Complex analysis Aug 2021

3. Note that $e^{e^{z_k}} = 1$ whenever $z_k = \ln|2\pi k| + \frac{\pi}{2} i$, $k \geq 1$.

Thus $f(z_k) = 0$ for all $k \geq 1$.

But, $z_k \to \infty$ so that ~~f(z_k)~~

$$0 = f(z_k) \to f(\infty)$$

Now, the only meromorphic functions in the extended complex plane are rational functions. Yet, f is entire so f must be a polynomial.

However, f is a non-constant polynomial if and only if $\lim_{|z| \to \infty} |f(z)| = \infty$.

Since this limit is zero, it must be that f is a constant function. We know ~~th~~ $f = 0$ for some z, so $f \equiv 0$.

[I feel this is wrong because I do not use the hypothesis $|f(z)| \leq e^{|z|^n}$].

1. I think the only such function is $f(z) = 1/z$.

To support this reasoning, consider a bijective conformal map $f: H \to H$. Then, with $g(z) = \frac{z-1}{z+1}$ we see that $g \circ f \circ g^{-1}: \mathbb{D} \to \mathbb{D}$. By the Schwarz Lemma, the only such functions are Möbius transformations.

Hence, f must be a Möbius transformation.

From here, I think we can algorithmically check that $f(z) = 1/z$ is the only possibility.

(my reasoning is to go through the possibilities of $g \circ f \circ g^{-1}: \mathbb{D} \to \mathbb{D}$, which include cz, $\frac{z-\omega}{1-\bar{\omega}z}$ with $|c| = |\omega| = 1$ ... The only one I could get to work was cz with $c = \pm 1$).

2. Similar to ~~#1~~ #3, I'm not 100% sure this method works.

Let $z \to \infty$ so that $f(\infty)^2 = 1$ and hence $f(\infty) = \pm 1$. So, f is meromorphic in the extended plane and thus is a rational function.

We now look at the zeros/poles of f. Evidently, if there is a zero at $z = z_i$ then there is a pole at $z = -z_i$

(Note: this makes sense since we assume $f(0) = 1$, so there is no issue ~~of~~ ~~wrong~~ ~~z.~~ concerning if $z_i = 0$).

The converse is also true. Hence, we can write

$$f(z) = \frac{\prod_{i=1}^{n} (z - z_i)}{\prod_{i=1}^{n} (z + z_i)} \cdot c$$

where $\{z_i\}_{i=1}^{n}$ are the zeros of

f (finitely many since Rational).

Now, $$f(0) = \frac{\prod_{i=1}^{n} (-z_i)}{\prod_{i=1}^{n} z_i} \cdot c$$

$$= (-1)^n \cdot c = 1$$

So $c = (-1)^n$ ~~I wonder if it's necessarily true that~~ n is an even integer. for non-negative

$\rightsquigarrow f(z) = \prod_{i=1}^{n} (z - z_i) \Big/ \prod_{i=1}^{n} (-z_i - z) = g(z)/g(-z)$.

4. Note that $f_n \neq 0$ for all n, so we can define $g_n(z) = \frac{1}{f_n(\frac{1}{z})}$.

The $g_n : \mathbb{D}^\circ \to \mathbb{D}^\circ$ are analytic and such that $g_n(\frac{1}{k})$ converges for all n.

The family $\{g_n\}$ is uniformly bounded (e.g. by 1) and hence normal.

So, up to a subsequence the $g_n \to g : \mathbb{D} \to \mathbb{D}$ which is analytic and converges uniformly on compact sets. Since g is an ~~open~~ analytic function, it is an open mapping and thus $g : \mathbb{D}^\circ \to \mathbb{D}^\circ$ also.

unless constant, in which case there is nothing to show.

~~Since~~ So... this at last does the job up to a subsequence...

The only thing I can think of (which seems rather silly) is the following: ~~Let $f(z) = f$~~

Let $f(z) = \lim_{n \to \infty} f_n(z) \quad z \in \Omega$.

(just defined as the pointwise limit)

Now let $h(z) = \frac{1}{g(\frac{1}{z})}$, which is analytic and agrees with $f(z)$ at $k = 2, 3, \ldots$

By the identity theorem in the extended plane, ∞ is an accumulation point and thus ~~$f(k) \neq g(k)$~~

$$f(k) - h(k) = 0 \quad \text{for } k > 1$$

implies $f \equiv h$ everywhere (may need f analytic for this, which I do not think is guaranteed).

~~The~~ and Since f is Möbius, it is determined by its action on three points. The conditions $f(1/2) = 2$, $f(2) = 1/2$ specify two, while $f \circ f = z$ accounts for the third.

1^*: Just noticed this, so I thought I'd add it in. Let $g(z) = f \circ f(z)$

Then $g(z)$ has two fixed points

Since H is simply connected, by the Riemann mapping theorem we can construct $\tilde{g}(z) = h^{-1} \circ g \circ h(z)$

st $\tilde{g} : \mathbb{D}^\circ \to \mathbb{D}^\circ$ and $\tilde{g}$ has two fixed points. Then, $\tilde{g} = \mathrm{Id}$ by Schwarz.

Follows that $g(z) = z$, so f must be st $f \circ f = z$