Analysis prelim 2021:

Problem 4: Suppose ~~all~~ none have empty interior then X_j^c is dense in X for each j.

By BCT, $\bigcap X_j^c$ is also dense

but $\bigcap X_j^c = (\bigcup X_j)^c = X^c = \emptyset$. $\lightning$

Problem 1: Set $E_{N,k} = \bigcup_{n=N}^{\infty} \{|f_n - f| \geq \frac{1}{k}\}$

Fix a $k \in \mathbb{N}$. Then $E_{N+1,k} \subseteq E_{N,k}$

Since $E_{N+1,k}$ is a union of fewer sets.

Since $\mu(X) < \infty$, $\mu(E_{1,k}) < \infty$ and thus DCT of sets applies.

It follows that

$$\lim_{N\to\infty} \mu(E_{N,k}) = \mu\left(\bigcap_{N=1}^{\infty} E_{N,k}\right)$$

$= 0$

~~Since f is measurable~~

Let $\varepsilon > 0$. Then $\exists N_k$ st

$$\mu(E_{N_k,k}) < \varepsilon 2^{-k}$$

(by above limit).

Hence, if $E = \bigcup_{k=1}^{\infty} E_{N_k,k}$ then

$$\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_{N_k,k}) < \varepsilon.$$

Now if $x \notin E$ then $x \in \bigcap_{k=1}^{\infty} E_{N_k,k}^c$

Fix k, Then $x \in E_{N_k,k}^c = \bigcap_{n=N_k}^{\infty} \{|f_n - f| < \frac{1}{k}\}$

Let $\eta > 0$. Then $\exists k$ st $\frac{1}{k} < \eta$.

Hence $|f_n(x) - f(x)| < \frac{1}{k} < \eta$ for all $n \geq N_k$, est. unif. conv.

Problem 2: Consider $(X, \mathcal{M}, \mu) = (\mathbb{R}, \mathcal{L}, m)$.

If $f_j = \chi_{[j,j+1]}$ then for $\varepsilon = \frac{1}{2}$ there is no set E w/ $m(E) < \frac{1}{2}$ where $f_j \to 0$ uniformly on E^c

(f_j is always 1 on a set of measure 1, ~~hence it is 1 on a set of positive~~

Hence, for any ~~$\varepsilon < \frac{1}{2}$~~ E w/ $m(E) < \frac{1}{2}$, $f_j = 1$ on a positive measure subset of E^c)

Problem 3: Let ν be a signed finite measure and μ a positive, ~~finite~~ measure.

We show that $\nu \ll \mu$ implies for every $\varepsilon > 0$ there exists a $\delta > 0$ st whenever $\mu(E) < \delta$ then $\nu(E) < \varepsilon$.

Suppose this is false. Then $\exists$ an $\varepsilon > 0$ st for all $n \in \mathbb{N}$ $\exists$ a set E_n where $\mu(E_n) < 2^{-n}$ but $\nu(E_n) \geq \varepsilon$.

Let $F_N = \bigcup_{n=N}^{\infty} E_n$, so that $\mu(F_N)$ is st

$$\mu(F_N) < 2^{-N+1} \quad \left(= \frac{1}{1-\frac{1}{2}} - \frac{1-(\frac{1}{2})^N}{1-\frac{1}{2}}\right)$$

Now let $F = \bigcap_{N=1}^{\infty} F_N$ so that $\mu(F) = 0$ by monotonicity.

By DCT (valid since ν is finite),

$$\nu(F) = \lim_{N\to\infty} \nu(F_N) \geq \lim_{N\to\infty} \nu(E_N) \geq \varepsilon$$

(since $F_N \supseteq E_N$)

Hence ν is not ac wrt μ.

~~Now; $\nu \ll \mu$ iff $|\nu| \ll \mu$. Let $|\nu|(E) = \int$~~

Now let

$$\nu(E) = \int_E |f| \, d\mu,$$

which is finite since $f \in L^1$.

Consider

$f_j = j\chi_{[0,1/j]}$ $\quad (X,\mu)=(\mathbb{R},m)$

~~Letn~~

Case 1: $\delta \geq 1$:

$\int_E |f_j| \, d\mu$ is maximized when $E \supseteq [0,1]$.

(since all f_j are certainly 0 off $[0,1]$).

In this case, $\int_E |f_j| = (\frac{1}{j}-0)\cdot j = 1.$

~~So that $\sup_{E \mid \nu(E)}$~~

Case 2: $\delta < 1$:

$\int_E |f_j| \, d\mu$ is maximized for

$E = [0,\delta]$, ~~with~~ in which case

$$\int_E |f_j| \, d\mu = \begin{cases} 1 & \frac{1}{j} < \delta \\ j\delta & \text{else } (\delta < \frac{1}{j}) \\ \leq 1 & \end{cases}$$

Hence $\sup_E \int_E |f_j| \, d\mu = 1$

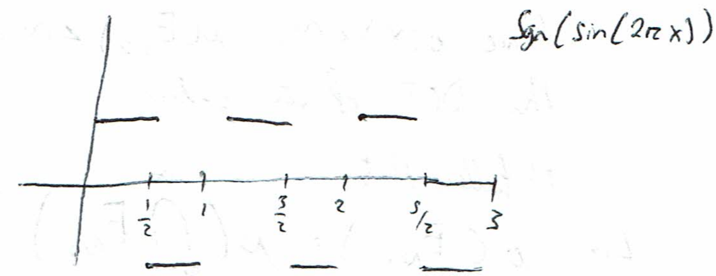
$\Rightarrow \sup_{E \mid \nu(E)<\delta} \sup_j \int_E |f_j| \, d\mu \geq 1$?

It feels like a counterexample (to the prior a.c. condition) should be constructed so that $f_j \to f \notin L^1$, ~~but I am not sure why the given example wouldn't work.~~ (reversed sups!)
Let me try and rewrite it

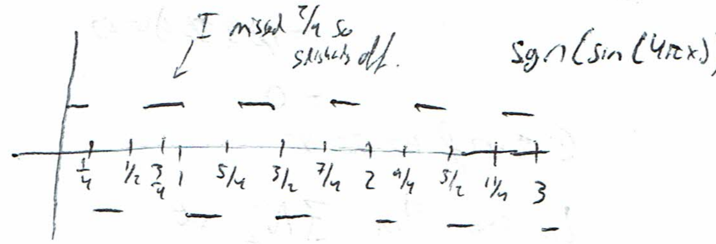
Problem 5:

It's been a little while since I studied something like this... I think ~~$f_n(x) = \text{sgn}(\sin$~~ $f_n(x) = \text{sgn}(\sin(2^n \pi x))$ is the standard example.

f_1



f_2

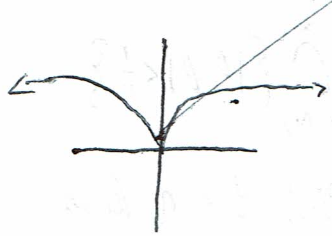


So, given x $f_n(x)$ oscillates between -1 & 1 a.e.

Converges weakly to 0... can't really be anything else intuitively. (I try to prove this later).

Problem 3 pt 2:

Consider $f_n(x) = 1 - e^{-x^2/n}$



Consider $f_n(x) = \chi_{[-n,n]}$

For any $E \subseteq \mathbb{R}$, $E \subseteq [-N,N]$ for N large enough.

Hence,

~~$\int_E |f_n| \, d\nu = \nu(E)$ for $n \ge N$.~~

If $|f_j| \nearrow |f| \notin L^1$ then

$$\sup_j \int_B |f_j| = \sup \int_E |f|$$

By MCT.

$$\sup_{\nu(E)<\delta} \int_E |f| \ge \varepsilon \ldots$$

Maybe do something ~~strange~~ silly like

$$f = \sum_{n=-\infty}^{\infty} \frac{2^{-n}}{(x-q_n)^2}$$

q_n enumeration of rationals.

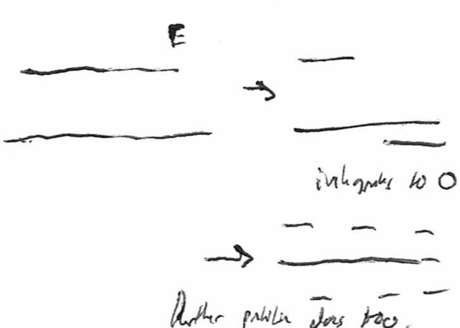
~~So that~~ $\infty >$

For $\nu(E)>0$, can approximate by intervals (Littlewood), which always contains a rational.

Hence $\int_E |f_j|$ blows up.

$$f_j = \sum_{n=-j}^{j} \frac{2^{-n}}{(x-q_n)^2} \cdot \chi_{\mathbb{Q}^c}$$

actually, not L^1. choose indicator functions which converge* to *isolate singularities.



Another idea:

$$f_n = n\chi_{[-n,n]} \qquad \int_{\mathbb{R}} f_n = n(2n) = 2n^2 < \infty$$

Since the f_n increase,

$$\sup_n \int_E f_n \ge \dots \ge \int_E f_N \ge \int_E f_{N-1} \ge \dots \ge \int_E f_1 .$$

For any E, $E \subseteq [-N,N]$ for N large enough w/ $\nu(E)<\infty$. (up to set of measure zero).

use outer approximation. or Littlewood

~~Hence $\sup_n \int_E f_n \ge \int_E f_N = \nu(E) > 0$.~~

Hence for $n \ge N$, $\int_E f_n = n\,\nu(E) \to \infty$ if $\nu(E) \neq 0$

Problem 5 (additional work):

$f_n \xrightarrow{w} f$ in L^2 if for every $g \in L^2$ then $\int f_n g \to \int f g$. Work in $[0,1] = E$

} Riesz-rep & general definition in normed spaces.

Note that $|f_n| = 1$ a.e. so that

$$\left(\int_E |f_n|^2\right)^{1/2} = 1.$$

By Hölder,

$$\int |f_n g| \le \|f\|_2 \|g\|_2 = \|g\|_2 .$$

Hence weak limit f is st $\int |fg| \le \|g\|_2$. ...

~~If g is a constant then $\int f_n g = 0$ for all n, hence $\int g f = 0$.~~

Let $g = \chi_E$ where E has finite measure. Then for N large enough $\int f_N g = 0 \Rightarrow \int_E f = 0$ for all E w/ positive measure

w/ approximation to intervals.

$\Rightarrow f = 0$ a.e.

f_N partitions E and computes a difference, for $n \ge N$ this partition is subordinate to that of N, hence is also 0.

Why is $\int f_N g = 0$ eventually?

By Littlewood, E is essentially the finite disjoint union of intervals.

For each such interval, perturb the endpoints slightly to make them dyadic

This new approximate E partitions nicely eventually, in the following sense:

If $E = [\frac{1}{3}, 1]$ for example, then we split at stage n $\mathbb{R}$ into intervals of length 2^{-n} w/ dyadic endpoints.

So $E \to [\frac{1}{3}, \frac{1}{2}] \cup [\frac{1}{2}, 1]$ $n=1$

positive negative

$\to [\frac{1}{3}, \frac{1}{2}] \cup [\frac{1}{2}, \frac{3}{4}] \cup [\frac{3}{4}, 1]$ $n=2$.

negative positive negative

Notice that $[\frac{1}{3}, \frac{1}{2}]$ has measure less than 2^{-n}.
So, the integral will certainly not be 0, it doesn't cancel nicely.

Instead, change $\frac{1}{3}$ to a dyadic rational close by, this should fix the above issue (in the limit).