Compactness of Stable Minimal Surfaces

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A Roadmap:

Want to prove the following theorem:

Theorem

Let $U \subset \mathbb{R}^n$ be open with $3 \leq n \leq 6$. Suppose $\Sigma_k \subset U$ is a sequence of stable minimal hypersurfaces with $\mathrm{Vol}(\Sigma_k) \leq a$. Then there exists a subsequence converging to Σ, a stable minimal hypersurface in U (possibly with multiplicity).

What does it mean to converge here?

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- Instead of asking for some global convergence of these surfaces, we instead turn to local convergence, and show that these patch nicely together.
- To handle local convergence, we'll need to show our surfaces can (locally) be written as graphs of functions.
- Of course, you can do that for any smooth surface, what we're really need is a uniform way of doing this.
- Idea: Maybe if our surface is not curving too badly, we can get uniform graphicality.
- Since the second fundamental form A dictates curvature, we should establish bounds on $|A|^2$.

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A Roadmap:

- Step 1: Fix a compact $K \subset U$. Given our sequence of stable minimal surfaces $\Sigma_k \subset U$, show that $|A_k|^2$ are uniformly bounded on K.
- Step 2: Show that each $\Sigma_k \cap K$ is uniformly graphical.
- Step 3: Transition to a local argument, use Arzela-Ascoli to find a limit $C^{1,\alpha}$ function satisfying the minimal surface equation.
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A Roadmap:

- We'll discuss the theorems involved in Steps 1 and 2. Our motto is "small curvature implies graphical".

Theorem

Let $\Sigma \subset \mathbb{R}^n$ be a stable, minimal hypersurface with $\partial\Sigma \subset \partial B_R(0)$, $\mathrm{Vol}(\Sigma \cap B_R(0)) \leq aR^{n-1}$ for some $a > 0$, and $3 \leq n \leq 6$. Then there exists some $C(n,a)$ (independent of Σ!) such that

$$\sup_{\Sigma \cap B_R(0)} d(x, \partial B_R(0))^2 |A|^2(x) \leq C(n,a).$$

In particular, if $0 \in \Sigma$ then

$$|A|^2(0) \leq \frac{C(n,a)}{R^2}.$$

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- We'll discuss the theorems involved in Steps 1 and 2. Our motto is "small curvature implies graphical".

Theorem (Small Curvature Implies Graphical)

Let $\Sigma \subset \mathbb{R}^n$ *be an immersed hypersurface with*

$$16s^2 \sup_{\Sigma} |A|^2(x) \leq 1.$$

If $x \in \Sigma$ *and* $d(x, \partial\Sigma) \geq 2s$*, then there is a function* u *defined on* $B_s^T(x) := B_s(x) \cap T_x\Sigma$ *such that* $B_s^{\Sigma}(x) \subset \mathrm{Graph}(u) \subset B_{2s}^{\Sigma}(x)$*. Moreover,* $|\nabla u| \leq 1$ *and* $\sqrt{2}|\mathrm{Hess}(u)| \leq 1$*.*

Minimal Surfaces:

What is a minimal surface?

Let $\Sigma^k \subset \mathbb{R}^n$ be a submanifold and $F : \Sigma \times (-\epsilon, \epsilon) \to \mathbb{R}^n$ such that $F = \mathrm{Id}$ outside a compact set and $F|_{\partial\Sigma} = \mathrm{Id}$. Then one can show the first variation of the volume satisfies

$$\delta \operatorname{Vol}(\Sigma, F) = \frac{d}{dt} \operatorname{Vol}(F(\Sigma, t))\Big|_{t=0} = -\int_{\Sigma} \langle F_t, H \rangle.$$

Definition

A submanifold $\Sigma^k \subset \mathbb{R}^n$ is a minimal surface if for all such variations F we have $\delta \operatorname{Vol}(\Sigma, F) = 0$. So, Σ is a critical point of the (area) functional Vol, defined by

$$\operatorname{Vol}(\Sigma) = \int_{\Sigma} 1 \; d\mathcal{H}^k$$

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The Monotonicity Formula:

Theorem (The Coarea Formula)

If Σ is a manifold and $h : \Sigma \to \mathbb{R}$ is a proper Lipschitz function on Σ, then for all $f \in L^1_{loc}(\Sigma)$ and $r \in \mathbb{R}$,

$$\int_{\{h \leq r\}} f \, |\nabla_\Sigma h| = \int_{-\infty}^{r} \int_{\{h=\tau\}} f \, d\tau.$$

Theorem (The Monotonicity Formula)

Let $\Sigma^k \subset \mathbb{R}^n$ be a minimal submanifold and $x_0 \in \mathbb{R}^n$. Then for all $0 < s < t$ we have

$$\frac{\text{Vol}(\Sigma \cap B_t(x_0))}{t^k} - \frac{\text{Vol}(\Sigma \cap B_s(x_0))}{s^k} = \int_{\Sigma \cap B_t(x_0) \setminus \Sigma \cap B_s(x_0)} \frac{|(x - x_0)^N|^2}{|x - x_0|^{k+2}}.$$

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Theorem (Generalized Monotonicity)

If $\Sigma^k \subset \mathbb{R}^n$ is a minimal submanifold and $f : \Sigma \to \mathbb{R}$ a function then

$$t^{-k} \int_{\Sigma \cap B_t} f - s^{-k} \int_{\Sigma \cap B_t} f = \int_{\Sigma \cap B_t \setminus \Sigma \cap B_s} f \frac{|x^N|^2}{|x|^{k+2}} + \frac{1}{2} \int_s^t \tau^{-k-1} \int_{\Sigma \cap B_\tau} (\tau^2 - |x|^2) \Delta_\Sigma f \; d\tau.$$

Theorem (The Mean Value Inequality)

Let $\Sigma^k \subset \mathbb{R}^n$ be a minimal submanifold, $x_0 \in \Sigma$, and $s > 0$ with $B_s(x_0) \cap \partial\Sigma = \emptyset$. If f is nonnegative on Σ with $\Delta_\Sigma f \geq -\lambda s^{-2} f$, then

$$f(x_0) \leq \frac{e^{\lambda/2}}{\mathrm{Vol}(\Sigma \cap B_s)} \int_{\Sigma \cap B_s(x_0)} f.$$

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The Second Fundamental Form:

Definition

Let $\Sigma^k \subset \mathbb{R}^n$ be an embedded submanifold. The second fundamental form A is defined pointwise by $A_x : T_x\Sigma \times T_x\Sigma \to \mathbb{R}^n$ where

$$A_x(X, Y) = (\nabla_X Y)^N.$$

Remark

One way to write the mean curvature vector is of Σ is as

$$H(x) = \sum_{i=1}^{k} A_x(E_i, E_i)$$

where $\{E_i\}_{i=1}^{k}$ is an orthonormal basis of $T_x\Sigma$.

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Theorem (Simon's inequality)

Let $\Sigma \subset \mathbb{R}^n$ be a minimal hypersurface. Then,

$$\Delta_\Sigma |A|^2 \geq -2|A|^4 + 2\left(1 + \frac{2}{n-1}\right)|\nabla_\Sigma |A||^2.$$

Remark

Since the right-most term is nonnegative, if $|A|^2 \leq 1$ then

$$\Delta_\Sigma |A|^2 \geq -2|A|^2.$$

In particular, we can use the Mean Value Inequality to estimate $|A|^2$. In fact, since $\Delta_\Sigma f^p \geq p f^{p-1} \Delta_\Sigma f$ for $p > 1$ we get $\Delta_\Sigma |A|^{2p} \geq -2p|A|^{2p}$. So, we can estimate $|A|^{2p}$ via the Mean Value Inequality.

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Stable Minimal Surfaces:

Definition

Let $\Sigma \subset \mathbb{R}^n$ be an orientable hypersurface. Then for smooth functions η we define the stability operator L by

$$L\eta = \Delta_\Sigma \eta + |A|^2 \eta.$$

Theorem

Let $\Sigma^k \subset \mathbb{R}^n$ be an orientable, minimal submanifold. If F is a normal variation of Σ with compact support then

$$\delta^2 \operatorname{Vol}(\Sigma, F) = \frac{d^2}{dt^2}\bigg|_{t=0} \operatorname{Vol}(F(\Sigma, t)) = -\int_\Sigma \langle F_t, LF_t \rangle$$

where $F_t = \eta_t N$ for some η_t, and $LF_t = (L\eta_t)N$.

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Stable Minimal Surfaces:

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We say that Σ is stable if for all compactly supported normal variations F which fix $\partial\Sigma$ that $\delta^2 \operatorname{Vol}(\Sigma, F) \geq 0$. In other words, Σ is a minimizer of the (area) functional.

Theorem (Stability Inequality)

Let $\Sigma \subset \mathbb{R}^n$ be a stable, orientable, minimal hypersurface. Then for all Lipschitz functions η with compact support,

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Is there an L^p analog of this?

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Stable Minimal Surfaces:

Yes, due to Schoen-Simon-Yau! For certain p at least.

Theorem (L^p Stability Inequality)

Let $\Sigma \subset \mathbb{R}^n$ be a stable, orientable, minimal hypersurface. Then for all compactly supported nonnegative Lipschitz functions ϕ and $p \in [2, 2 + \sqrt{2/(n-1)}]$ we have

$$\int_\Sigma |A|^{2p}\phi^{2p} \leq C(n,p) \int_\Sigma |\nabla \phi^{2p}|.$$

Summary:

Minimal surfaces are critical points of the area functional, and give us monotonicity of

$$\frac{\text{Vol}(\Sigma \cap B_t(x_0))}{t^k}$$

in t. Given a lower bound on the Laplacian of f in terms of f, the mean value inequality gives a pointwise bound in terms of an integral

$$f(x_0) \leq C(n,s) \int_{\Sigma \cap B_s(x_0)} f.$$

Stable minimal surfaces are area minimizers, and give us L^p stability

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Proof of Estimate:

We're now ready to prove our first estimate theorem

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$$\sup_{\Sigma \cap B_R(0)} d(x, \partial B_R(0))^2 |A|^2(x) \leq C(n, a).$$

Method: A proof by contradiction using a blowup argument.

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Suppose there does not exist such a $C(n, a)$. Then there exists a sequence of stable minimal hypersurfaces Σ_k such that

$$\sup_{\Sigma_k \cap B_R(0)} |A_k|^2 (R - |x|)^2 \geq k^2.$$

Now fix a $k \in \mathbb{N}$ and choose $p_k \in \Sigma_k \cap B_R(0)$ such that the sup is attained. If $y \in \Sigma_k \cap B_R(0)$ then

$$|A_k(y)|^2 = \frac{|A_k(y)|^2 (R - |y|)^2}{(R - |y|)^2} \leq \frac{|A_k(p_k)|^2 (R - |p_k|)^2}{(R - |y|)^2}.$$

So, to control $|A_k(y)|^2$ within $B_R(0)$, we need control on $(R - |p_k|)/(R - |y|)$.

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Can't do this on all of $B_R(0)$, need to shrink to a smaller ball. Define σ_k by

$$\sigma_k = \frac{k}{2|A_k(p_k)|}.$$

We immediately have

$$\frac{k^2(R-|p_k|)^2}{4\sigma_k^2} = |A_k(p_k)|^2(R-|p_k|)^2 \geq k^2$$

This implies $R - |p_k| \geq 2\sigma_k$, and so $B_{2\sigma_k}(p_k) \subset B_R(0)$. Next, for $y \in B_{\sigma_k}(p_k)$,

$$\frac{R-|y|}{R-|p_k|} \geq \frac{R-|p_k|-\sigma_k}{R-|p_k|} = 1 - \frac{\sigma_k}{R-|p_k|} \geq \frac{1}{2}.$$

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So, if $y \in \Sigma_k \cap B_{\sigma_k}(p_k)$ then

$$|A_k(y)|^2 \leq \frac{|A_k(p_k)|^2(R - |p_k|)^2}{(R - |y|)^2} \leq 4|A_k(p_k)|^2.$$

Thus, $|A_k|^2$ on $\Sigma_k \cap B_{\sigma_k}(p_k)$ is controlled by $|A_k(p_k)|^2$.

With this information, we can now begin our blowup argument.

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Proof Continued

So, if $y \in \Sigma_k \cap B_{\sigma_k}(p_k)$ then

$$|A_k(y)|^2 \leq \frac{|A_k(p_k)|^2(R - |p_k|)^2}{(R - |y|)^2} \leq 4|A_k(p_k)|^2.$$

Thus, $|A_k|^2$ on $\Sigma_k \cap B_{\sigma_k}(p_k)$ is controlled by $|A_k(p_k)|^2$.

With this information, we can now begin our blowup argument.

Proof of Estimate:

Proof Continued

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Proof Continued

Blowup Step 1: Define $\mu_k : B_R(0) \to \mathbb{R}^n$ by $\mu_k(x) = 2|A_k(p_k)|(x - p_k)$. Let $\tilde{\Sigma}_k = \mu_k(\Sigma_k \cap B_{\sigma_k}(p_k))$. Recalling $\sigma_k = k/(2|A_k(p_k)|)$, μ_k maps $B_{\sigma_k}(p_k)$ to $B_k(0)$.

Note that the second fundamental form scales by $1/(2|A_k(p_k)|)$. So, if $\tilde{A}_k$ is the second fundamental form of $\tilde{\Sigma}_k$ then for all $y \in \tilde{\Sigma}_k$,

$$|\tilde{A}_k(y)|^2 = \frac{|A_k(\mu_k^{-1}(y))|^2}{4|A_k(p_k)|^2} \leq 1.$$

Importantly, the mean value inequality applies! Noting that $|\tilde{A}_k(0)| = 1/2$,

$$\frac{1}{2^{2p}} = |\tilde{A}_k(0)|^{2p} \leq C(n) \int_{\tilde{\Sigma}_k \cap B_1(0)} |\tilde{A}_k|^{2p} \leq C(n) \int_{\tilde{\Sigma}_k \cap B_k(0)} |\tilde{A}_k|^{2p}.$$

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Proof of Estimate:

Proof Continued

Blowup Step 2: Define a cutoff function ϕ_k where $\phi_k|_{B_k(0)} \equiv 1$, $\phi_k|_{B^c_{2k}(0)} \equiv 0$, and ϕ_k decreases radially from k to $2k$ (so $|\nabla \phi_k| = 1/r$).

Applying our L^p stability inequality with this cutoff function and $2p = 4 + \sqrt{7/5}$,

$$\int_{\tilde{\Sigma}_k \cap B_{2k}(0)} |\tilde{A}_k|^{2p} \phi_k^{2p} \leq C(n) \int_{\tilde{\Sigma}_k \cap B_{2k}(0)} |\nabla \phi_k|^{2p}$$
$$\leq C(n) k^{-4-\sqrt{7/5}} \mathrm{Vol}(\tilde{\Sigma}_k \cap B_{2k}(0)).$$

Proof of Estimate:

Proof Continued

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Proof of Estimate:

Proof Continued

In total,

$$\frac{1}{2^{4+\sqrt{7/5}}} \leq C(n)\int_{\tilde{\Sigma}_k \cap B_k(0)} |\tilde{A}_k|^{2p} \leq C(n)\int_{\tilde{\Sigma}_k \cap B_{2k}(0)} |\tilde{A}_k|^{2p}\phi_k^{2p}$$
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Blowup Step 3: Use volume bound to arrive at a contradiction.

Proof of Estimate:

Proof Continued.

Blowup Step 3: Use volume bound to arrive at a contradiction. Recall $\mathrm{Vol}(\Sigma_k \cap B_R(0)) \leq aR^{n-1}$ for some $a > 0$. Since the volume scales like $n-1$, we get

$$\begin{aligned}\mathrm{Vol}(\tilde{\Sigma}_k \cap B_{2k}(0)) &= (2|A_k(p_k)|)^{n-1}\,\mathrm{Vol}(\Sigma_k \cap B_{2\sigma_k}(p_k)) \\ &= \frac{(2k)^{n-1}\,\mathrm{Vol}(\Sigma_k \cap B_{2\sigma_k}(p_k))}{(2\sigma_k)^{n-1}} \\ &\leq \frac{(2k)^{n-1}\,\mathrm{Vol}(\Sigma_k \cap B_R(0))}{R^{n-1}} \leq a(2k)^{n-1}\end{aligned}$$

Finally,

$$\frac{1}{2^{4+\sqrt{7/5}}} \leq C(n,a)k^{-4-\sqrt{7/5}+n-1}.$$

Proof of Estimate:

Proof Continued.

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