Real Analysis prelim Aug. 2021

5. First note that $\rightsquigarrow$ assume $f \in C_c^1(\mathbb{R})$

$$f(\lambda x) - f(x) = \int_1^\lambda f'(tx) \cancel{dx}\, dt$$

Hence,

$$|f(\lambda x) - f(x)|^q \leq \left[\int_1^\lambda |f'(tx)| dt\right]^q$$

Now, by Jensen's inequality (which is valid since we're on a finite measure space $[1, \lambda]$)

$$|f(\lambda x) - f(x)|^q \leq \int_1^\lambda |f'(tx)|^q dt\, (\lambda - 1)^q$$

we have to normalize.

Finally,

$$\int_{\mathbb{R}} |f(\lambda x) - f(x)|^q \leq (\lambda-1)^q \int_1^\lambda \int_{\mathbb{R}} |f'(tx)|^q dx\, dt$$

Fubini

$$= (\lambda-1)^q \int_1^\lambda \int_{\mathbb{R}} \frac{1}{t} |f'(x)|^q dx\, dt$$

$$= (\lambda-1)^q \|f'\|_q^q \int_1^\lambda \frac{1}{t} dt$$

$$= (\lambda-1)^q \|f'\|_q^q \log(\lambda)$$

Note that $\|f'\|_q < \infty$ since $f \in C_c^1(\mathbb{R})$.

Evidently, as $\lambda \to 1^+$ we get $\int_{\mathbb{R}} |f(\lambda x) - f(x)|^q \to 0$.

Now, given arbitrary $f \in L^q(\mathbb{R})$, let $g \in L^q(\mathbb{R})$ be such that $g \in C_c^1(\mathbb{R})$ and $\|f-g\|_q < \varepsilon$.

Then,

$$\int_{\mathbb{R}} |f(\lambda x) - f(x)|^q \leq \underbrace{\int_{\mathbb{R}} |f(\lambda x) - g(\lambda x)|^q}_{I_1} + \underbrace{\int_{\mathbb{R}} |g(\lambda x) - g(x)|^q}_{I_2} + \underbrace{\int_{\mathbb{R}} |g(x) - f(x)|^q}_{I_3}$$

We have

$$I_3 = \|f-g\|_q^q < \varepsilon^q$$

$$I_1 = \int_{\mathbb{R}} |f(\lambda x) - g(\lambda x)|^q$$

$$= \int_{\mathbb{R}} \frac{1}{\lambda} |f(x) - g(x)|^q = \frac{1}{\lambda} \|f-g\|_q^q < \varepsilon^q/\lambda$$

$$I_2 = \int_{\mathbb{R}} |g(\lambda x) - g(x)|^q < \varepsilon^q$$

for λ ~~small~~ close to 1 (by argument to left)

Hence, for λ close to 1:

$$\int_{\mathbb{R}} |f(\lambda x) - f(x)|^q < \varepsilon^q/\lambda + \varepsilon^q + \varepsilon^q$$

$$= \left(\frac{1}{\lambda} + 2\right) \varepsilon^q.$$

1. Since f is a.c., its derivative f' exists a.e. Now suppose f is Lipschitz w/ $\mathrm{Lip}(f) < \infty$. Then for all $x, y \in [a,b]$

$$|f(x) - f(y)| \leq \mathrm{Lip}(f)|x-y|$$

or, equivalently,

$$\left|\frac{f(x) - f(y)}{x-y}\right| \leq \mathrm{Lip}(f).$$

By letting $y \to x$, we see that

$$|f'(x)| \leq \mathrm{Lip}(f) \text{ a.e.}$$

So that $f' \in L^\infty([a,b])$.

Now assume $f:[a,b]\to\mathbb{R}$ is a.c. and that $f' \in L^\infty[a,b]$.

Then,

$$|f(x)-f(a)| = \left|\int_a^x f'\right| \le \int_a^x |f'| \le \int_a^x \|f'\|_\infty = \|f'\|_\infty (x-a) \le \|f'\|_\infty |x-a|.$$

I think this only works as written for $x \ge a$. Otherwise, just change sign as necessary.

~~But~~ If $x>a$, ~~we immediately see~~ ~~$|f(x)-f(a)|$~~

So that f is Lipschitz with $Lip(f) \le \|f'\|_\infty$

2 i) We say that $f \in L^{p,\omega}(B_1^n(0), dx)$ if ~~for every~~

$$N_p(f) := \sup_{\lambda>0} |\{x\in B_1^n(0) \mid |f(x)|>\lambda\}|\cdot\lambda^p < \infty.$$

ii) Consider the function $f(x) = |x|^{-n/p}$. The level set $f(x)=\lambda$ is precisely $|x|^{-n/p} = \lambda \rightsquigarrow |x| = \lambda^{-p/n}$ or, $\partial B^n_{\lambda^{-p/n}}(0)$. Since f is radially monotone decreasing, $\{x\in B_1^n(0) \mid f(x)>\lambda\} = B^n_{\lambda^{-p/n}}(0)$. Letting $\omega_n = |B_1^n(0)|$ we have $|B^n_{\lambda^{-p/n}}(0)| = \lambda^{-p}|B_1^n(0)| = \lambda^{-p}\omega_n$.

Thus $N_p(f) = \sup_{\lambda>0} [\lambda^{-p}\cdot\omega_n]\cdot\lambda^p = \omega_n$.

Thus, $f \in L^{p,\omega}(B_1^n(0))$.

On the other hand, the integral of such a function classically diverges. ($\Rightarrow$ not in L^p)

To see this,

$$\int_{B_1^n(0)} |f|^p = \int_{B_1^n(0)} |x|^{-n} = \int_0^1\int_{S^{n-1}} r^{-n} r^{n-1} dS^{n-1} dr = C\cdot\int_0^1 \frac{1}{r}dr = \infty.$$

3. I think the easiest way to do this is via a discrete Fourier Transform, but I'm not 100% sure on the details. In particular, let $e_n = \cos(nx)$ so that $\{e_n\}_{n=1}^\infty$ forms an orthonormal basis (perhaps after normalizing it). Regardless, in $L^2[0,\pi]$ we could probably apply the fact it is a Hilbert space to conclude

$$f = \sum_{n=1}^\infty \langle f, e_n\rangle e_n$$

Then by Parseval's identity,

$$\|f\|_2^2 = \sum_{n=1}^\infty |\langle f, e_n\rangle|^2 < \infty$$

This implies the coefficients $\langle f, e_n\rangle \to 0$ so that $\cos(nx)\to 0$.

Here's maybe how you do it in the general setting:

Let $f \in L^q[0,\pi]$ where $\frac{1}{q}+\frac{1}{p}=1$. Then, since we are on a finite measure space, $f \in L^1[0,\pi]$. Now, considering its Fourier transform, we know that by Riemann-Lebesgue this implies $\mathcal{F}[f(x)](t) \to 0$ as $|t| \to \infty$. Then, appealing to Euler's identity we should obtain the result [and a corresponding one for $\sin(nx)$].

To show no strong convergence, if $\|\cos(nx)\|_p \to 0$ for some $p \geq 1$, then by Hölder (ie, since we're on a finite measure space) we'd have $\|\cos(nx)\|_1 \to 0$.

Observe that

$$\int_0^{\pi} |\cos(nx)|\, dx = 2n \int_0^{\pi/2n} \cos(nx)\, dx$$

$$= 2\int_0^{\pi/2} \cos(x)\, dx$$

$$= 2$$

The first equality can be seen graphically by counting the number of maxima and using periodicity.

4. I'm not confident in this proof, but it seems to be the best I have. It suffices to show that Σ_i^+ is a closed subset of Σ_i. To this end, let $\{f_n^+\} \subseteq \Sigma^+$ be a convergent sequence to some $g \in L^p[0,1]$. We wish to show $g = f_g^+$ with $f \in \Sigma_i$.

To this end, consider $\{f_n\} \subset \Sigma_i$ where f_n^+ are the terms in previous sequence. By compactness, passing to a subsequence if necessary, we have $f_n \xrightarrow{L^p} f$. We thus aim to show $f^+ = g$ a.e.

Note that

i) $$\int |f-g|^p \leq \int |f-f_n|^p + \int |f_n - f_n^+|^p + \int |f_n^+ - g|^p$$

where, by assumption, $\int |f-f_n|^p \to 0$ and $\int |f_n^+ - g|^p \to 0$

So, $$\int |f-g|^p \leq \liminf_{n\to\infty} \int |f_n - f_n^+|^p = \int |f^-|^p.$$

ii) $$\int |f_n^-|^p = \int |f_n - f_n^+|^p \leq \int |f_n - f|^p + \int |f-g|^p + \int |f_n^+ - g|$$

So, $$\int |f^-|^p \leq \int |f-g|^p$$

(I feel like one of these inequalities is wrong, but I cannot remember if one can strengthen Fatou by being on a finite measure space)

So In total, $\int |f^-|^p = \int |f-g|^p$.

Actually, maybe this is it:

Since $f_n^+ \xrightarrow{L^p} g$ and $f_n \to f$,

there exists ~~a~~ a subsequence such that

$$f_n^+ \xrightarrow{a.e.} g \quad f_n \xrightarrow{a.e.} f.$$

So,

i) $|f-g| \leq |f-f_n| + |f_n - f_n^+| + |f_n^+ - g|$

$$\to_a \quad |f-f^+| = |f^-| = f^-$$

ii) ~~$f^- = |f| \leq$~~

$$f_n^- = |f_n^+ - f_n| \leq |f_n^+ - g| + |g-f| + |f_n - f|$$

Taking limits,

$$f^- \leq |g-f|.$$

This convergence may not hold a priori...

Combining the two,

$|g-f| = f^-$, which implies $g = f^+$?