Applied Math 1 prelim.

3. ($\Leftarrow$) It is a THM that if X is a NLS and Y is f.d. (possibly Banach?) then $T: X \to Y$ is compact for any linear T.

So, since the A_n have finite rank, they are compact.

But $C(X,Y)$ is closed, so the limit is compact. Hence $A \in C(X,Y)$.

($\Rightarrow$) Let $\{e_n\}_{n=1}^{\infty}$ be a maximal onb for Y. Define $P_n = \text{span}\{e_1, \dots, e_n\}$.

Consider $A_n : X \to Y$ by

$$A_n(x) = \sum_{i=1}^{n} \langle Ax, e_i \rangle e_i.$$

Then $\|A_n - A\| \to 0$.

2. ~~The dual space X^* is separable, so $\exists$ a dense countable subset $\{f_n\}$.~~
~~Let $x_n \in X$ be st~~
~~X is separable. Let $D \subseteq X$ be a countable dense subset.~~
~~Take $\{x_n\} \subseteq D$ normalized.~~

$x_n \rightharpoonup 0$ means that for any $f \in X^*$ we have $f(x_n) \to f(0) = 0$.

Suppose $f_i(x_n) \to 0$ on countable dense subset of X^{*}.

~~By Mazur sep. Thm, $\exists f \in X$~~
~~st $f|_{\text{span}\{x_1, \dots, x_n, \dots\}} = 0$~~
~~$f(w) \neq 0$ for some~~

Proof that $B_1(0)$ is not compact:

Let $x_1 \in B_1(0)$. Consider $\text{span}\{x_1\}$, use Riesz lemma to find $x_2 \in B_1(0) \setminus \text{span}\{x_1\}$ st $d(x_1, x_2) > 1/2$.

Repeat inductively.

Let $f \in X^*$. Since X^* is sep. $\exists$ f_n st $f_n \to f$.

For each f_n let x_n w/ norm 1 be st

$f_n(x_n) \geq \frac{1}{2}\|f_n\|$

For each $f_i \in D \subseteq X^*$ countable dense subset,

4. By closed graph THM, P is continuous iff P is a closed operator. That is, for each $x_n \to x$, $Tx_n \to y$ we have ~~$x = Ty$~~ $y = Tx$.

If P is continuous, then for $x_n \in \text{Null}(P)$ and $x_n \to x$

$0 = Px_n \to Px$

So $x \in \text{Null}(P)$.

Suppose range is closed. Let y_n

~~Continuous if for $x_n \to x$, $Px_n \to Px$.~~

~~Suppose $x_n \to x$. Since the range is closed, $Px_n \to y = Px'$ for some x'.~~

~~Now $Px_n = P(Px_n)$~~

Let $y_n \in$ Range(P) be st $y_n \to y$

Then $\exists x_n$ st $Px_n = y_n$.

~~WTS $\exists x$ st $Px = y$.~~

~~But trivially P is id on its range~~

So $x = y$ works.

P is the identity on its range (since $P^2 = P$).

By continuity

$$Py_n = P(Px_n) \to Py$$

OTOH,

$$P(Px_n) = Px_n = y_n$$

So

$$\begin{array}{l} y_n \longrightarrow Py \\ \downarrow \\ y \end{array}$$

Thus $Py = y$.

Null space of P is all x st $Px = 0$.

Want to use Closed graph thm ~~to~~ (show if

$$x_n \to x, \quad Px_n \to y$$

Then $\underbrace{y = Px'}_{\text{closedness of Range}}$ for some x'.

Then use null space hypothesis to show $x' = x$.

For 2, I imagine this has something to do with non-compactness of infinite dim Banach spa

Tools that come to mind are H-B & the Riesz Lemma

↳ Mazur se