Calculus in Metric Measure Spaces

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Outline

i) How do we translate derivatives into a metric setting?
ii) Defining $W^{1,2}(X)$ via the Cheeger energy.
iii) Existence of a Laplace operator.

A General Heuristic

Let's say we have a concept which we want to define in a more abstract setting. It could be that the current definition doesn't make any sense in abstraction. So, we need to find equivalent formulations.

Example

We say a function $f \in C^2(\mathbb{R}^n)$ is *convex* if $\nabla^2 f \geq 0$.
Equivalently, f is convex if for all $x, y \in \mathbb{R}^n$ and $t \in [0, 1]$ it holds

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y).$$

The first formulation has the advantage of easily testing for convexity. The second formulation allows us to drop the regularity assumption.

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It really is useful to consider this equivalent formulation. Consider the Dirichlet energy $E : W^{1,2}(\mathbb{R}^n) \to \mathbb{R}$ defined as

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx.$$

The first variation is computable: for $u, v \in W^{1,2}(\mathbb{R}^n)$

$$\delta E[u]v = \frac{d}{dt}\bigg|_{t=0} E[u + tv] = \int_{\mathbb{R}^n} \langle \nabla u(x), \nabla v(x) \rangle \, dx = -\int_{\mathbb{R}^n} v(x) \Delta u(x) \, dx.$$

Riesz representation then allows us to define the gradient of E as

$$\delta E[u]v = \langle v, \nabla E(u) \rangle \quad \Rightarrow \quad \nabla E(u) = -\Delta u.$$

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doesn't admit a good definition for $\nabla^2 E(u)$. How can we possibly test for convexity? We must appeal to the pointwise definition. For $u, v \in W^{1,2}(\mathbb{R}^n)$ we have

$$\begin{aligned} E((1-t)u+tv) &= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla((1-t)u+tv)(x)|^2 \, dx \\ &= \frac{1}{2}\int_{\mathbb{R}^n} |(1-t)\nabla u(x) + t\nabla v(x)|^2 \, dx \\ &\leq \frac{1}{2}\int_{\mathbb{R}^n} |(1-t)\nabla u(x)|^2 + |t\nabla v(x)|^2 \, dx \\ &= (1-t)^2 E(u) + t^2 E(v) \leq (1-t)E(u) + tE(v). \end{aligned}$$

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Metric Measure Spaces

Definition

A *metric measure space* is a triple (X, d, m) where (X, d) is a complete and separable metric space and m is a Borel, non-negative measure such that $m(X) > 0$ and $m(B) < \infty$ for any bounded $B \subset X$.

Why study metric measure spaces? Over the past 20 years they've proven remarkably useful in Riemannian geometry. Particularly, metric measure spaces are the right framework to talk about non-smooth geometry.

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Asymptotic Lipschitz Constant

Idea: $W^{1,2}(\mathbb{R}^n)$ can be defined as $\{E < \infty\}$, the domain of the Dirichlet energy, so we just need to define the Dirichlet energy. However, it involves a term $|\nabla u|^2$, which doesn't make sense in a metric space.

Recall however that if $u : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz then $\|\nabla u\|_{L^\infty(\mathbb{R}^n)} \leq \mathrm{Lip}(u)$. This is promising, except the fact that it is global. To this end we define the following local alternative.

Definition (Asymptotic Lipschitz Constant)

For $f : X \to \mathbb{R}$ we define the *asymptotic Lipschitz constant*, $\mathrm{Lip}_a f : X \to [0, \infty]$, via

$$\mathrm{Lip}_a f(x) = \inf_{R>0} \mathrm{Lip}(f|_{B_R(x)}) = \limsup_{y,z \to x} \frac{|f(y) - f(z)|}{d(y,z)}.$$

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Characterizing the Dirichlet Energy

Note that if $u \in C^\infty(\mathbb{R}^n)$ then $\mathrm{Lip}_a\, u(x) = |\nabla u(x)|$. However, if $u \in W^{1,2}(\mathbb{R}^n)$ then it is typically discontinuous everywhere and thus $\mathrm{Lip}_a\, u(x) = \infty$. Hence we cannot just consider $\int_{\mathbb{R}^n} \mathrm{Lip}_a\, f(x)^2\, dx$ as this would be $+\infty$.

So we must resort to some kind of approximation. If $u \in W^{1,2}(\mathbb{R}^n)$ then we can find $\{u_n\}_{n=1}^{\infty} \subset \mathrm{Lip}_{\mathrm{BS}}(\mathbb{R}^n)$ a sequence of Lipschitz functions with bounded support such that $u_n \xrightarrow{W^{1,2}} u$. By lower semi-continuity of the Dirichlet energy it follows that

$$E(u) \leq \liminf_{n\to\infty} E(u_n) = \liminf_{n\to\infty} \left(\frac{1}{2} \int_{\mathbb{R}^n} \mathrm{Lip}_a(u_n)^2\, dx \right).$$

It stands to reason that taking the infinum over all possible sequences gives

$$E(u) = \inf \left\{ \liminf_{n\to\infty} \left(\frac{1}{2} \int_{\mathbb{R}^n} \mathrm{Lip}_a(u_n)^2\, dx \right) \right\}.$$

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By the above discussion, the Cheeger energy coincides with the Dirichlet energy on $\mathbb{R}^n$. So, we may simply define $W^{1,2}(X) = D(\mathrm{Ch}) = \{\mathrm{Ch} < \infty\}$, the domain of the Cheeger energy.

Since

$$\mathrm{Lip}_a(\alpha_1 f_1 + \alpha_2 f_2) \leq |\alpha_1| \, \mathrm{Lip}_a(f_1) + |\alpha_2| \, \mathrm{Lip}_a(f_2)$$

for $\alpha_1, \alpha_2 \in \mathbb{R}$ and $f_1, f_2 \in L^2(X)$ it follows that $W^{1,2}(X)$ is a vector space.

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The Sobolev Space $W^{1,2}(X)$

Endowing $W^{1,2}(X)$ with the norm

$$\|f\|^2_{W^{1,2}(X)} = \|f\|^2_{L^2(X)} + 2\,\mathrm{Ch}(f),$$

one can prove that $(W^{1,2}(X), \|\cdot\|_{W^{1,2}(X)})$ is a Banach space. The proof relies on lower semi-continuity of the Cheeger energy.

Remark

In general $W^{1,2}(X)$ is *NOT* a Hilbert space. One of the defining features of the so-called Ricci limit spaces $RCD(K, N)$, a class of metric measure spaces, is that $W^{1,2}(X)$ is a Hilbert space. This property is called *infinitesmially Hilbertian*.

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Relaxed Slopes

Coupled with the concept of Cheeger energy is the concept of relaxed slope.

Definition (Relaxed Slope)

We say $G \geq 0$ an $L^2(X)$ function is a *relaxed slope* for f if there exists $\{f_n\}_{n=1}^{\infty}$ as in the definition of $\mathrm{Ch}(f)$ such that $\mathrm{Lip}_a(f_n) \rightharpoonup G'$ and $G' \leq G$ a.e.

Heuristically the relaxed slope is like a modulus of the gradient. The following proposition is easily deduced

Proposition

Let $f \in W^{1,2}(X)$. Then the set $\{G \in L^2(X) \mid G$ is a relaxed slope for $f\}$ is closed, convex, and stable by taking $\min(\cdot)$.

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Modulus of the Distributional Derivative

In particular, the convexity and closedness guarantee that there is a unique minimal norm element of the set $\{G \in L^2(X) \mid G \text{ is a relaxed slope for } f\}$. Stability by taking $\min(\cdot)$ guarantees that it is actually unique pointwise a.e.

Definition

We call this minimal element $|Df| \in L^2(X)$.

Remark

In the smooth setting we recover the norm of the distributional derivative, as anticipated.

Does $|Df|$ really behave like the modulus of a distributional derivative?

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Yes, the properties are summarized below:

Proposition

i) If $f_n \xrightarrow{L^2} f$ and $|Df_n| \rightharpoonup G$ then $f \in W^{1,2}(X)$ and $|Df| \leq G$ a.e.

ii) A chain rule holds: $|D(\varphi \circ f)| = (|\varphi'| \circ f)|Df|$ for all $\varphi : \mathbb{R} \to \mathbb{R}$ Lipschitz and C^1 (the latter is not necessary but makes life easier).

iii) A Leibniz rule: $|D(fg)| \leq |f||Dg| + |g||Df|$.

iv) A locality principle: $|Df| = |Dg|$ a.e. on $\{f = g\}$.

v) $|Df|$ is the strong L^2-limit of $\{\mathrm{Lip}_a f_n\}_{n=1}^{\infty}$ for any "optimal" sequence $\{f_n\}_{n=1}^{\infty}$ in the definition of $\mathrm{Ch}(f)$. In particular,

$$\mathrm{Ch}(f) = \frac{1}{2}\int |Df|^2 \, dm.$$

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ii) *A chain rule holds:* $|D(\varphi \circ f)| = (|\varphi'| \circ f)|Df|$ *for all* $\varphi : \mathbb{R} \to \mathbb{R}$ *Lipschitz and* C^1 *(the latter is not necessary but makes life easier).*

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Subdifferentials

Our next goal is to define the Laplacian on (a subset of) $W^{1,2}(X)$. To this end we need to concept of a subdifferential.

Definition (Subdifferential of a functional)

Let $E : H \to [0, \infty]$ be any convex, lower semi-continuous functional and $x \in H$ such that $E(x) < \infty$. Then the *subdifferential* of E at x is

$$\partial E(x) := \{v \in H \mid E(x) + \langle v, y - x\rangle \leq E(y) \text{ for every } y \in H\}.$$

The subdifferential is always closed and convex.

Example

Let $f : \mathbb{R}^n \to \mathbb{R}$ be any convex function. Then if f is differentiable at x it follows that $\partial f(x) = \{\nabla f(x)\}$.

Subdifferentials

Our next goal is to define the Laplacian on (a subset of) $W^{1,2}(X)$. To this end we need to concept of a subdifferential.

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The Laplacian

If $E : W^{1,2}(\mathbb{R}^n) \to \mathbb{R}$ is the Dirichlet energy then using the ansatz from the previous example,

$$\partial E(u) = \{\nabla E(u)\} = \{-\Delta u\}.$$

So to obtain the second order quantity Δu it suffices to study the first order set $\partial E(u)$.

Since we just defined the Cheeger energy on the Hilbert space $L^2(X)$ and it is supposed to extend the Dirichlet energy, we define the Laplacian as:

Definition

The Laplace operator Δ is defined on

$$\{f \in W^{1,2}(X) \mid \partial \operatorname{Ch}(f) \neq \emptyset\}$$

and for such f, the Laplacian $-\Delta f$ is defined as the element of smallest norm in $\partial \operatorname{Ch}(f)$.

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The Laplacian

The following lemma is immediate and shows the previous definition is good

Lemma

Let $f \in D(\Delta)$ and $g \in W^{1,2}(X)$ then

$$\left| \int g \Delta f \; dm \right| \leq \int |Df||Dg| \; dm.$$

Moreover, if $\varphi \in C^1(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$ then

$$\int (\varphi \circ f) \Delta f \; dm = -\int (\varphi' \circ f) |Df|^2 \; dm.$$

Concluding Remarks

i) Can define a heat flow H_t on $L^2(X)$ as the gradient flow of the Cheeger energy. Has all the same useful properties of a heat flow on $\mathbb{R}^n$ (minus regularization!)

ii) We really didn't use the measure much. There is an entire alternative approach, more related to the geometry of the Wasserstein space. The two are equivalent.