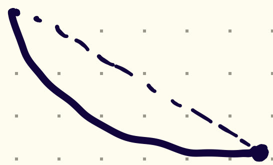
The calculus of variations:

Dates back to 17th century, ex Bernoulli & the brachistochrone problem (curve of fastest descent), later by Euler & Lagrange



to develop Lagrangian mechanics

Essential idea: In calculus we learn that derivatives of functions let us find critical points (min, max, etc)

We want to adapt this idea to a more generic setting.

Def: A *functional* is a map $F: X \to \mathbb{R}$, where X is some (topological) space.

As the name suggests, functionals take the place of functions.

E.g. $F: C([0,1]) \to \mathbb{R}$

$$f \mapsto \int_0^1 f(x)\,dx$$

$P: \{\text{sets w/ smooth boundary}\} \to \mathbb{R}^+$

$$E \mapsto \text{Perimeter of } E$$

Lagrangian

$L: \text{System} \to \mathbb{R}$

$$p \mapsto K(p) - V(p)$$

(kinetic) (potential)

The latter ex suggests that our functionals should heuristically represent energy.

General goal: Minimize energy

↪ derivatives of functionals

Method 1:

Let's work with $F(f) = \int_0^1 f(x)\,dx$

Want to mimic definition for functions.

Let $g : \mathbb{R}^n \to \mathbb{R}$ be differentiable.

Then $(D_v g)(x) = \langle \nabla g(x), v \rangle = \lim_{t \to 0} \dfrac{g(x+tv) - g(x)}{t}$

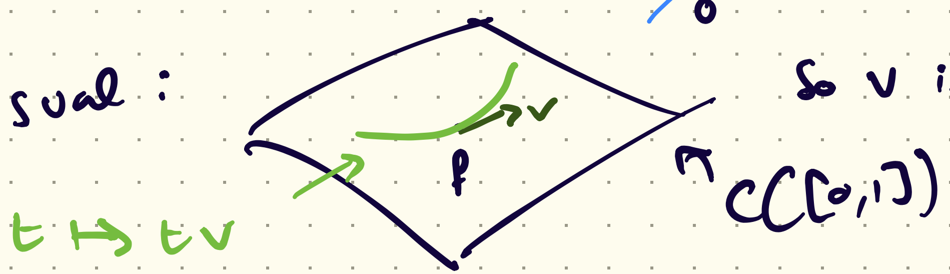
(directional derivative)

$$\rightsquigarrow \lim_{t \to 0} \frac{F(f+tv) - F(f)}{t}$$

where $v \in C(C[0,1])$ is a <u>function</u>.

$$= \lim_{t \to 0} \cancel{\frac{1}{t}} \left[\int_0^1 (\cancel{f(x)} + \cancel{t}\, v(x))\,dx - \int_0^1 \cancel{f(x)}\,dx \right] = \int_0^1 v(x)\,dx.$$

Visual:



So v is like the velocity.

Denote by $\delta F[f](v)$ the "first variation."
f is a critical point iff $\delta F[f](v) = 0$ for all v.

Method 2: Can sometimes reformulate to make easier

L: System $\to \mathbb{R}$

particles in space,
each having fixed mass m,
position x, velocity v

$P \longleftrightarrow (x, v)$

KE: $K(v) = \frac{1}{2} m|v|^2$

PE: $V(x)$ (electrostatic, gravity, etc.)

$L(x,v) = \frac{m}{2}|v|^2 - V(x)$

So $L : \underset{pos}{\mathbb{R}^3} \times \underset{vel}{\mathbb{R}^3} \to \mathbb{R}$,

can differentiate as usual: $\frac{\partial L}{\partial x} = -\nabla V(x)$

$$\frac{\partial L}{\partial v} = mv$$

Critical points of functionals, when they exist, satisfy certain conditions (Euler-Lagrange equations).

For $x_0, x_1 \in \mathbb{R}^3$ let $X = \left\{ x(t) \;\middle|\; \begin{array}{l} x : [0,1] \to \mathbb{R}^3 \\ x(0) = x_0, \quad x(1) = x_1 \end{array} \right\}$.

Notice if $\gamma : [0,1] \to \mathbb{R}^3$ is st $\gamma(0) = \gamma(1) = 0$ then

$x + s\gamma \in X$ for any $s \in \mathbb{R}$.

Let $A : X \to \mathbb{R}$ be the <u>action</u>

$$x(t) \mapsto \int_0^1 L(x(t), x'(t))\, dt = \int_0^1 \frac{m}{2} |x'(t)|^2 - V(x(t))\, dt$$

and compute $$\delta A(\gamma) = \lim_{s \to 0} \frac{A(x(t) + s\gamma(t)) - A(x(t))}{s}$$

$$= \lim_{s \to 0} \left[\frac{1}{s} \int_0^1 \frac{m}{2} \left(|x'(t) + s\gamma'(t)|^2 - |x'(t)|^2 \right) - V(x(t) + s\gamma(t)) + V(x(t))\, dt \right]$$

$$= \lim_{s\to 0}\left[\frac{1}{s}\int_0^1 \frac{m}{2}\left(2s\langle \gamma'(t), x'(t)\rangle + s^2|\gamma'(t)|^2\right) - V(x(t)+s\gamma(t)) + V(x(t))\,dt\right]$$

$$= \int_0^1 \underbrace{m\langle \gamma'(t), x'(t)\rangle}_{\text{IBP}} - \langle \nabla V(x(t)), \gamma(t)\rangle\,dt$$

$$\int_0^1 \langle \gamma'(t), x'(t)\rangle\,dt = \langle \cancelto{0}{\gamma(1)-\gamma(0)}, x'(1)-x'(0)\rangle - \int_0^1 \langle \gamma(t), x''(t)\rangle\,dt$$

Contd...

$$= \int_0^1 -m\langle \gamma(t), x''(t)\rangle - \langle \nabla V(x(t)), \gamma(t)\rangle\,dt$$

$$= -\int_0^1 \langle \gamma(t), m x''(t) + \nabla V(x(t))\rangle\,dt$$

So $x(t) \in \mathcal{X}$ is a crit. pt. of $\mathcal{A}$ iff

$$m x''(t) + \nabla V(x(t)) \equiv 0. \quad \text{(E-L eqn)}$$

$$\rightsquigarrow \quad m x''(t) = -\nabla V(x(t)), \text{ for all } t \in [0,1].$$

Since $-\nabla V$ is the force, we're saying $F = ma$.
Thus Newton's 2^{nd} law is the Euler-Lagrange eqn associated to the action.

↳ "Principle of least action."

So far we've discussed the relationship of c.o.v. w/physics...
What about geometry?

The isoperimetric problem:

Consider the variational problem

$$I(m) = \inf\{\underbrace{P(E)}_{\text{perimeter}} \mid E \subseteq \mathbb{R}^n, |E| = m\}$$

Does a minimizer exist? What E-L eqn does it satisfy?

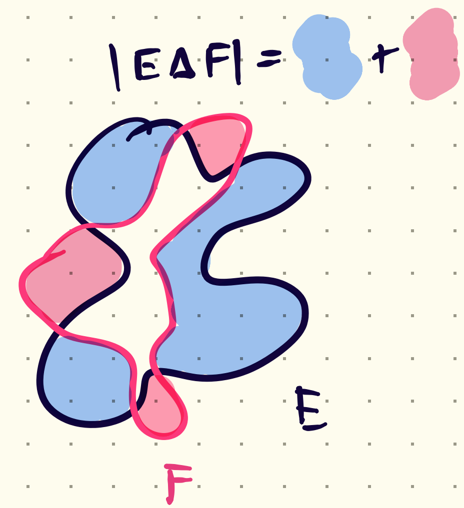
Existence:

Uses the direct method.

Consider the space $\mathcal{E}^m = \{E \subseteq \mathbb{R}^n \mid |E| = m\}$

Need a topology on $\mathcal{E}^m$, most common is L^1:

$$B_r(E) = \{F \in \mathcal{E}^m \mid |E \Delta F| < \varepsilon\}$$



$\mathcal{E}$ can contain wild sets, so we consider $\mathcal{E}_{reg}$, with some regularity (eg smooth boundary).

We can compute $P(E) = \int_{\partial E} 1$ for $E \in \mathcal{E}^m_{reg}$.

In the L^1 topology, P is lower semi-continuous
can only decrease in limit.

ie $P(E_\infty) \leq \liminf_{j \to \infty} P(E_j)$

when $E_j \xrightarrow{L^1} E_\infty$. + technicalities, $\subseteq B_R$

Furthermore, $\mathcal{E}^m_{reg}$ w/ L^1 topology is compact.

So, for $m \in \mathbb{R}^+$ let $\{E_j\}_{j=1}^\infty \subseteq \mathcal{E}^m_{reg}$ be st $P(E_j) \to I(m)$

By compactness there is a convergent subsequence $\{E_{j_K}\}_{K=1}^\infty$ st $E_{j_K} \xrightarrow{L^1} E_\infty$. Then,

$$I(m) \overset{\text{def}}{\leq} P(E_\infty) \overset{\text{lsc}}{\leq} \liminf_{K \to \infty} P(E_K) \overset{\text{converges}}{=} \lim_{K \to \infty} P(E_K) = I(m)$$

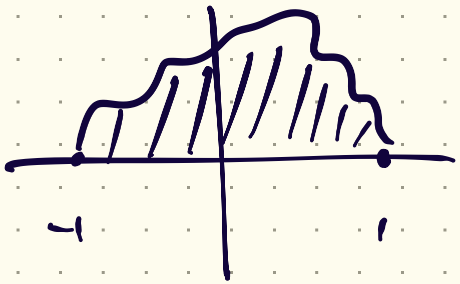
Hence E_∞ attains the minimum.

How do we characterize the minima?

Let $\mathcal{U} = \{u \in C^1([-1,1]) \mid u \geq 0,\ u(-1) = u(1) = 0\}$

Define $A(u) = \int_{-1}^{1} u(x)\,dx$

$$L(u) = \int_{-1}^{1} \sqrt{1 + u'(x)^2}\,dx$$



Want to minimize L wrt constraint $A(u) = m$

Consider $H(u, \lambda) = L(u) + \lambda \underbrace{[A(u) - m]}_{\text{Penalty term}}$

↳ Lagrange multipliers

For $v \in \mathcal{U}$, notice that $u + tv \in \mathcal{U}$ for all t.

Hence $\delta H[u,\lambda](v,\Lambda) = \delta L[u](v) + \left.\frac{d}{dt}\right|_{t=0} \left([\lambda + t\Lambda][A(u+tv) - m] \right)$

Former: $\delta L[u](v) = \int_{-1}^{1} \left.\frac{d}{dt}\right|_{t=0} \sqrt{1 + (u'(x) + tv'(x))^2}\, dx$

$$= \int_{-1}^{1} \frac{u'(x)\, v'(x)}{\sqrt{1 + u'(x)^2}}\, dx \overset{\text{IBP}}{=} \int_{-1}^{1} -\frac{d}{dx}\left(\frac{u'(x)}{\sqrt{1+u'(x)^2}} \right) v(x$$

Latter: $\left.\frac{d}{dt}\right|_{t=0} \left([\lambda + t\Lambda][A(u+tv) - m] \right)$

$$= \Lambda (A(u) - m) + \lambda \int_{-1}^{1} \left.\frac{d}{dt}\right|_{t=0} (u(x) + t v(x))\, dx$$

$$= \Lambda (A(u) - m) + \lambda \int_{-1}^{1} v(x)\, dx$$

So: $\delta H[u,\lambda](v,\Lambda) = \int_{-1}^{1} \left(\lambda - \frac{d}{dx}\frac{u'(x)}{\sqrt{1+u'(x)^2}}\right) v(x)\,dx$

$+ \Lambda\,(A(u)-m)$

Need $\delta H[u,\lambda] \equiv 0$, so

$$\frac{d}{dx}\frac{u'(x)}{\sqrt{1+u'(x)^2}} = \lambda \quad \text{and} \quad A(u)=m \qquad \text{E-L equation}$$

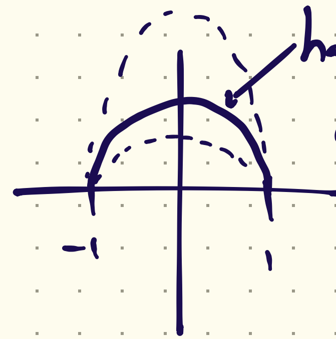
Penalty term gave us correct constraint!

Let's take $m = \pi/2$.

The first gives

$$\rightsquigarrow \frac{u'(x)}{\sqrt{1+u'(x)^2}} = \lambda x + c_1 \iff u'(x) = \frac{\pm(\lambda x + c_1)}{\sqrt{1-(\lambda x+c_1)^2}}$$

$\rightsquigarrow$ $u(x) = \mp\sqrt{1-(\lambda x+c_1)^2}+c_2$



Family of ellipses parametrized by λ

c_1, c_2 determined by $u(\pm 1)=0$

Isoperimetric inequality:

Let $E \subseteq \mathbb{R}^n$ have $|E|<\infty$. Then,

$$P(E) \geq n|B_1|^{1/n}|E|^{n-1/n}$$

Heuristic:

Minimizing functional $\longleftrightarrow$ Solving PDE (E-L eqn) $\longleftrightarrow$ Equality case of inequality

Eg. Minimize perimeter $\longleftrightarrow$ constrained MSE $\longleftrightarrow$ Isoperimetric ineq.