3. \( (\Leftarrow) \) It is a Hahn Banach that if \( X \) is a NLS and \( Y \) is f.d. (possibly Banach) then 
\( T: X \to Y \) is compact if and only if 
\( \text{span}(x_n \implies x) = 0 \) 
\( f(x) > 0 \) for some 

Proof: Let \( B, C, x \) be not compact. 

2. Let \( \{e_i \} \) be a maximal and 
\( \text{span}\{e_1, \ldots, e_n\} \) is closed. Hence \( A \to C(X, Y) \).

3. (\Rightarrow) Let \( \{e_i \} \) be a maximal and 
\( \text{span}\{e_1, \ldots, e_n\} \) be Y. Define \( A_n = \text{span}\{e_1, \ldots, e_n\} \).

Consider \( A_n: X \to Y \) by 
\[
A_n(x) = \sum_{i=1}^{n} \langle Ax, e_i \rangle e_i.
\]

Then \( \|A_n A - A\| \to 0 \).

By Mazur's Theorem, if \( f \in X \) 
\[
\text{span}(x_n, \ldots, x) = 0
\]
\( f(x) > 0 \) for some 

Proof: Let \( B, C, x \) be not compact. 

2. Let \( x \in B, C \). Consider \( \text{span}\{e_1, \ldots, e_n\} \) 
\( \text{span}\{e_1, \ldots, e_n\} \) is closed. Hence \( A \to C(X, Y) \).

3. (\Rightarrow) Let \( \{e_i \} \) be a maximal and 
\( \text{span}\{e_1, \ldots, e_n\} \) be Y. Define \( A_n = \text{span}\{e_1, \ldots, e_n\} \).

Consider \( A_n: X \to Y \) by 
\[
A_n(x) = \sum_{i=1}^{n} \langle Ax, e_i \rangle e_i.
\]

Then \( \|A_n A - A\| \to 0 \).

For each \( f \in \text{D}(X) \) 
\( \text{span}\{e_1, \ldots, e_n\} \) is closed. Hence \( A \to C(X, Y) \).

4. By closed graph Thm, 
\( P \) is the solution if \( P \) is a closed 
operator. That is, for each 
\( x_n \to x \), \( P x_n \to y \) 
we have \( Ax = y \). 

If \( P \) is continuous, then \( f \in \text{null}(P) \) 
and \( x \to x \) 
\( 0 = P x_n \to P x \) 
so \( x \in \text{null}(P) \).
Suppose range is closed. Let \( y_n \)

Continuity of \( P, x_n \to x, P x_n \to P x \).

Suppose \( x_n \to x \). Since the range is closed, \( P x_n \to P x \). Let some \( x' \).

Now \( P x_n \to P (P x_n) \)

Let \( y_n \in \text{range}(P) \) be st \( y_n \to y \).

Then \( 3 x_n \to P x_n \).

WTS: \( 3 x \to P x \).

but obviously \( P x = x \) is its range.

So \( x = y \) works.

\( P \) is the identity on its range.

(since \( P^2 = P \)).

By continuity

\( P y_n \to P (P x_n) \to P y \)

So

\( P (P x_n) = P x_n \to y_n \)

Therefore

\( P y_n \to P y \)

\( \therefore y_n \to P y \)

So \( y = y \).

For 2, I imagine this has something to do with non-computeness of infinite dim Banach space.

Tools that come to mind are H-B & the Baire property

by Mazur set.