_0)}}{z_0 - \overline{z_0}}\right| = \left|\frac{2 \operatorname{Im} f(z_0)}{2 \operatorname{Im} z_0}\right| = \frac{\operatorname{Im} f(z_0)}{y_0}.$$

For arbitrary z_0, this results in the desired inequality.

4.3.4.3. In 4.3.4.1 and 4.3.4.2, prove that equality implies that $f(z)$ is a linear transformation.

Solution: Observe that both proofs use the same method; apply the Schwarz lemma with $F(\zeta) = Sf(T^{-1}(\zeta))$ for some linear transformations S and T. If equality holds, then we have that $Sf(T^{-1}(\zeta)) = c\zeta$ for some $c \in \mathbb{C}$. If $z = T^{-1}(\zeta)$, then $Sf(z) = cT(z)$. Then, by composing with S^{-1} we have $f(z) = S^{-1}(cTz)$. As a composition of linear transformations, f is one too.

4.3.4.4. Derive corresponding inequalities if $f(z)$ maps $|z| < 1$ into the upper half plane.

Solution: Consider the linear transformation $Sw = (w - f(0))/(w - \overline{f(0)})$. As observed, Sw maps the upper half plane onto $|Sw| < 1$. Hence, the composition $F(z) = Sf(z)$ satisfies $|F(z)| < 1$ for $|z| < 1$. Moreover, $Sf(0) = 0$ so that $F(0) = 0$ and F is analytic on $|z| < 1$. By the Schwarz lemma,

$$\left|\frac{f(z) - f(0)}{f(z) - \overline{f(0)}}\right| = |Sf(z)| \leq |z|.$$

Rearranging this yields

$$\left|\frac{f(z) - f(0)}{z}\right| \leq |f(z) - \overline{f(0)}|$$

so that taking $z \to 0$ gives

$$|f'(0)| \leq |f(0) - \overline{f(0)}| = 2|\operatorname{Im} f(0)| = 2 \operatorname{Im} f(0).$$

4.3.4.5. Prove by use of Schwarz's lemma that every one-to-one conformal mapping of a disk onto another (or a half plane) is given by a linear transformation.

Solution: First, the problem reduces to considering only those biholomorphic maps f from the unit disc to itself. Suppose $f : D_1 \to D_2$ (where D_1 and D_2 are discs or half planes) and f is injective. Then, f is biholomorphic onto D_2. Next, we can consider two linear transformations T and S which map the unit disc $\mathbb{D}$ to D_1 and D_2 to $\mathbb{D}$. We may choose these to also be biholomorphic. Thus, the composition $S \circ f \circ T : \mathbb{D} \to \mathbb{D}$ is biholomorphic. If we showed all such maps are linear transformations, then by considering appropriate inverses of S and T we can write f as the composition of three linear transformations.

So, we only consider biholomorphic $f : \mathbb{D} \to \mathbb{D}$. Let S be given by $Sw = [w - f(0)]/[1 - \overline{f(0)}w]$ and define $F(z) = Sf(z)$. Note that $|Sw| < 1$ for $|w| < 1$ (since also $|f(0)| < 1$ by hypothesis), see 1.1.5.1 for details. It follows that $F : \mathbb{D} \to \mathbb{D}$ is such that $F(0) = 0$. Consequently, we can apply the Schwarz lemma to F and obtain $|F(z)| \leq |z|$. On the other hand, since $F : \mathbb{D} \to \mathbb{D}$ is biholomorphic, we can apply the Schwarz lemma to F^{-1} and obtain $|F^{-1}(\zeta)| \leq |\zeta|$. Letting $\zeta = F(z)$, which is valid since F maps into $\mathbb{D}$, we get

$$|z| = |F^{-1}(F(z))| \leq |F(z)| \leq |z|.$$

Hence, $|F(z)| = |z|$ and $F(z) = cz$ for some $c \in \mathbb{C}$. It follows that $f(z) = S^{-1}(cz)$, and thus f is a linear transformation.

4.5. **The Calculus of Residues.**

4.5.2. *The Argument Principle.*
4.5.2.1. How many roots does the equation $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ have in the disk $|z| < 1$? *Hint*: Look for the biggest term when $|z| = 1$ and apply Rouché's theorem.

Solution: Let $f(z) = 6z^3$ and $g(z) = z^7 - 2z^5 + 6z^3 - z + 1$. Then on $|z| = 1$,

$$|f(z) - g(z)| = |z^7 - 2z^5 - z + 1| \leq |z|^7 + 2|z|^5 + |z| + 1 = 5 < 6 = |f(z)|.$$

By Rouché's theorem, $f(z)$ and $g(z)$ have the same number of zeros in $|z| < 1$. Clearly $f(z) = 6z^3$ has three (a zero of order three at the origin), so that $g(z)$ has three zeros in $|z| < 1$.

4.5.2.2. How many roots of the equation $z^4 - 6z + 3 = 0$ have their modulus between 1 and 2?

Solution: We simply need to apply Rouché's theorem twice with $|z| = 1$ and $|z| = 2$. To find the number of zeros in $|z| < 1$, let $f(z) = 6z$ and $g(z) = z^4 - 6z + 3$. Then on $|z| = 1$,

$$|f(z) - g(z)| = |z^4 + 3| \leq |z|^4 + 3 = 4 < 6 = |f(z)|.$$

Thus, $f(z)$ and $g(z)$ have the same number of zeros in $|z| < 1$. Clearly $f(z)$ has one so that $g(z)$ has one zero. To find the number of zeros in $|z| < 2$, choose $f(z) = z^4$ instead. Then on $|z| = 2$,

$$|f(z) - g(z)| = |6z + 3| \leq 6|z| + 3 = 15 \leq 16 = |z|^4 = |f(z)|.$$

Thus, $g(z)$ has four zeros in $|z| < 2$. We've already counted one of these, so that $g(z)$ has three zeros in $1 \leq |z| < 2$.

4.5.2.3. How many roots of the equation $z^4 + 8z^3 + 3z^2 + 8z + 3 = 0$ lie in the right half plane? *Hint*: Sketch the image of the imaginary axis and apply the argument principle to a large half disk.

Solution: We parametrize the imaginary axis by $z(t) = it$ for $t \in \mathbb{R}$. Then, the image of the imaginary axis is given by

$$z(t) = (it)^4 + 8(it)^3 + 3(it)^2 + 8(it) + 3 = t^4 - 8t^3 i - 3t^2 + 8ti + 3.$$

Note that $\operatorname{Re} f(it) = t^4 - 3t^2 + 3 = (t^2 - 3/2)^2 + 3/4 > 0$. It follows that travelling along the imaginary axis produces 0 turns. Considering this, we take a semicircle of radius R with diameter along the imaginary axis, whose circular arc extends into the right half-plane. We need only analyze the change in argument along the circular arc – that is, for $\theta = -\pi/2$ to $\theta = \pi/2$. Finally, observe that for large R the z^4 term dominates, and therefore $f(Re^{i\theta}) \approx R^4 e^{4i\theta}$. As θ ranges from $-\pi/2$ to $\pi/2$, we see that $f(Re^{i\theta})$ ranges from -2π to 2π, or over an interval of length 4π. This tells us that f winds twice, and therefore there are two roots.

4.5.3. *Evaluation of Definite Integrals.*
4.5.3.1. Find the poles and residues of the following functions:

a) $1/(z^2+5z+6)$, b) $1/(z^2-1)^2$, c) $1/\sin z$, d) $\cot z$,

e) $1/\sin^2 z$, f) $1/(z^m(1-z)^n)$

where n, m are positive integers.

Solution:

a) We can factor z^2+5z+6 as $(z+3)(z+2)$, so that there are two poles at $z = -3$ and $z = -2$. The corresponding residues are

$$\operatorname{Res}_{z=-3} \frac{1}{z^2+5z+6} = \frac{1}{-3+2} = -1, \qquad \operatorname{Res}_{z=-2} \frac{1}{z^2+5z+6} = \frac{1}{-2+3} = 1.$$

b) Clearly the poles of $1/(z^2-1)^2$ will be at $z = \pm 1$. Both of these poles are of order 2, so we apply the formula

$$\operatorname{Res}_{z=a} f(z) = \frac{1}{(n-1)!} \lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}}((z-a)^n f(z))$$

where $f(z)$ has a pole of order n at $z = a$. In this case,

$$\operatorname{Res}_{z=1} \frac{1}{(z-1)^2(z+1)^2} = \lim_{z\to 1} \frac{d}{dz}\left(\frac{(z-1)^2}{(z-1)^2(z+1)^2}\right) = -\lim_{z\to 1} \frac{2}{(z+1)^3} = -\frac{1}{4},$$

$$\operatorname{Res}_{z=-1} \frac{1}{(z-1)^2(z+1)^2} = \lim_{z\to -1} \frac{d}{dz}\left(\frac{(z+1)^2}{(z-1)^2(z+1)^2}\right) = -\lim_{z\to -1} \frac{2}{(z-1)^3} = \frac{1}{4}.$$

c) The zeros of $\sin z$ occur at $z = n\pi$ for integers n. Since $d/dz \sin z = \cos z$ is nonzero at these points, the poles are simple. Thus,

$$\operatorname{Res}_{z=n\pi} \frac{1}{\sin z} = \lim_{z\to n\pi} \frac{z-n\pi}{\sin z} = \lim_{z\to 0} \frac{z}{\sin(z+n\pi)} = (-1)^n \lim_{z\to 0} \frac{z}{\sin z} = (-1)^n.$$

d) Since $\cot z = \cos z/\sin z$, the poles are still $z = n\pi$ for integers n. Since $\cos z$ is nonzero at these points, and the order of the zero is 1 for $\sin z$, we see that they are still simple poles. By the same logic as above,

$$\operatorname{Res}_{z=n\pi}(z-n\pi)\cot z = \lim_{z\to 0} z\cot(z-n\pi) = \lim_{z\to 0} z\cot z = [\lim_{z\to 0}\cos z] \lim_{z\to 0} \frac{z}{\sin z} = 1.$$

e) The poles are still at $z = n\pi$ for integers n, but since we have $\sin^2 z$ the order of each pole is 2. Let us compute the residue at $z = 0$. Observe that

$$\int_{|z|=1} \frac{dz}{\sin^2 z} = 0$$

since $1/\sin^2 z$ takes the same values on opposite sides of the circle. But, by the residue theorem

$$\frac{1}{2\pi i}\int_{|z|=1}\frac{dz}{\sin^2 z} = \sum_j \mathrm{Res}_{z=a_j}\left(\frac{1}{\sin^2 z}\right)$$

where a_j are all the isolated singularities in $|z| < 1$. There is only one, namely $z = 0$. Thus, we find $\mathrm{Res}_{z=0}\, 1/\sin^2 z = 0$. The other poles are equal by periodicity (e.g., the same technique used in part d)).

f) There are two poles at $z = 0$ and $z = 1$ with orders m and n, respectively. Hence,

$$\begin{aligned}
\mathrm{Res}_{z=0}\frac{1}{z^m(1-z)^n} &= \frac{1}{(m-1)!}\lim_{z\to 0}\frac{d^{m-1}}{d^{m-1}}\frac{z^m}{z^m(1-z)^n} \\
&= \frac{n(n+1)(n+2)...(n+m-2)}{(m-1)!} = \binom{n+m-2}{m-1} \\
\mathrm{Res}_{z=1}\frac{1}{z^m(1-z)^n} &= \frac{1}{(n-1)!}\lim_{z\to 1}\frac{d^{n-1}}{d^{n-1}}\frac{(z-1)^n}{z^m(1-z)^n} \\
&= \frac{(-1)^{n-1}m(m+1)...(m+n-2)}{(-1)^n(n-1)!} = -\binom{m+n-2}{n-1}
\end{aligned}$$

Interestingly, the two residues are equal up to a sign.

4.5.3.2. Show that in Sec. 5.3, Example 3, the integral may be extended over a right-angled isosceles triangle. (Suggested by a student.)

Solution: Details for the semicircle method. Let $\rho > 0$, let γ be the semicircle of radius ρ whose diameter lies on the real axis and arc lies in the upper half plane, and let γ' be the corresponding arc. Then,

$$\int_{-\rho}^{\rho} R(x)e^{ix}\ dx = \int_\gamma R(z)e^{iz}\ dz - \int_{\gamma'} R(z)e^{iz}\ dz.$$

We wish to show the final integral tends to zero in absolute value as $\rho \to \infty$. Note that $e^{iz} = e^{i(x+iy)} = e^{-y+ix}$, so that $|e^{iz}| = e^{-y}$. Since $0 \le y \le \rho$, we have that $|e^{iz}| \le 1$. Suppose our rational function is given by

$$R(z) = c\frac{(z-a_1)...(z-a_n)}{(z-b_1)...(z-b_m)}$$

with $m \ge n+2$. For any $1 \le i \le n$, along γ' we have that

$$|z - a_i| \le |z| + |a_i| = \rho + |a_i|.$$

For any $1 \le j \le m$, along γ' we have

$$|z - b_j| \ge ||z| - |b_j|| = |\rho - |b_j|| = \rho - |b_j|$$

when ρ is large enough (e.g. so that all the a_i and b_j are in $|z| < \rho$). Consequently, along γ' we have

$$|R(z)| \le |c|\frac{(\rho+|a_1|)...(\rho+|a_n|)}{(\rho-|b_1|)...(\rho-|b_m|)}.$$

As soon as $m \ge n+1$ we have that $|R(z)| \to 0$ as $\rho \to \infty$. Hence, the integral over γ' tends to zero. We need to have $m \ge n+2$ because the $|dz|$ (coming from the absolute value) contributes a factor of $\pi\rho$.

When using a right-hand isosceles triangle, we situate the hypotenuse so it lies on the real axis, and so that its two legs lie in the upper half plane. Suppose the hypotenuse is the interval $[-\rho, \rho]$; denote the two legs by γ'. Then,

$$\int_0^{\rho} R(x)e^{ix}\ dx = \int_\gamma R(z)e^{iz}\ dz - \int_{\gamma'} R(z)e^{iz}\ dz.$$

We now need to estimate the last integral. To estimate $|R(z)e^{iz}|$ along γ' we use the maximum principle. Indeed, the two legs can be bounded by a semicircular arc. It follows that the modulus

of $R(z)e^{iz}$ along γ' is bounded by that along the arc. But, as above, we showed that this tends to zero. I'm not sure if this is what Ahlfors intended for this problem – he might've wanted a direct computation along the two legs, but I think that would be tedious.

4.5.3.3. Evaluate the following integrals by the method of residues:

a) $\displaystyle\int_0^{\pi/2} \frac{dx}{a+\sin^2 x}$, $|a|>1$, b) $\displaystyle\int_0^{\infty} \frac{x^2\,dx}{x^4+5x^2+6}$, c) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,dx$,

d) $\displaystyle\int_0^{\infty} \frac{x^2\,dx}{(x^2+a^2)^3}$, a real, e) $\displaystyle\int_0^{\infty} \frac{\cos x}{x^2+a^2}\,dx$, a real, f) $\displaystyle\int_0^{\infty} \frac{x\sin x}{x^2+a^2}\,dx$, a real,

g) $\displaystyle\int_0^{\infty} \frac{x^{1/3}}{1+x^2}\,dx$, h) $\displaystyle\int_0^{\infty} \frac{\log x}{1+x^2}\,dx$, i) $\displaystyle\int_0^{\infty} \frac{\log(1+x^2)}{x^{1+\alpha}}\,dx$, $0<\alpha<2$.

Solution:

a) First, recall the trig identity

$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$

so the integral becomes

$$\int_0^{\pi/2} \frac{dx}{a+\sin^2 x} = \int_0^{\pi/2} \frac{dx}{a+(1-\cos(2x))/2} = \int_0^{\pi} \frac{dx}{2a+1-\cos x}$$

after a simple substitution. The above integral is similar to Example 1 in the text. Letting $b=-(2a+1)$, we see that

$$2<|2a|\le|2a+1|+1=|b|+1$$

so that $|b|>1$. Now, suppose $b>1$. This is exactly the integral in the text, so

$$\int_0^{\pi} \frac{dx}{b+\cos x} = \frac{\pi}{\sqrt{b^2-1}} = \frac{\pi}{\sqrt{(2a+1)^2-1}} = \frac{\pi}{2\sqrt{a}\sqrt{a+1}}.$$

If $b<-1$, we can use essentially the same method in the text. When computing the residue, we choose instead however $\beta=-b-\sqrt{b^2-1}$. The residue then is $-1/(2\sqrt{b^2-1}$, and the integral becomes

$$\int_0^{\pi} \frac{dx}{b+\cos x} = -\frac{\pi}{\sqrt{b^2-1}} = -\frac{\pi}{2\sqrt{|a|}\sqrt{|a+1|}}.$$

In total,

$$\int_0^{\pi} \frac{dx}{b+\cos x} = \frac{-\operatorname{sgn}(a)\pi}{2\sqrt{|a|}\sqrt{|a+1|}}.$$

The integral is thus

$$\int_0^{\pi/2} \frac{dx}{a+\sin^2 x} = -\int_0^{\pi} \frac{dx}{b+\cos x} = \frac{\operatorname{sgn}(a)\pi}{2\sqrt{|a|}\sqrt{|a+1|}}.$$

b) We are integrating a rational function over $-\infty$ to ∞, and the numerator and denominators have the appropriate degrees. Therefore we can apply the same method as in Example 2 of the text. Factoring the denominator gives

$$z^4+10z^2+9=(z^2+9)(z^2+1)=(z+3i)(z-3i)(z+i)(z-i).$$

Note that all of these are poles since substituting $z=\pm 3i,\pm i$ into z^2-z+2 is clearly nonzero. The necessary residues are at $z=3i,i$ since these have positive imaginary part. Then,

$$\operatorname{Res}_{z=3i} \frac{z^2-z+2}{(z+3i)(z-3i)(z+i)(z-i)} = \frac{-9-3i+2}{6i(-9+1)} = \frac{7+3i}{48i}$$

$$\operatorname{Res}_{z=i} \frac{z^2-z+2}{(z+3i)(z-3i)(z+i)(z-i)} = \frac{-1-i+2}{(-1+9)(2i)} = \frac{1-i}{16i} = \frac{3-3i}{48i}.$$

Appealing to the residue theorem gives

$$\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\, dx = 2\pi i \left(\mathrm{Res}_{z=3i} \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} + \mathrm{Res}_{z=i} \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} \right) = \frac{5\pi}{12}.$$

c) As with the previous rational integrands, we can apply the same theory. First note that if $a = 0$, the integral is infinite. Indeed, in this case the pole lies on the real axis. So, assume $a \neq 0$. The denominator factors over $\mathbb{C}$ as

$$(z^2 + a^2)^3 = (z - ai)^3 (z + ai)^3$$

so that $z = \pm ai$ are poles of order 3. WLOG assume that $a > 0$, hence the only pole in the upper half plane is $z = ai$. The residue at this point is

$$\begin{aligned} \mathrm{Res}_{z=ai} \frac{z^2}{(z - ai)^3 (z + ai)^3} &= \frac{1}{2!} \lim_{z \to ai} \frac{d^2}{dz^2} \left(\frac{z^2 (z - ai)^3}{(z - ai)^3 (z + ai)^3} \right) = \frac{1}{2} \lim_{z \to ai} \frac{d^2}{dz^2} \left(\frac{z^2}{(z + ai)^3} \right) \\ &= \frac{1}{2} \lim_{z \to ai} \left(\frac{2}{(z + ai)^3} - \frac{12z}{(z + ai)^4} + \frac{12z^2}{(z + ai)^5} \right) = \frac{1}{16a^3 i}. \end{aligned}$$

Then,

$$\int_0^{\infty} \frac{x^2\, dx}{(x^2 + a^2)^3} = \frac{1}{2} \int_0^{\infty} \frac{x^2\, dx}{(x^2 + a^2)^3} = \pi i\, \mathrm{Res}_{z=ai} \left(\frac{z^2}{(z^2 + a^2)^3} \right) = \frac{\pi}{16a^3}.$$

d) Observe that

$$\int_0^{\infty} \frac{\cos x}{x^2 + a^2}\, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2}\, dx = \frac{1}{2} \mathrm{Re} \left(\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + a^2}\, dx \right).$$

Let $R(z) = 1/(z^2 + a^2)$, which has a zero of order 2 at ∞. So, we can apply the method used in Example 3 to conclude

$$\frac{1}{2} \int_{-\infty}^{\infty} R(x) e^{ix}\, dx = \pi i \sum_{y>0} \mathrm{Res}\, R(z) e^{iz}.$$

The poles of $R(z)$ occur at $z = \pm ai$, so we only compute the residue at $z = ai$. Doing this yields

$$\mathrm{Res}_{z=ai} \frac{e^{iz}}{z^2 + a^2} = \lim_{z \to ai} \frac{e^{iz}}{(z + ai)} = \frac{e^{-a}}{2ai}.$$

Hence,

$$\int_0^{\infty} \frac{\cos x}{x^2 + a^2}\, dx = \frac{\pi e^{-a}}{2a}.$$

Note that the integral diverges for $a = 0$.

e) Similar to the previous example, we have

$$\int_0^{\infty} \frac{x \sin x}{x^2 + a^2}\, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2}\, dx = \frac{1}{2} \mathrm{Im} \left(\int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2 + a^2}\, dx \right).$$

Letting $R(z) = z/(z^2 + a^2)$, we see there is only a simple zero at infinity. A priori, we can only compute the principal value via

$$\mathrm{pv} \int_{-\infty}^{\infty} R(x) e^{ix}\, dx = 2\pi i \sum_{y>0} \mathrm{Res}\, R(z) e^{iz} + \pi i \sum_{y=0} \mathrm{Res}\, R(z) e^{iz}.$$

The poles of $R(z)$ are at $z = \pm ai$, and the residue at $z = ai$ is

$$\mathrm{Res}_{z=ai} \frac{z e^{iz}}{z^2 + a^2} = \lim_{z \to ai} \frac{z e^{iz}}{(z + ai)} = \frac{e^{-a}}{2}.$$

Hence,

$$\mathrm{pv} \int_{-\infty}^{\infty} R(x) e^{ix}\, dx = \pi e^{-a} i.$$

It follows that

$$\mathrm{pv}\int_0^\infty \frac{x\sin x}{x^2+a^2}\,dx = \frac{1}{2}\,\mathrm{Im}\left(\pi e^{-a}i\right) = \frac{\pi e^{-a}}{2}.$$

f) Our integrand is of the form $x^\alpha R(x)$ with $0<\alpha<1$ and $R(x)=1/(1+x^2)$ a rational function. Therefore we follow the procedure in the text, without making the initial substitution. Considering $z^{1/3}$, we exclude the negative imaginary axis and form a branch cut – the argument should lie between $-\pi/2(1/3)=-\pi/6$ and $3\pi/2(1/3)=\pi/2$. We now show that the integral along semicircular arcs vanishes. Consider $z(t)=Re^{it}$, with $0\le t\le\pi$. Then,

$$\left|\int_{\gamma_R}\frac{z^{1/3}}{1+z^2}\,dz\right| \le \int_{\gamma_R}\frac{|z|^{1/3}}{||z|^2-1|}\,|dz| = \frac{\pi R^{4/3}}{|R^2-1|}$$

where γ_R is the image of $z(t)$. The right hand side tends to zero as R goes to infinity or 0. Using the contour described in the text, we obtain

$$2\pi i\sum_{y>0}\mathrm{Res}\left(\frac{z^{1/3}}{1+z^2}\right) = \int_{-\infty}^\infty \frac{x^{1/3}}{1+x^2}\,dx = \int_0^\infty \frac{x^{1/3}}{1+x^2}\,dx + \int_0^\infty \frac{(-x)^{1/3}}{1+x^2}\,dx.$$

We note that $(-1)^{1/3}=(e^{i\pi})^{1/3}=e^{i\pi/3}$, so that

$$\int_0^\infty \frac{x^{1/3}}{1+x^2}\,dx = \frac{2\pi i}{1+e^{\pi i/3}}\sum_{y>0}\mathrm{Res}\left(\frac{z^{1/3}}{1+z^2}\right).$$

The poles of $z^{1/3}/(1+z^2)$ are at $z=\pm i$, so we need only compute the residue at $z=i$. It is

$$\mathrm{Res}_{z=i}\left(\frac{z^{1/3}}{1+z^2}\right) = \frac{i^{1/3}}{2i} = \frac{e^{i\pi/6}}{2i}.$$

Then, the integral is

$$\int_0^\infty \frac{x^{1/3}}{1+x^2}\,dx = \frac{2\pi i}{1+e^{\pi i/3}}\left(\frac{e^{i\pi/6}}{2i}\right) = \frac{\pi e^{i\pi/6}}{1+e^{i\pi/3}} = \frac{\pi}{e^{-i\pi/6}+e^{i\pi/6}} = \frac{\pi}{\sqrt{3}}$$

g) Let γ be the contour given by the rightward oriented intervals $[-R,-r]$ and $[r,R]$, as well as the counterclockwise oriented upper semicircle of radius R and clockwise oriented upper semicircle of radius $-r$. This produces a semicircle-like contour avoiding the origin. With our typical branch cut of $\log z$, this contour meets the cut. Hence we choose a different one, namely along the negative imaginary axis. That is, we choose $\arg z$ so that $-\pi/2<\arg z<3\pi/2$.

Note that $f(z)=\log z/(z^2+1)$ has poles at $z=\pm i$, but only one of these is bounded by the contour for R large enough. Hence,

$$\begin{aligned}2\pi i\,\mathrm{Res}_{z=i}\left(\frac{\log z}{z^2+1}\right) &= \int_\gamma \frac{\log z}{z^2+1}\,dz\\ &= \int_{-R}^{-r}\frac{\log x}{1+x^2}\,dx + \int_r^R \frac{\log x}{1+x^2}\,dx + \int_{\gamma_r}\frac{\log z}{1+z^2}\,dz + \int_{\gamma_R}\frac{\log z}{1+z^2}\,dz\end{aligned}$$

where γ_R and γ_r are the semicircular arcs of radius R and r respectively. We wish to show that the second two vanish and the first can be absorbed into the second. For the first integral,

$$\int_{-R}^{-r}\frac{\log x}{1+x^2}\,dx = \int_r^R \frac{\log(-x)}{1+x^2}\,dx = \int_r^R \frac{\log x}{1+x^2}\,dx + \int_r^R \frac{\pi i}{1+x^2}\,dx.$$

In the limit, the second integral becomes $\pi^2 i/2$.

Computing the second to last integral, we parametrize it by $z(t)=re^{it}$ with $0\le t\le\pi$, of course with the opposite orientation from usual. Hence,

$$\int_{\gamma_r}\frac{\log z}{1+z^2}\,dz = -ir\int_0^\pi \frac{(\log r+it)e^{it}}{1+r^2e^{2it}}\,dt.$$

Bounding this gives

$$\left|\int_{\gamma_r} \frac{\log z}{1+z^2}\,dz\right| \le \frac{r|\log r|}{1-r^2}\int_0^\pi dt + \frac{r}{1-r^2}\int_0^\pi t\,dt = \frac{r|\log r|\pi}{1-r^2} + \frac{r\pi^2}{2(1-r^2)}.$$

Letting $r \to 0$ we see the integral vanishes. Observe that bounding the integral over γ_R provides the same estimate with r replaced by R. Then, letting $R \to \infty$, we also see the integral vanishes.

Finally, we compute the residue of $\log z/(z^2+1)$ at $z = i$. This is easily found as

$$\mathrm{Res}_{z=i}\frac{\log z}{z^2+i} = \lim_{z\to i}\frac{\log|z| + i\theta}{z+i} = \frac{i\pi/2}{2i} = \frac{\pi}{4}.$$

$$\int_\gamma \frac{\log z}{z^2+1}\,dz = \frac{\pi^2 i}{2}$$

and on the other hand,

$$\int_\gamma \frac{\log z}{z^2+1}\,dz = 2\int_r^R \frac{\log x}{1+x^2}\,dx + \frac{\pi^2 i}{2} + \int_{\gamma_r}\frac{\log z}{1+z^2}\,dz + \int_{\gamma_R}\frac{\log z}{1+z^2}\,dz.$$

In the limit, the last two integrals vanish. It follows that

$$\int_0^\infty \frac{\log x}{1+x^2}\,dx = 0.$$

h) We begin by applying integration by parts:

$$\begin{aligned}\int_0^\infty \frac{\log(1+x^2)}{x^{1+\alpha}}\,dx &= -\frac{\log(1+x^2)}{\alpha x^\alpha}\bigg|_0^\infty + \frac{1}{\alpha}\int_0^\infty \frac{d/dx\log(1+x^2)}{x^\alpha}\,dx \\ &= \frac{2}{\alpha}\int_0^\infty \frac{x}{x^\alpha(1+x^2)}\,dx.\end{aligned}$$

There are now three cases. First, if $\alpha = 1$ then the integral easily evaluates to π. Now assume $0 < \alpha < 1$. We wish to find

$$\int_0^\infty \frac{x^\beta}{1+x^2}\,dx$$

where $0 < \beta = 1 - \alpha < 1$. We apply the same method as in part g) and find

$$\int_0^\infty \frac{x^\beta}{1+x^2}\,dx = \frac{2\pi i}{1+e^{\beta\pi i}}\sum_{y>0}\mathrm{Res}\left(\frac{z^\beta}{1+z^2}\right).$$

There is only one necessary residue to compute, at $z = i$. It is found as

$$\mathrm{Res}_{z=i}\left(\frac{z^\beta}{1+z^2}\right) = \frac{e^{i\beta\pi/2}}{2i}.$$

Hence, the integral is

$$\int_0^\infty \frac{x^\beta}{1+x^2}\,dx = \frac{\pi e^{i\beta\pi/2}}{1+e^{\beta\pi i}} = \frac{\pi}{2\cos(\beta\pi/2)} = \frac{\pi}{2\sin(\alpha\pi/2)}.$$

Now we assume $2 > \alpha > 1$, so we wish to find

$$\int_0^\infty \frac{1}{x^\beta(1+x^2)}\,dx$$

where $0 < \beta = \alpha - 1 < 1$. Note the additional pole at $z = 0$. Let γ_R be a semicircle in the upper half plane of radius R. Then,

$$\left|\int_{\gamma_R}\frac{1}{z^\beta(1+z^2)}\,dz\right| \le \frac{\pi R}{R^{\alpha-1}|R^2-1|} = \frac{\pi R^{2-\alpha}}{|R^2-1|}.$$

Since $1 < \alpha < 2$ we have $0 < 2 - \alpha < 1$, and thus the integral tends to zero as $R \to 0$ or $R \to \infty$. Hence we can use the same keyhole contour method as before. The only residue we capture is at $z = i$, and is computed as

$$\mathrm{Res}_{z=i}\left(\frac{1}{z^\beta(1+z^2)}\right) = \frac{e^{-i\beta\pi/2}}{2i}.$$

Hence, the integral is Hence, the integral is

$$\int_0^\infty \frac{x^\beta}{1+x^2}\,dx = \frac{\pi e^{-i\beta\pi/2}}{1+e^{-\beta\pi i}} = \frac{\pi}{2\cos(\beta\pi/2)} = \frac{\pi}{2\sin(\alpha\pi/2)}.$$

In total,

$$\int_0^\infty \frac{\log(1+x^2)}{x^{1+\alpha}}\,dx = \frac{\pi}{\alpha\sin(\alpha\pi/2)}.$$

This is consistent when $\alpha = 1$.

4.5.3.4. Compute

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2}$$

where $|a| \neq \rho$. *Hint*: Use $z\overline{z} = \rho^2$ to convert the integral to a line integral of a rational function.

Solution: As seen in Exercise 4.2.2.3., we have that $|dz| = -i\rho dz/z$. Thus,

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = \int_{|z|=\rho} \frac{-i\rho\,dz}{z(z-a)(\overline{z}-\overline{a})} = i\rho\int_{|z|=\rho} \frac{dz}{(z-a)(\overline{a}z-\rho^2)}.$$

There are now two cases. First, if $|a| < \rho$, then $\rho^2/|\overline{a}| > \rho$, and thus there is only one pole in $|z| < \rho$. The residue at this pole is

$$\mathrm{Res}_{z=a}\frac{1}{(z-a)(\overline{a}z-\rho^2)} = \frac{1}{\overline{a}a-\rho^2} = \frac{1}{|a|^2-\rho^2}.$$

Hence, by the residue theorem

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = i\rho\int_{|z|=\rho} \frac{dz}{(z-a)(\overline{a}z-\rho^2)} = \frac{2\pi\rho}{\rho^2-|a|^2}.$$

On the other hand, if $|a| > \rho$ then $\rho^2/|\overline{a}| < \rho$. Thus we still have one pole in $|z| < \rho$, but at a different location. The residue here is

$$\mathrm{Res}_{z=\rho^2/\overline{a}}\frac{1}{(z-a)(\overline{a}z-\rho^2)} = \lim_{z\to\rho^2/\overline{a}}\frac{(z-\rho^2/\overline{a})}{\overline{a}(z-a)(z-\rho^2/\overline{a})} = \frac{1}{\rho^2-|a|^2}.$$

So, by the residue theorem

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = i\rho\int_{|z|=\rho} \frac{dz}{(z-a)(\overline{a}z-\rho^2)} = \frac{2\pi\rho}{|a|^2-\rho^2}.$$

Note that in both cases, the integral is positive, as expected.

4.6. **Harmonic Functions.**

4.6.2. *The Maximum Principle.*

4.6.2.1. If u is harmonic and bounded in $0 < |z| < \rho$, show that the origin is a removable singularity in the sense that u becomes harmonic in $|z| < \rho$ when $u(0)$ is properly defined.

Solution: Recall that the arithmetic mean of u can be written as

$$\frac{1}{2\pi}\int_{|z|=r} u\ d\theta = \alpha \log r + \beta$$

for $0 < r < \rho$ and constants α, β. Since u is bounded, we have that

$$|\alpha \log r + \beta| \leq Mr$$

where $|u| \leq M$ on $0 < |z| < \rho$. Taking $r \to 0^+$, we see that $\alpha = 0$. Now, recall that

$$0 = \alpha = \frac{1}{2\pi}\int_{|z|=r} r\frac{\partial u}{\partial r}\ d\theta = \frac{1}{2\pi}\int_{|z|=r} \star du$$

where $\star du$ is the conjugate differential. Since $|z| = r$ is a homology basis for the punctured disc, it follows that $\int_\gamma \star du = 0$ for any closed curve γ in the punctured disc. It follows that the conjugate harmonic function v is single valued.

Define now $f(z) = u + iv$, which is analytic in the punctured disc. Since $\operatorname{Re} f(z)$ is bounded, by Exercise 4.3.2.5 it follows that any isolated singularity of $f(z)$ is removable. In this way, u has a well defined harmonic extension.

4.6.2.2. Suppose that $f(z)$ is analytic in the annulus $r_1 < |z| < r_2$ and continuous on the closed annulus. If $M(r)$ denotes the maximum of $|f(z)|$ for $|z| = r$, show that

$$M(r) \leq M(r_1)^\alpha M(r_2)^{1-\alpha}$$

where $\alpha = \log(r_2/r)/\log(r_2/r_1)$ (Hadamard's three-circle theorem). Discuss cases of equality. *Hint*: Apply the maximum principle to a linear combination of $\log|f(z)|$ and $\log|z|$.

Solution: By taking logarithms of both sides, we are equivalently asked to prove

$$\log M(r) \leq \alpha \log M(r_1) + (1-\alpha)\log M(r_2).$$

Let $a > 0$ and consider

$$|\log|f(z)| + a\log|z|| \leq \log|f(z)| + a\log|z| = \log|z^a f(z)|.$$

We can find the maximum of this by looking at the maximum of $z^a f(z)$ instead. On the annulus $\Omega = \{r_1 \leq |z| \leq r_2\}$,

$$|z^a f(z)| \leq \max\{r_1^a M(r_1), r_2^a M(r_2)\}.$$

Evidently, for some a we achieve equality so that

$$r_1^a M(r_1) = r_2^a M(r_2).$$

Taking logarithms,

$$a\log(r_1) + \log(M(r_1)) = a\log(r_2) + \log(M(r_2)),$$

and solving for a yields

$$a = \frac{\log(M(r_2)/M(r_1))}{\log(r_1/r_2)}.$$

Now, we have

$$|z^a f(z)| \leq r^a M(r) \leq r_1^a M(r_1)$$

owing to the maximum modulus principle and equality in the max. Taking logarithms,

$$a\log r + \log M(r) \leq a\log r_1 + \log M(r_1).$$

Note that

$$a\log r - a\log r_1 = a\log(r/r_1) = \frac{\log(r/r_1)}{\log(r_2/r_1)}\log M(r_1) - \frac{\log(r/r_1)}{\log(r_2/r_1)}\log M(r_2).$$

Hence,

$$\log M(r) \leq \left(1 - \frac{\log(r/r_1)}{\log(r_2/r_1)}\right) \log M(r_1) + \frac{\log(r/r_1)}{\log(r_2/r_1)} \log M(r_2)$$

Now, if $\alpha = \log(r_2/r)/\log(r_2/r_1)$, we see that

$$\alpha = \frac{\log(r_2/r)}{\log(r_2/r_1)} = -\frac{\log(r/r_1)}{\log(r_2/r_1)}$$
$$1 - \alpha = 1 - \frac{\log(r_2/r)}{\log(r_2/r_1)} = \frac{\log(r/r_1)}{\log(r_2/r_1)}$$

so

$$\log M(r) \leq \alpha \log M(r_1) + (1-\alpha) \log M(r_2)$$

as desired.

5. Series and Product Developments

5.1. **Power Series Expansions.**

5.1.1. *Weierstrass' Theorem.*

5.1.1.1. Using Taylor's theorem applied to a branch of $\log(1 + z/n)$, prove that

$$\lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n = e^z$$

uniformly on all compact sets.

Solution: We equivalently show that

$$\lim_{n\to\infty} n \log\left(1 + \frac{z}{n}\right) = z$$

uniformly on all compact sets. The finite Taylor expansion of $n\log(1 + z/n)$ is

$$n \log\left(1 + \frac{z}{n}\right) = z - \frac{z^2}{2n} + \frac{z^3}{3n^2} + ... + R\left(\frac{z}{n}\right) z^m$$

where $R(z/n)$ is analytic. By the Cauchy integral formula

$$R\left(\frac{z}{n}\right) = \frac{1}{2\pi i} \int_{C_r} \frac{R(\zeta/n)}{\zeta - z}\, d\zeta$$

for all $|z| < r$, where C_r is a circle of radius r centered at the origin. By a change of variables,

$$R\left(\frac{z}{n}\right) = \frac{1}{2\pi i} \int_{C_{r/n}} \frac{R(\zeta)}{n(n\zeta - z)}\, d\zeta$$

Since $R(\zeta)$ is analytic, and $C_{r/n}$ is compact, $|R(\zeta)| \leq M$ on $C_{r/n}$. Thus,

$$\left|R\left(\frac{z}{n}\right)\right| \leq \frac{M}{2n\pi} \int_{C_{r/n}} \frac{1}{|n^2 r - |z||}\, |d\zeta| = \frac{MR}{n^2 |r - |z||}.$$

Choosing, say, $|z| < r/2$ we find

$$\left|R\left(\frac{z}{n}\right)\right| \leq \frac{M}{2n\pi} \int_{C_{r/n}} \frac{1}{|n^2 r - |z||}\, |d\zeta| = \frac{2M}{n^2} \to 0$$

where the convergence is uniform since we bound $R(z/n)$ independently of z.

Now let Ω be compact, so that $|z| < r$ for all $z \in \Omega$ and some $r > 0$. It follows that

$$\left|n \log\left(1 + \frac{z}{n}\right)\right| \leq r + \frac{r^2}{2n} + \frac{r^3}{3n^2} + ... + \left|R\left(\frac{z}{n}\right)\right| r^m.$$

Taking $n \to \infty$, we see that $n\log(1 + z/n)$ grows like $|z|$.

5.1.1.2. Show that the series

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$$

converges for $\operatorname{Re} z > 1$, and represent its derivative in series form.

Solution: First observe that

$$\left|n^{-z}\right| = \left|e^{-\operatorname{Re} z \log n}\right|\left|e^{-i \operatorname{Im} z \log n}\right| = \frac{1}{n^{\operatorname{Re} z}}.$$

Let $\epsilon = \operatorname{Re} z > 1$. Then,

$$|\zeta(z)| \leq \sum_{n=1}^{\infty} |n^{-z}| \leq \sum_{n=1}^{\infty} \frac{1}{n^{\epsilon}} < \infty.$$

So, we know that $\zeta(z)$ converges on $\operatorname{Re} z > 1$. Moreover, if K is any compact subset of $\operatorname{Re} z > 1$, then the series converges. Because each term $f_n(z) = n^{-z}$ is analytic, it follows by Weierstrass' theorem that $\zeta(z)$ is analytic on $\operatorname{Re} z > 1$. Not only this, but we can differentiate the series term by term. Hence,

$$\zeta'(z) = \sum_{n=1}^{\infty} \frac{d}{dz} n^{-z} = \sum_{n=1}^{\infty} \frac{d}{dz} e^{-z \log n} = \sum_{n=1}^{\infty} -\frac{\log n}{n^z}.$$

5.1.1.3. Prove that

$$(1 - 2^{1-z})\zeta(z) = 1^{-z} - 2^{-z} + 3^{-z} - \ldots$$

and that the latter series represents an analytic function for $\operatorname{Re} z > 0$.

Solution: There is an interesting trick to this which I'm not sure will be useful anywhere else. Recalling the alternating series test, we cannot apply it here since our terms do not monotonically decrease (indeed, there is compatible total ordering of the complex numbers with both multiplication and addition). Instead, we appeal to a generalized alternating series test which states if a_n and b_n are complex sequences, then $\sum_{n=1}^{\infty} a_n b_n$ converges when:

(1) There exists an M, independent of N, such that

$$\left|\sum_{n=1}^{N} b_n\right| \leq M.$$

(2) The terms a_n tend to zero as $n \to \infty$.

(3) The sequence of a_n is of bounded variation, that $\sum_{n=1}^{\infty} |a_{n+1} - a_n| \leq L < \infty$.

With this in hand, let $a_n = n^{-z}$ and $b_n = (-1)^{n+1}$. Clearly the series $\sum_{n=1}^{\infty} a_n b_n$ represents the series on the right hand side. The two conditions are clearly satisfied, since the partial sums of b_n alternate between zero and one, and since $|n^{-z}| = n^{-\operatorname{Re} z}$, with $\operatorname{Re} z > 0$. The third criteria is the most interesting. Indeed, we have

$$\sum_{n=1}^{\infty} \left|\frac{1}{(n+1)^z} - \frac{1}{n^z}\right| \leq \int_1^{\infty} \left|\frac{d}{dt}\left(\frac{1}{t^z}\right)\right| dt = \int_1^{\infty} \left|\frac{-z}{t^{z+1}}\right| dt = \int_1^{\infty} \frac{|z|}{t^{1+\operatorname{Re} z}}\, dt = \frac{|z|}{\operatorname{Re} z}$$

where we recognize the first series as an approximation of the arc length of the curve $1/t^z$. Since $\operatorname{Re} z > 0$, this is finite. Hence, the series in question converges for $\operatorname{Re} z > 0$, and also on compact subsets. Since each term is analytic, it represents an analytic function.

We are thus left to prove equality, which only makes sense on the domain $\operatorname{Re} z > 1$. In this case,

$$(1 - 2^{1-z}) \sum_{n=1}^{\infty} n^{-z} = \left[\sum_{n \text{ odd}} n^{-z} + \sum_{n \text{ even}} n^{-z}\right] - 2\sum_{n=1}^{\infty} (2n)^{-z} = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-z}$$

as desired.

5.1.1.4. As a generalization of Theorem 2, prove that if the $f_n(z)$ have at most m zeros in Ω, then $f(z)$ is either identically zero or has at most m zeros.

Solution: Suppose f is not identically zero. Let $\{z_1, z_2, ...\}$ be the distinct zeros of f. Fix an $i \in \mathbb{N}$. We form a circle C of radius ρ around z_i such that $f(z) \neq 0$ on $0 < |z - z_i| < \rho$. By the maximum modulus principle, $|f(z)| > 0$ on C. Let $\delta > 0$ be such that $|f(z)| > \delta$ on C. Since $f_n(z) \to f(z)$ uniformly on compact sets, there exists an N such that if $n \geq N$ then $|f_n(z) - f(z)| < \delta/2$ on C. Hence,

$$\delta < |f(z)| \leq |f_n(z) - f(z)| + |f_n(z)| < \delta/2 + |f_n(z)|$$

and $|f_n(z)| > \delta/2$ on C. Since $g(z) = 1/z$ is uniformly continuous on $|z| > \delta/2$, $|f_n(z)| > \delta$ and $|f(z)| > \delta/2$, and the f_n converge uniformly to f on C, we have that $1/f_n(z)$ converges uniformly to $1/f(z)$ on C. For full details, see the end of this solution. By Weierstrass' theorem, we additionally have $f_n' \to f'$ uniformly on C. Finally, because $1/f$ and f' are bounded on C, the product f_n'/f_n also uniformly converges. Consequently,

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\lim_{n\to\infty}\int_C \frac{f_n'(z)}{f_n(z)}\,dz$$

Let $\{z_1^n, ..., z_{m_n}^n\}$ be the distinct zeros of f_n, where $m_n \leq m$ for all n.

We now show that if g is a uniformly continuous function on E and $f_n \to f$ uniformly on Ω, then $g \circ f_n \to g \circ f$ uniformly on Ω. Let $\epsilon > 0$, then there exists a $\delta > 0$ such that $|g(z) - g(w)| < \epsilon$ whenever $|z - w| < \delta$. With this δ, by uniform convergence of the f_n there exists an N such that if $n \geq N$ then $|f_n(z) - f_n(w)| < \delta$. Combining the two, we see that if $n \geq N$ then $|g(f_n(z)) - g(f_n(w))| < \epsilon$ for all $z, w \in \Omega$, asserting uniform convergence. We now show that $g(z) = 1/z$ is uniformly continuous on subsets of $|z| > m > 0$. Clearly we have

$$\left|\frac{1}{z} - \frac{1}{w}\right| = \frac{|z - w|}{|z||w|} < \frac{|z - w|}{m^2}$$

whenever $|z| > m$ and $|w| > m$. So, if $\delta = \epsilon/m^2$, we are done.

5.1.1.5. Prove that

$$\sum_{n=1}^{\infty} \frac{nz^n}{1 - z^n} = \sum_{n=1}^{\infty} \frac{z^n}{(1 - z^n)^2}$$

for $|z| < 1$. (Develop in a double series and reverse the order of summation.)

Solution: We recognize in the first sum a term that looks like $1/(1 - z)$, which can be written as a geometric series. Since $|z| < 1$, we have $|z^n| < 1$ for any fixed n. Exploiting this allows us to rewrite the series on the left as a doubly indexed sum:

$$\sum_{n=1}^{\infty} \frac{nz^n}{1 - z^n} = \sum_{n=1}^{\infty} nz^n \sum_{k=0}^{\infty} z^{nk} = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} nz^{n(k+1)} = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} nz^{nk}.$$

It is not immediately obvious to me that we can interchange the order of summation – indeed, the series is not absolutely convergent. However, proceeding forth we get

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} nz^{nk} = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} nz^{nk} = \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} nz^{nk}.$$

Now, notice that

$$\frac{z}{k}\frac{d}{dz}\sum_{n=0} z^{kn} = z\sum_{n=0}^{\infty} nz^{kn-1} = \sum_{n=0}^{\infty} nz^{kn}$$

for any $k \geq 1$. We can also evaluate the left hand side via a geometric series,

$$\frac{z}{k}\frac{d}{dz}\sum_{n=0} z^{kn} = \frac{z}{k}\frac{d}{dz}\left(\frac{1}{1-z^k}\right) = \frac{z}{k}\left(\frac{kz^{k-1}}{(1-z^k)^2}\right) = \frac{z^k}{(1-z^k)^2}.$$

In total,

$$\sum_{n=1}^{\infty}\frac{nz^n}{1-z^n} = \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} nz^{nk} = \frac{z}{k}\frac{d}{dz}\sum_{n=0} z^{kn} = \frac{z}{k}\frac{d}{dz}\left(\frac{1}{1-z^k}\right) = \frac{z^k}{(1-z^k)^2}.$$

5.1.2. *The Taylor Series.*
5.1.2.1. Develop $1/(1+z^2)$ in powers of $z-a$, a being a real number. Find the general coefficient and for $a=1$ reduce to simplest form.

Solution: Recall by the geometric series that

$$\frac{1}{b+(z-a)} = \frac{1}{b}\left(\frac{1}{1+(z-a)/b}\right) = \frac{1}{b}\sum_{n=0}^{\infty}\left(-\frac{z-a}{b}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n(z-a)^n}{b^{n+1}}$$

which converges when $|z-a| < b$. Now, notice that

$$\frac{1}{1+z^2} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right) = \frac{1}{2i}\left(\frac{1}{(a-i)+(z-a)} - \frac{1}{(a+i)+(z-a)}\right).$$

So, we apply the above twice with $b = a-i$ then $b = a+i$. This gives

$$\begin{aligned}
\frac{1}{1+z^2} &= \frac{1}{2i}\left(\sum_{n=0}^{\infty}\frac{(-1)^n(z-a)^n}{(a-i)^{n+1}} - \sum_{n=0}^{\infty}\frac{(-1)^n(z-a)^n}{(a+i)^{n+1}}\right) \\
&= \frac{1}{2i}\left(\sum_{n=0}^{\infty}\left(\frac{(a+i)^{n+1}-(a-i)^{n+1}}{(a-i)^{n+1}(a+i)^{n+1}}\right)(-1)^n(z-a)^n\right) \\
&= \frac{1}{2i}\left(\sum_{n=0}^{\infty}\left(\frac{(a+i)^{n+1}-(a-i)^{n+1}}{(a^2+1)^{n+1}}\right)(-1)^n(z-a)^n\right)
\end{aligned}$$

By the binomial theorem we have

$$\begin{aligned}
(a+i)^{n+1}-(a-i)^{n+1} &= \sum_{k=0}^{n+1}\binom{n+1}{k} i^k a^{n+1-k} - \sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k i^k a^{n+1-k} \\
&= 2\sum_{1\leq k\leq n+1,\ \text{odd}}\binom{n+1}{k} i^k a^{n+1-k} \\
&= 2\sum_{k=0}^{N}\binom{n+1}{2k+1} i^{2k+1}a^{n-2k} = 2i\sum_{k=0}^{N}\binom{n+1}{2k+1}(-1)^k a^{n-2k}
\end{aligned}$$

where N is the largest integer such that $2N+1 \leq n+1$. In the case $a=1$, this simplifies dramatically to $2i[2^{(n+1)/2}\sin(\pi(n+1)/4)]$. One way of showing this is to probably write out the difference in terms of exponentials instead. I found this by trial and error. Thus, for $a=1$ we have

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty}\frac{(-1)^n\sin(\pi(n+1)/4)}{2^{(n+1)/2}}(z-1)^n$$

5.1.2.2. The Legendre polynomials are defined as the coefficients $P_n(\alpha)$ in the development

$$(1-2\alpha z+z^2)^{-1/2} = 1 + P_1(\alpha)z + P_2(\alpha)z^2 + \ldots$$

Find P_1, P_2, P_3, and P_4.

Solution: The simplest way to do this is likely by differentiation. Letting $f(z) = (1-2\alpha z+z^2)^{-1/2}$, we have

$$\begin{aligned}
f'(z) &= \frac{d}{dz}\left(\frac{1}{(1-2\alpha z+z^2)^{1/2}}\right) = \frac{\alpha - z}{(1-2\alpha z+z^2)^{3/2}} \\
f''(z) &= \frac{-(1-2\alpha z+z^2)^{3/2} - 3(\alpha-z)(1-2\alpha z+z^2)^{1/2}(z-\alpha)}{(1-2\alpha z+z^2)^3} \\
&= -\frac{1}{(1-2\alpha z+z^2)^{3/2}} + \frac{3(\alpha-z)^2}{(1-2\alpha z+z^2)^{5/2}} \\
f'''(z) &= \frac{3(z-\alpha)}{(1-2\alpha z+z^2)^{5/2}} + \frac{6(z-\alpha)(1-2\alpha z+z^2)^{5/2} - 15(z-\alpha)^3(1-2\alpha z+z^2)^{3/2}}{(1-2\alpha z+z^2)^5} \\
&= \frac{9(z-\alpha)}{(1-2\alpha z+z^2)^{5/2}} - \frac{15(z-\alpha)^3}{(1-2\alpha z+z^2)^{7/2}} \\
f''''(z) &= \frac{9(1-2\alpha z+z^2)^{5/2} - 45(z-a)^2(1-2\alpha z+z^2)^{3/2}}{(1-2\alpha z+z^2)^5} \\
&\quad - \left[\frac{45(z-\alpha)^2(1-2\alpha z+z^2)^{7/2} - 105(z-\alpha)^4(1-2\alpha z+z^2)^{5/2}}{(1-2\alpha z+z^2)^7}\right] \\
&= \frac{9}{(1-2\alpha z+z^2)^{5/2}} - \frac{90(z-\alpha)^2}{(1-2\alpha z+z^2)^{7/2}} + \frac{105(z-\alpha)^4}{(1-2\alpha z+z^2)^{9/2}}
\end{aligned}$$

Consequently,

$$f'(0) = \alpha, \qquad f''(0) = 3\alpha^2 - 1, \qquad f'''(0) = 15\alpha^3 - 9\alpha, \qquad f''''(0) = 105\alpha^4 - 90\alpha^2 + 9.$$

The associated Legendre polynomials are thus

$$P_1(\alpha) = \alpha, \qquad P_2(\alpha) = \frac{3\alpha^2-1}{2}, \qquad P_3(\alpha) = \frac{5\alpha^3-3\alpha}{2}, \qquad P_4(\alpha) = \frac{35\alpha^4 - 30\alpha^2 + 3}{8}.$$

5.1.2.3. Develop $\log(\sin z/z)$ in powers of z up to the term z^6.

Solution: The Taylor expansion of $\sin z$ is

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + ...$$

Next, we write $\log(\sin z/z)$ as $\log(1+[\sin z/z-1])$, where the Taylor expansion for $f(z) = \sin z/z - 1$ is easily derived from the above as

$$\sin z/z - 1 = -\frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + ...$$

Set $P(z) = -z^2/3! + z^4/5! - z^6/7!$. Now, the Taylor expansion of $\log(1+z)$ (choosing the branch which is zero at the origin) is

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + ...$$

Since we want up to the z^6 term, we can neglect anything beyond it. In particular, since the expansion of $f(z)$ starts with a z^2 term, we can ignore everything beyond the z^3 term in $\log(1+z)$. Thus we let $Q(z) = z - z^2/2 + z^3/3$. Furthermore, we included the z^6 term in $P(z)$ since $\log(1+z)$ starts with a linear term. Now, expanding $Q(P(z))$ gives

$$\begin{aligned}
Q(P(z)) &= \left(-\frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!}\right) - \frac{1}{2}\left(-\frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!}\right)^2 + \frac{1}{3}\left(-\frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!}\right)^3 \\
&= \left(-\frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!}\right) - \frac{1}{2}\left(\frac{z^4}{3!^2} - \frac{2z^6}{3!5!} + [z^7]\right) + \frac{1}{3}\left(-\frac{z^6}{3!^3} + [z^7]\right) \\
&= -\frac{z^2}{6} - \frac{z^4}{180} - \frac{z^6}{2835} + [z^7]
\end{aligned}$$

It follows that the first three terms form the Taylor expansion of $\log(\sin z/z)$ in powers of z up to the z^6 term.

5.1.2.4. What is the coefficient of z^7 in the Taylor development of $\tan z$?

Solution: We continue the development presented in the text. Namely, since

$$\arctan z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + [z^9],$$

we can find the Taylor expansion of $z = \tan w$ by repeated substitution of

$$z = w + \frac{z^3}{3} - \frac{z^5}{5} + \frac{z^7}{7} - [z^9]$$

into itself. Consider the relevant integer partitions of 7:

$$7 = 7, 1+1+5, 1+2+4, 2+2+3, 1+3+3, 1+1+1+1+3, 1+1+1+2+2$$

Only four of these are relevant: $7, 1+1+5, 1+3+3, 1+1+1+1+3$, since these contain only odd terms; so too does our expansion for z. The w^7 terms corresponding to these are

$$\frac{w^7}{7}, \quad -\frac{w^7}{15}, \quad \frac{w^7}{27}, \quad -\frac{w^7}{15}$$

taking into account permutations of these sums. XXX

5.1.2.5. The Fibonacci numbers are defined by $c_0 = 0$, $c_1 = 1$,

$$c_n = c_{n-1} + c_{n-2}.$$

Show that the c_n are Taylor coefficients of a rational function, and determine a closed expression for c_n.

Solution: Consider the function $P(z) = \sum_{n=0}^{\infty} c_n z^n$. We then have

$$\begin{aligned} P(z) &= \sum_{n=0}^{\infty} c_n z^n = c_0 z + c_1 z + \sum_{n=2}^{\infty} c_n z^n \\ &= z + \sum_{n=2}^{\infty} (c_{n-1} + c_{n-2}) z^n = z + z \sum_{n=0}^{\infty} c_n z^n + z^2 \sum_{n=0} c_n z^n \\ &= z + zP(z) + z^2 P(z) \end{aligned}$$

Rearranging this gives

$$P(z) = \frac{z}{1 - z - z^2}.$$

Next, note the sequence obeys the matrix product

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_{n-1} \\ c_{n-2} \end{pmatrix} = \begin{pmatrix} c_n \\ c_{n-1} \end{pmatrix}$$

Repeated application of this then yields

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} c_n \\ c_{n-1} \end{pmatrix}$$

for $n \geq 1$. We now diagonalize the 2×2 matrix. The eigenvalues are roots of

$$\det \begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = (1-\lambda)(-\lambda) - 1 = \lambda^2 - \lambda - 1 = 0,$$

i.e. $\lambda = 1/2(1 \pm \sqrt{5})$. Note that if $\varphi = 1/2(1+\sqrt{5})$ then $1/2(1-\sqrt{5}) = -\varphi^{-1}$. Manual verification shows the eigenvectors respectively are

$$e_1 = \begin{pmatrix} \varphi \\ 1 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} -\varphi^{-1} \\ 1 \end{pmatrix}$$

and since

$$\begin{pmatrix} c_1 \\ c_0 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}} \begin{pmatrix} -\varphi^{-1} \\ 1 \end{pmatrix}$$

it follows that

$$\begin{aligned} \begin{pmatrix} c_n \\ c_{n-1} \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} c_1 \\ c_0 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} \varphi \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} -\varphi^{-1} \\ 1 \end{pmatrix} \\ &= \frac{\varphi^{n-1}}{\sqrt{5}} \begin{pmatrix} \varphi \\ 1 \end{pmatrix} - \frac{(-\varphi^{-1})^{n-1}}{\sqrt{5}} \begin{pmatrix} -\varphi^{-1} \\ 1 \end{pmatrix} \end{aligned}$$

owing to the fact the two vectors are eigenvectors. Hence,

$$c_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$$

5.2. **Partial Fractions and Factorization.**

5.2.1. *Partial Fractions.*
5.2.1.1.
5.2.1.2. Express

$$\sum_{n=-\infty}^{\infty} \frac{1}{z^3 - n^3}$$

in closed form.

Solution: We can factor $z^3 - n^3$ as

$$(z^3 - n^3) = (z - n)(z - ne^{2\pi i/3})(z - ne^{4\pi i/3}).$$

Let $\alpha = e^{2\pi i/3}$. Notice that

$$(z - n\alpha)(z - n/\alpha) + (z - n)(z - n/\alpha) + (z - n)(z - n\alpha) = 3z^2 - 2n(1/\alpha + 1 + \alpha)z + n^2(1/\alpha + 1 + \alpha)$$

but since α is a root of unity, we have $1 + \alpha + \alpha^2 = 0$. Thus,

$$(z - n\alpha)(z - n/\alpha) + (z - n)(z - n/\alpha) + (z - n)(z - n\alpha) = 3z^2.$$

We therefore obtain the partial fraction decomposition

$$\frac{3z^2}{z^3 - n^3} = \frac{1}{z - n} + \frac{1}{z - n\alpha} + \frac{1}{z - n/\alpha} = \frac{1}{z - n} + \frac{1/\alpha}{z/\alpha - n} + \frac{\alpha}{\alpha z - n}.$$

Recalling that

$$\pi \cot(\pi z) = \lim_{m \to \infty} \sum_{n=-m}^{m} \frac{1}{z - n},$$

by summing the above identity from $n = -m$ to m and taking limits, we obtain

$$\begin{aligned} \lim_{m \to \infty} \sum_{n=-m}^{m} \frac{3z^2}{z^3 - n^3} &= \lim_{m \to \infty} \sum_{n=-m}^{m} \frac{1}{z - n} + \frac{1}{\alpha} \lim_{m \to \infty} \sum_{n=-m}^{m} \frac{1}{z/\alpha - n} + \alpha \lim_{m \to \infty} \sum_{n=-m}^{m} \frac{1}{\alpha z - n} \\ &= \pi \cot(\pi z) + \frac{\pi}{\alpha} \cot(\pi z/\alpha) + \pi\alpha \cot(\alpha\pi z) \end{aligned}$$

Consequently,

$$\sum_{n=-\infty}^{\infty} \frac{1}{z^3 - n^3} = \frac{\pi \cot(\pi z) + \pi/\alpha \cot(\pi z/\alpha) + \pi\alpha \cot(\alpha\pi z)}{3z^2}.$$

Note: We technically must, in each of the three sums prior, add a constant term like $1/n$, $1/(\alpha n)$, and α/n respectively to ensure the sums converge absolutely. Then, we can rearrange them into the valid cotangent terms.

5.2.1.3. Use (13) to find the partial fraction development of $1/\cos(\pi z)$, and show that it leads to $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + ...$

Solution: Recall (13) states

$$\frac{\pi}{\sin(\pi z)} = \lim_{m\to\infty} \sum_{n=-m}^{m} \frac{(-1)^n}{z-n}.$$

Since $\cos(\pi z) = \sin(\pi(z+1/2))$, we obtain

$$\frac{\pi}{\cos(\pi z)} = \lim_{m\to\infty} \sum_{n=-m}^{m} \frac{(-1)^n}{z+1/2-n} = 2\lim_{m\to\infty} \sum_{n=-m}^{m} \frac{(-1)^n}{2z+1-2n}.$$

Thus, the partial fraction decomposition of $1/\cos(\pi z)$ is

$$\frac{1}{\cos(\pi z)} = \lim_{m\to\infty} \sum_{n=-m}^{m} \frac{(-1)^n}{z+1/2-n} = \frac{2}{\pi}\lim_{m\to\infty} \sum_{n=-m}^{m} \frac{(-1)^n}{2z+1-2n}.$$

Letting $z = 0$, we obtain

$$\begin{aligned}
\frac{\pi}{2} &= \lim_{m\to\infty}\left(\sum_{n=-m}^{m} \frac{(-1)^n}{1-2n}\right) = \lim_{m\to\infty}\left(\sum_{n=1}^{m} \frac{(-1)^n}{1-2n} + \sum_{n=-m}^{0} \frac{(-1)^n}{1-2n}\right) \\
&= \lim_{m\to\infty}\left(\sum_{n=0}^{m-1} \frac{-(-1)^n}{1-2(n+1)} + \sum_{n=0}^{m} \frac{(-1)^n}{2n+1}\right) \\
&= \lim_{m\to\infty}\left(\sum_{n=0}^{m-1} \frac{(-1)^n}{2n+1} + \sum_{n=0}^{m} \frac{(-1)^n}{2n+1}\right) = \lim_{m\to\infty}\left(2\sum_{n=0}^{m-1} \frac{(-1)^n}{2n+1} + \frac{(-1)^m}{2m+1}\right)
\end{aligned}$$

Since $|(-1)^m/(2(2m+1))| \to 0$ as $m \to \infty$, we finally get

$$\frac{\pi}{4} = \lim_{m\to\infty}\left(\sum_{n=0}^{m-1} \frac{(-1)^n}{2n+1} + \frac{(-1)^m}{2(2m+1)}\right) = \lim_{m\to\infty} \sum_{n=0}^{m-1} \frac{(-1)^n}{2n+1}$$

as desired.

5.2.1.4. What is the value of

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2+a^2}?$$

Solution: Note the factorization

$$(z+n)^2 + a^2 = (z+n+ia)(z+n-ia)$$

which leads to the partial fraction decomposition

$$\frac{1}{(z+n)^2+a^2} = \frac{1}{2ia}\left(\frac{1}{z-ia+n} - \frac{1}{z+ia+n}\right).$$

Summing from $n = -m$ to m, applying an index change, and taking limits gives

$$\begin{aligned}
\lim_{m\to\infty} \sum_{n=-m}^{m} \frac{1}{(z+n)^2+a^2} &= \frac{1}{2ia}\lim_{m\to\infty} \sum_{n=-m}^{m} \frac{1}{(z-ia)-n} - \frac{1}{2ia}\lim_{m\to\infty} \sum_{n=-m}^{m} \frac{1}{(z+ia)-n} \\
&= \frac{\pi}{2ia}\left(\cot(\pi(z-ia)) - \cot(\pi(z+ia))\right).
\end{aligned}$$

5.2.2. *Infinite Products.*

5.2.2.1. Show that

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)=\frac{1}{2}.$$

Solution: Let $a_n=-1/n^2$ for $n\geq 1$. We therefore have an infinite product of the following form

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)=\prod_{n=2}^{\infty}\left(1+a_n\right).$$

Hence, the series converges simultaneously with the series

$$\sum_{n=2}^{\infty}\log(1+a_n)=\sum_{n=2}^{\infty}\log\left(1-\frac{1}{n^2}\right)=\sum_{n=2}^{\infty}\log\left(\frac{(n-1)(n+1)}{n^2}\right).$$

At this point, we evaluate the partial sums directly:

$$\begin{aligned}\sum_{n=2}^{N}\log\left(\frac{(n-1)(n+1)}{n^2}\right)&=\sum_{n=2}^{N}\log(n-1)+\sum_{n=2}^{N}\log(n+1)-2\sum_{n=2}^{N}\log(n)\\&=\sum_{n=1}^{N-1}\log(n)+\sum_{n=3}^{N+1}\log(n)-2\sum_{n=2}^{N}\log(n)\\&=\log(1)-\log(2)+\log(N+1)+2\sum_{n=3}^{N-1}\log(n)-2\sum_{n=3}^{N}\log(n)\\&=\log(N+1)-\log(N)-\log(2)=\log\left(1-\frac{1}{N}\right)-\log(2)\end{aligned}$$

Evidently, this tends to $-\log(2)$ as $N\to\infty$. We therefore establish the infinite product converges, and moreover that

$$\log\left(\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)\right)=\sum_{n=2}^{\infty}\log\left(1-\frac{1}{n^2}\right)=-\log(2)$$

as desired.

5.2.2.2. Prove that for $|z|<1$

$$(1+z)(1+z^2)(1+z^4)(1+z^8)...=\frac{1}{1-z}.$$

Solution: Consider the infinite product

$$P=\prod_{n=0}^{\infty}(1+z^{2^n}).$$

Let $N>0$ and consider the partial product P_N. We claim that

$$P_N=\prod_{n=0}^{N}(1+z^{2^n})=\sum_{n=0}^{2^{N+1}-1}z^n.$$

Clearly this holds when $N=1$. Suppose it holds for N. Then by definition of P_{N+1} and the inductive hypothesis,

$$\begin{aligned}P_{N+1}=\left(1+z^{2^{N+1}}\right)P_N=\left(1+z^{2^{N+1}}\right)\sum_{n=0}^{2^{N+1}-1}z^n&=\sum_{n=0}^{2^{N+1}-1}z^n+\sum_{n=0}^{2^{N+1}-1}z^{n+2^{N+1}}\\=\sum_{n=0}^{2^{N+1}-1}z^n+\sum_{n=2^{N+1}}^{2^{N+1}+2^{N+1}-1}z^n&=\sum_{n=0}^{2^{N+1}-1}z^n+\sum_{n=2^{N+1}}^{2^{N+2}-1}z^n=\sum_{n=0}^{2^{N+2}-1}z^n\end{aligned}$$

as desired. Hence the partial products form a geometric series, and it follows easily from this that $P=1/(1-z)$ when $|z|<1$.

5.2.2.3. Prove that

$$\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}$$

converges absolutely and uniformly on every compact set.

Solution: Let a_n be defined by

$$a_n = e^{-z/n} + \frac{ze^{-z/n}}{n} - 1$$

so that the above product can be written as

$$\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n} = \prod_{n=1}^{\infty}(1+a_n).$$

It follows that this converges absolutely so long as $\sum_{n=1}^{\infty}|a_n|$ does. Let $r = \operatorname{Re} z$; if $r = 0$ then we easily have convergence, since the product is just of complex numbers on the unit circle. As $n \to \infty$, the polar angle tends to zero, and thus for large n there is little rotation. If $r \neq 0$ we have

$$\left|e^{-z/n}\right| = e^{-r/n} \to 1$$

as $n \to \infty$. It follows that $e^{-z/n} - 1 \to 0$ as $n \to \infty$. Furthermore,

$$\left|\frac{ze^{-z/n}}{n}\right| \leq \frac{|z|e^{-r/n}}{n} \leq \frac{C}{n} \to 0$$

where C is some constant making the inequality valid for large n. XXX

5.2.2.4. Prove that the value of an absolutely convergent product does not change if the factors are reordered.

Solution: Recall a product $\prod_{n=1}^{\infty}(1+a_n)$ converges absolutely if and only if the corresponding series $\sum_{n=1}^{\infty}|a_n|$ converges (absolutely). If $\sigma(n)$ is a permutation of $\mathbb{N}$, we therefore have by the Riemann Rearrangement theorem that $\sum_{n=1}^{\infty}|a_{\sigma(n)}|$ converges. It follows that $\prod_{n=1}^{\infty}(1+a_{\sigma(n)})$ converges. Denote the original product's value by L and this new product's value by L'. We wish to show $L = L'$. XXX

5.2.3. *Canonical Products.*

5.2.3.1. Suppose that $a_n \to \infty$ and that the A_n are arbitrary complex numbers. Show that there exists an entire function $f(z)$ which satisfies $f(a_n) = A_n$. *Hint*: Let $g(z)$ be a function with simple zeros at the a_n. Show that

$$\sum_{n=1}^{\infty} g(z)\frac{e^{\gamma_n(z-a_n)}}{z-a_n}\frac{A_n}{g'(a_n)}.$$

converges for some choice of the numbers γ_n.

Solution: First let us see how the hint is useful. Indeed, if $g(z)$ has simple zeros at the a_n, then each $g(z)/(z-a_n)$ is an entire function. Consequently, each term in the series is an entire function, and by Weierstrass' theorem the series itself is entire. Next, by L'hopitals rule we have

$$\lim_{z\to a_n}\frac{g(z)}{z-a_n} = \lim_{z\to a_n}\frac{g'(z)}{1} = g'(a_n)$$

so if $f(z)$ denotes the series, then $f(a_n) = A_n$. Thus we need only show convergence of the above series for a choice of γ_n. To achieve this, we show convergence on every disc of radius R. Let $h_n(z)$ denote the function

$$h_n(z) = \frac{g(z)A_n}{g'(z)(z-a_n)}$$

so that the series is

$$\sum_{n=1}^{\infty} e^{\gamma_n(z-a_n)}h_n(z).$$

If we choose γ_n appropriately so the series is bounded by a geometric series, then we show uniform convergence. Let D_R denote the disc of radius R; we want the γ_n to be such that

$$\left|e^{\gamma_n(z-a_n)}\right| \leq \frac{2^{-n}}{\sup_{z\in D_R}|h_n(z)|} = \frac{2^{-n}}{\sup_{z\in C_R}|h_n(z)|}$$

where C_R is the circle $|z| = R$. What this tells is is the real part of $\gamma_n(z-a_n)$ must be negative, since $|e^z| = e^{\operatorname{Re} z}$, which must be small. Since the $a_n \to \infty$, we can conceivably hope for this. Let N be such that for all $n \geq N$, $|a_n| \geq 2R$. Set $\theta_n = \arg(a_n)$ and let $\gamma_n = \eta_n e^{-\theta_n}$ for $\eta_n > 0$. Write $z \in C_R$ as $z = Re^{i\varphi}$. We find then that

$$\gamma_n(z-a_n) = R\eta e^{i(\varphi-\theta_n)} - |a_n|\eta_n.$$

Notice first that $R\eta e^{i(\varphi-\theta_n)}$ is a point on $|w| = R\eta$, and as φ varies we traverse this circle. By subtracting $|a_n|\eta_n$, we translate this circle to the left. Moreover, $|a_n|\eta_n \geq 2R\eta_n$ so that

$$\operatorname{Re}(\gamma_n(z-a_n)) \leq R\eta_n - |a_n|\eta_n \leq -R\eta_n.$$

Hence, if η_n is large enough so that

$$e^{-R\eta_n} \leq \frac{2^{-n}}{\sup_{z\in C_R}|h_n(z)|}$$

then we are done. Choosing η_n for $n \geq N$ obeying this, we split the series into a finite sum and an infinite series dominated by a convergent geometric series.

5.2.3.2. Prove that

$$\sin(\pi(z+\alpha)) = e^{\pi z\cot(\pi\alpha)} \prod_{n=-\infty}^{\infty}\left(1+\frac{z}{n+\alpha}\right)e^{-z/(n+\alpha)}$$

whenever α is not an integer. *Hint*: Denote the factor in front of the canonical product by $g(z)$ and determine $g'(z)/g(z)$.

Solution: The zeros of $\sin(\pi(z+\alpha))$ are given by $z = n-\alpha$ for $n \in \mathbb{Z}$. Since α is not an integer, the $\sin(\pi(z+\alpha))$ is nonzero at $z = 0$. Letting $a_n = 1/(n-\alpha)$, we see that

$$\sum_{n=-\infty}^{\infty}\frac{1}{|a_n|} \approx \sum_{n=-\infty}^{\infty}\frac{1}{n} = \infty$$

whereas

$$\sum_{n=-\infty}^{\infty}\frac{1}{|a_n|^2} \approx \sum_{n=-\infty}^{\infty}\frac{1}{n^2} < \infty.$$

Hence, we write

$$\sin(\pi(z+\alpha)) = e^{h(z)} \prod_{n=-\infty}^{\infty}\left(1-\frac{z}{n-\alpha}\right)e^{z/(n-\alpha)}.$$

By an index change $n \mapsto -n$ we get

$$\sin(\pi(z+\alpha)) = e^{h(z)} \prod_{n=-\infty}^{\infty}\left(1+\frac{z}{n+\alpha}\right)e^{-z/(n+\alpha)},$$

which is almost what we want. We show now that $h(z) = e^{\pi z\cot(\pi\alpha)}$. For fixed n, we have that

$$\begin{aligned}\frac{d}{dz}\left(e^{-z/(n+\alpha)}\left(1+\frac{z}{n+\alpha}\right)\right) &= e^{-z/(n+\alpha)}\left(\frac{1}{n+\alpha}\right) - \frac{e^{-z/(n+\alpha)}}{n+\alpha}\left(1+\frac{z}{n+\alpha}\right)\\ &= \frac{e^{-z/(n+\alpha)}}{n+\alpha}\left(1-\left(1+\frac{z}{n+\alpha}\right)\right) = -\frac{ze^{-z/(n+\alpha)}}{(n+\alpha)^2}\end{aligned}$$

So, taking the derivative on both sides in the Weierstrass expansion of $\sin(\pi(z+\alpha))$, we get

$$\pi\cos(\pi(z+\alpha)) = h'(z)\sin(\pi(z+\alpha)) - e^{h(z)}\sum_{k=-\infty}^{\infty}\frac{ze^{-z/(k+\alpha)}}{(k+\alpha)^2}\prod_{n\neq k}\left(1+\frac{z}{n+\alpha}\right)e^{-z/(n+\alpha)}.$$

The logarithmic derivative is

$$\begin{aligned}
\pi\cot(\pi(z+\alpha)) &= \frac{\pi\cos(\pi(z+\alpha))}{\sin(\pi(z+\alpha))} \\
&= \frac{h'(z)\sin(\pi(z+\alpha))}{\sin(\pi(z+\alpha))} - e^{-h(z)}e^{h(z)}\sum_{k=-\infty}^{\infty}\frac{ze^{-z/(k+\alpha)}}{(k+\alpha)^2}\frac{\prod_{n\neq k}\left(1+\frac{z}{n+\alpha}\right)e^{-z/(n+\alpha)}}{\prod_{n=-\infty}^{\infty}\left(1+\frac{z}{n+\alpha}\right)e^{-z/(n+\alpha)}} \\
&= h'(z) - \sum_{k=-\infty}^{\infty}\frac{ze^{-z/(k+\alpha)}}{(k+\alpha)^2}\frac{1}{\left(1+\frac{z}{k+\alpha}\right)e^{-z/(k+\alpha)}} \\
&= h'(z) - \sum_{k=-\infty}^{\infty}\frac{z}{(k+\alpha)}\frac{1}{(k+\alpha)+z} = h'(z) + \sum_{k=-\infty}^{\infty}\left(\frac{1}{z+(k+\alpha)} - \frac{1}{k+\alpha}\right)
\end{aligned}$$

Recall the series expansion for $\pi\cot(\pi z)$ is

$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{k\neq 0}\left(\frac{1}{z-k}+\frac{1}{k}\right) = \frac{1}{z} + \sum_{k\neq 0}\left(\frac{1}{z+k}-\frac{1}{k}\right).$$

Hence, the series for $\pi\cot(\pi(z+\alpha))$ is

$$\pi\cot(\pi(z+\alpha)) = \frac{1}{z+\alpha} + \sum_{k\neq 0}\left(\frac{1}{z+(\alpha+k)}-\frac{1}{k}\right).$$

We therefore see that

$$\begin{aligned}
\frac{1}{z+\alpha} + \sum_{k\neq 0}\left(\frac{1}{z+(\alpha+k)}-\frac{1}{k}\right) &= h'(z) + \sum_{k=-\infty}^{\infty}\left(\frac{1}{z+(k+\alpha)}-\frac{1}{k+\alpha}\right) \\
&= h'(z) + \frac{1}{z+\alpha} - \frac{1}{\alpha} + \sum_{k\neq 0}\left(\frac{1}{z+(k+\alpha)}-\frac{1}{k+\alpha}\right)
\end{aligned}$$

and, consequently

$$h'(z) = \frac{1}{\alpha} + \sum_{k\neq 0}\left(\frac{1}{k+\alpha}-\frac{1}{k}\right).$$

Recalling our formula for $\pi\cot(\pi z)$, we find that $h'(z) = \pi\cot(\pi\alpha)$. Thus, $h(z) = \pi\cot(\pi\alpha) + c$ for some complex number c, and

$$\sin(\pi(z+\alpha)) = Ae^{\pi z\cot(\pi\alpha)}\prod_{n=-\infty}^{\infty}\left(1+\frac{z}{n+\alpha}\right)e^{-z/(n+\alpha)}$$

with $A = e^c$. This is actually a mistake in the book – we must further determine this constant A. When $z = 0$, we easily find $A = \sin(\pi\alpha)$. Thus,

$$\sin(\pi(z+\alpha)) = \sin(\pi\alpha)e^{\pi z\cot(\pi\alpha)}\prod_{n=-\infty}^{\infty}\left(1+\frac{z}{n+\alpha}\right)e^{-z/(n+\alpha)}.$$

5.2.3.3. What is the genus of $\cos\sqrt{z}$?

Solution: We first find the zeros of $\cos\sqrt{z}$, and to do so it suffices to find the zeros of $\cos z$. Recall that

$$|\cos z|^2 = \cosh^2(y) - \sin^2(x)$$

where $z = x + iy$ as usual. Note that $0 \leq \sin^2(x) \leq 1$ and $\cosh^2 \geq 1$ so that $|\cos z|^2 = 0$ if and only if $\cosh^2(y) = 1$ and $\sin^2(x) = 1$. The former only happens when $y = 0$, while the latter occurs when $x = \pi/2 + n\pi = \pi/2(2n+1)$ for integers k. So, if $\cos\sqrt{z} = 0$ then $\sqrt{z} = \pi/2(2n+1)$ and hence $z = \pi^2/4(2n+1)^2$. However, these roots are double counted for $n \leq -1$, so we require $n \geq 0$. Then,

$$\sum_{n=0}^{\infty}\frac{4}{\pi^2(2n+1)^2} \leq \sum_{n=1}^{\infty}\frac{4}{\pi^2 n^2} < \infty$$

so the canonical product for $\cos\sqrt{z}$ has genus zero. We can therefore write $\cos\sqrt{z}$ as

$$\cos\sqrt{z} = e^{g(z)}\prod_{n=0}^{\infty}\left(1-\frac{4z}{\pi^2(2n+1)^2}\right).$$

To find the genus, we must find $g(z)$. By a change of variables we get

$$\begin{aligned}\sin z = \cos(z-\pi/2) = \cos\sqrt{(z-\pi/2)^2} &= e^{h(z)}\prod_{n=0}^{\infty}\left(1-\frac{4(z-\pi/2)^2}{\pi^2(2n+1)^2}\right)\\
&= e^{h(z)}\prod_{n=0}^{\infty}\left(1-\frac{2z-\pi}{\pi(2n+1)}\right)\left(1+\frac{2z-\pi}{\pi(2n+1)}\right) = e^{h(z)}\prod_{n=0}^{\infty}\left(\frac{2\pi(n+1)-2z}{\pi(2n+1)}\right)\left(\frac{2n\pi+2z}{\pi(2n+1)}\right)\\
&= e^{h(z)}\prod_{n=0}^{\infty}\left(\frac{2\pi(n+1)-2z}{\pi(2n+1)}\right)\prod_{n=0}^{\infty}\left(\frac{2n\pi+2z}{\pi(2n+1)}\right) = \frac{ze^{h(z)}}{\pi}\prod_{n=0}^{\infty}\left(\frac{2\pi(n+1)-2z}{\pi(2n+1)}\right)\prod_{n=1}^{\infty}\left(\frac{2n\pi+2z}{\pi(2n+1)}\right)\end{aligned}$$

where $h(z) = g((z-\pi/2)^2)$. All the above product manipulations are valid since the sum $\sum_n |z|/|a_n|$ is convergent.

5.2.3.4. If $f(z)$ is of genus h, how large and how small can the genus of $f(z^2)$ be?

Solution: If the zeros of $f(z)$ are denoted $z_k = R_k e^{i\theta_k}$, then the zeros of $f(z)$ are $a_k = \sqrt{R_k}e^{i\theta_k/2}$. So, if $f(z)$ is of genus h then

$$\sum_{k=1}^{\infty}\frac{1}{|z_k|^{\eta+1}} = \sum_{k=1}^{\infty}\frac{1}{R_k^{\eta+1}}$$

converges for $\eta = h$ but diverges for $\eta < h$. The corresponding series for $f(z^2)$ is

$$\sum_{k=1}^{\infty}\frac{1}{|a_k|^{\eta+1}} = \sum_{k=1}^{\infty}\frac{1}{R_k^{\eta/2+1/2}}.$$

We want to find the smallest integer η such that $\eta/2+1/2 \geq h+1$. Evidently, this is $\eta = 2h+1$. Thus the genus of the canonical product is $2h-1$. Now, if $g(z)$ in the canonical product representation of $f(z)$ is a degree h polynomial, then $g(z^2)$ is a degree $2h$ polynomial. It follows that the genus of $f(z^2)$ is either $2h+1$ or $2h$.

5.2.3.5. Show that if $f(z)$ is of genus 0 or 1 with real zeros, and if $f(z)$ is real for real z, then all zeros of $f'(z)$ are real. *Hint*: Consider $\operatorname{Im} f'(z)/f(z)$.

Solution: Suppose first $f(z)$ is of genus zero. Then we can write $f(z)$ as

$$f(z) = Cz^m\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)$$

with $\sum 1/|a_n| < \infty$, $m \geq 0$, and all the a_n are real. The logarithmic derivative is

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{Cmz^{m-1}\prod_{n=1}^{\infty}(1-z/a_n)}{Cz^m\prod_{n=1}^{\infty}(1-z/a_n)} - \frac{Cz^m\sum_{k=1}^{\infty}1/a_k\prod_{n=1,\neq k}^{\infty}(1-z/a_n)}{Cz^m\prod_{n=1}^{\infty}(1-z/a_n)}\\
&= \frac{m}{z} - \sum_{k=1}^{\infty}\frac{1}{a_k}\frac{1}{1-z/a_k} = \frac{m}{z} + \sum_{k=1}^{\infty}\frac{1}{z-a_k} = \frac{m\overline{z}}{|z|^2} + \sum_{k=1}^{\infty}\frac{\overline{z}-a_k}{|z-a_k|^2}\end{aligned}$$

Since the a_k are real, $\operatorname{Im}(\overline{z}-a_k) = \operatorname{Im}(\overline{z}) = -\operatorname{Im}(z)$. Thus,

$$\operatorname{Im}\left(\frac{f'(z)}{f(z)}\right) = -\left(\frac{m}{|z|^2} + \sum_{k=1}^{\infty}\frac{1}{|z-a_k|^2}\right)\operatorname{Im}(z).$$

Notice the term in the parenthesis is strictly positive. Hence, if $\operatorname{Im}(f'(z)/f(z)) = 0$ then $\operatorname{Im}(z) = 0$. Suppose z is a common zero of $f(z)$ and $f'(z)$ – then z is real. On the other hand, if $f'(z) = 0$ but $f(z) \neq 0$ then $\operatorname{Im}(f'(z)) = \operatorname{Im}(f'(z)/f(z)) = 0$. Hence, by the above, $\operatorname{Im}(z) = 0$ and z is real.

Suppose now $f(z)$ is of genus one. Then we can write $f(z)$ as

$$f(z) = Cz^m e^{\alpha\beta z} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\gamma z/a_n}$$

where all the a_n are real, $\alpha, C \in \mathbb{C}$, and $\beta, \gamma \in \{0, 1\}$. The logarithmic derivative is

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{Cmz^{m-1}e^{\alpha\beta z}\prod_{n=1}^{\infty}(1-z/a_n)e^{\gamma z/a_n}}{Cz^m e^{\alpha\beta z}\prod_{n=1}^{\infty}(1-z/a_n)e^{\gamma z/a_n}} + \frac{\alpha\beta C z^m e^{\alpha\beta z}\prod_{n=1}^{\infty}(1-z/a_n)e^{\gamma z/a_n}}{Cz^m e^{\alpha\beta z}\prod_{n=1}^{\infty}(1-z/a_n)e^{\gamma z/a_n}} \\ &\quad + \frac{Cz^m e^{\alpha\beta z}\sum_{k=1}^{\infty}(\gamma - 1 - \gamma z/a_k)e^{\gamma z/a_k}/a_k \prod_{n=1,\neq k}^{\infty}(1-z/a_n)e^{\gamma z/a_n}}{Cz^m e^{\alpha\beta z}\prod_{n=1}^{\infty}(1-z/a_n)e^{\gamma z/a_n}} \\ &= \frac{m}{z} + \alpha\beta + \sum_{k=1}^{\infty}\frac{\gamma - 1 - \gamma z/a_k}{a_k(1-z/a_k)} = \frac{m}{z} + \alpha\beta + \sum_{k=1}^{\infty}\frac{\gamma z/a_k + 1 - \gamma}{z - a_k}\end{aligned}$$

We must now make the distinction between the $\gamma = 0$ and $\gamma = 1$ cases:

$$\frac{f'(z)}{f(z)} = \begin{cases} \frac{m}{z} + \alpha\beta + \sum_{k=1}^{\infty}\frac{1}{z-a_k} & \gamma = 0 \\ \frac{m}{z} + \alpha\beta + \sum_{k=1}^{\infty}\left(\frac{1}{a_k} + \frac{1}{z-a_k}\right) & \gamma = 1 \end{cases}$$

where the convergence of $\sum_k 1/a_k$ is assumed in the $\gamma = 1$ case.

5.5. **Normal Families.**

5.5.5. *The Classical Definition.*
5.5.5.1. Prove that in any region Ω the family of analytic functions with positive real part is normal. Under what added condition is it locally bounded? *Hint*: Consider the functions e^{-f}.

Solution: Let $\mathscr{F}$ be the family defined by

$$\mathscr{F} = \{f : \Omega \to \mathbb{C} \mid \operatorname{Re}(f) > 0,\ f \text{ is analytic}\}.$$

We wish to show $\mathscr{F}$ is normal in the classical sense. By Theorem 17, this amounts to showing $\rho(f) = 2|f'(z)|/(1 + |f(z)|^2)$ is locally bounded. Consider the map