A Strong Form of the Quantitative Wulff Inequality for Crystalline Norms

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Outline

- Introduction
  - Anisotropy and crystals
  - Quantitative stability
- Main Result
- Overview of Argument

Anisotropy

Many physical phenomena can be explained in terms of energy minimization. E.g., soap bubbles are spheres because they need to minimize surface tension with a constrained volume.

The formation of crystals in the small mass regime can be explained similarly. Thermodynamically, crystals at equilibrium should minimize Gibbs free energy:

$$\Delta G := \sum_i \gamma_i A_i = \lambda \sum_i h_i A_i,$$

where γ_i is the surface energy per unit area and A_i is the area of the ith face. The equality is due to Wulff, where he interpreted the problem in terms of a Lagrange multiplier $\lambda > 0$, and h_i is the distance to each face.

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Anisotropy: Wulff Shapes

These optimal configurations are called Wulff shapes. In general,

Definition

A *Wulff shape* is an open, bounded, convex set $K \subset \mathbb{R}^n$ containing the origin.

There are two important 1-homogeneous non-negative functions naturally associated to K:

- The *surface tension* $f : \mathbb{R}^n \to [0, \infty)$, for which $f(\nu)$ is the distance from the origin to the supporting hyperplane of K with normal ν. Typically view f as a function on S^{n-1}.
- The *gauge function* $f_* : \mathbb{R}^n \to [0, \infty)$, for which $K = \{f_* < 1\}$.

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Anisotropy: The Surface Tension and Gauge

The surface tension and gauge function are always semi-norms on $\mathbb{R}^n$, norms when K is symmetric about the origin. In this case, f and f_* are dual to each other.

For example, if $f_* = \ell^p$ then $f = \ell^q$, for p, q conjugate exponents.

In fact, we always have

$$f(\nu) = \sup\{\langle x, \nu \rangle \mid f_*(x) < 1\}$$
$$f_*(x) = \sup\{\langle x, \nu \rangle \mid f(\nu) < 1\},$$

so that for any $x \in \mathbb{R}^n$ and $\nu \in S^{n-1}$,

$$\langle x, \nu \rangle \leq f(\nu) f_*(x).$$

This is known as the *Fenchel inequality.*

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Anisotropy: The Anisotropic Perimeter

Given a Wulff shape K we can ask the following question: For what energy functional Φ is K the volume-constrained minimizer? I.e., find Φ such that

$$E \in \arg\min\{\Phi(F) \mid |F| = v\} \quad \text{if and only if} \quad E = rK + x_0, |rK| = v.$$

It turns out the following energy is appropriate

Definition

The *anisotropic perimeter* (associated to K) is given by

$$\Phi(E) = \int_{\partial^* E} f(\nu_E(x)) \, d\mathcal{H}^{n-1}(x).$$

The isotropic perimeter is recovered when $f = \ell^2$, for which the Wulff shape is a ball.

Anisotropy: Crystalline Setting

When K is a polytope we say that Φ is crystalline. We denote by N the number of facets of K, by F_i a generic $n-1$ dimensional facet, and by ν_i the outer unit normal of this facet.

In this setting we have that $f(\nu_i) = h_i$, the distance from the origin to the supporting hyperplane of F_i. In particular,

$$\Phi(K) = \int_{\partial^* K} f(\nu_K(x))\, d\mathcal{H}^{n-1}(x) = \sum_{i=1}^{N} h_i \mathcal{H}^{n-1}(F_i),$$

which is precisely the form the minimum Gibbs free energy takes for a crystal at equilibrium.

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Anisotropy: The Wulff Inequality

Φ is the right generalization to use because we have an anisotropic version of the isoperimetric inequality known as the *Wulff inequality*:

$$\Phi(E) \geq n|K|^{1/n}|E|^{(n-1)/n}$$

with equality if and only if $|E\Delta(rK + x_0)| = 0$ for some $r > 0$ and $x_0 \in \mathbb{R}^n$.

This is the same as the isoperimetric inequality with Φ in place of P and $K = \{f_* < 1\}$ in place of B_1. We have a rigidity statement, so we can ask about stability.

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Quantitative Stability: The isotropic setting

To discuss quantitative stability we introduce the following scale invariant quantities.

- Closeness to equality: Define the *isoperimetric deficit* δ as

$$\delta(E) = \frac{P(E)}{n|B_1|^{1/n}|E|^{(n-1)/n}} - 1$$

  which is always non-negative owing to the isoperimetric inequality, and is zero precisely when E is essentially a ball.

- Closeness to a ball: Use an *asymmetry index* α. Supposed to capture the geometry and is also such that $\alpha(E) = 0$ iff E is essentially a ball.

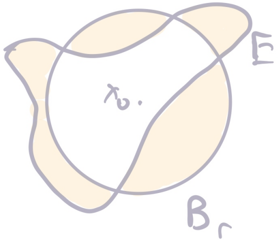
Qualitative stability says given $\{E_j\}_{j=1}^{\infty}$, if $\delta(E_j) \to 0$ then $\alpha(E_j) \to 0$. Quantitative stability quantifies this control, e.g. $\alpha(E)^p \leq \delta(E)$ for all sets of finite perimeter.

Quantitative Stability: Asymmetry Indexes

Many kinds, heuristically measure the distance to the set of minimizers $\{B_r(x_0) \mid x_0 \in \mathbb{R}^n, r > 0\}$. For ex. the Hausdorff distance.

The most common asymmetry index is the *Fraenkel asymmetry*

$$\alpha(E) = \inf_{x_0 \in \mathbb{R}^n} \left\{ \frac{|E \Delta B_r(x_0)|}{|E|} \;\middle|\; |B_r| = |E| \right\}.$$



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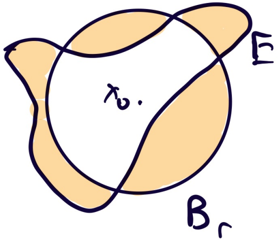

Quantitative Stability: Previous Results

Theorem (Fusco-Maggi-Pratelli, '08)

There exists $C(n) > 0$ *such that for any set of finite perimeter* $E \subset \mathbb{R}^n$ *with* $0 < |E| < \infty$,

$$\alpha(E)^2 \leq C(n)\delta(E). \qquad \text{(Q.S.)}$$

The power of 2 *in* (Q.S.) *is sharp.*

The proof exploits symmetrization techniques a la De Giorgi.

Previous results by Fuglede '89, Hall-Hayman-Weitsman '91, and Hall '92 prove (Q.S.) under various other hypotheses, e.g. if E is convex, nearly spherical, smooth, and/or axially symmetric.

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Quantitative Stability: Selection Principle

In '12 Cicalese-Leonardi showed sharp quantitative stability for the isoperimetric inequality by exploiting the regularity of almost minimizers. This technique became known as the *selection principle.*

Idea: proof by contradiction

- Use selection principle to replace original sequence with a new one with upgraded regularity, while still maintaining the contradictory hypothesis.
- Prove directly sharp stability with upgraded regularity.
- Derive a contradiction.

Does not use symmetrization!

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Quantitative Stability: A Strong Form

In '14 Fusco-Julin, using the selection principle, proved the following strong form of (Q.S.).

Theorem (Fusco-Julin)

There exists $C(n) > 0$ *such that for any set of finite perimeter* E *with* $0 < |E| < \infty$,

$$\alpha(E)^2 + \beta(E)^2 \leq C(n)\delta(E)$$

where $\beta(E)$ is the *oscillation index.*

Quantitative Stability: Oscillation Index

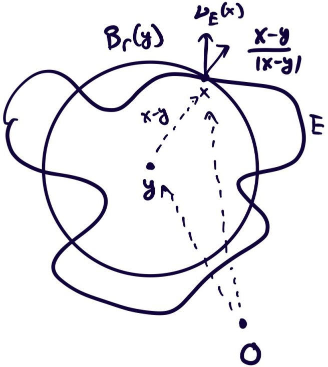
The oscillation index β is defined as

$$\beta(E) = \inf_{y \in \mathbb{R}^n} \left\{ c(n,E) \int_{\partial^* E} \left[1 - \frac{\langle x - y, \nu_E(x) \rangle}{|x-y|} \right] d\mathcal{H}^{n-1}(x) \right\}^{1/2}$$

where $c(n,E) = 1/(n|B_1|^{1/n}|E|^{(n-1)/n})$. It measures the deviation from equality in Cauchy-Schwarz:

$$\langle (x-y)/|x-y|, \nu_E(x) \rangle \leq 1,$$

with equality if and only if $(x-y)/|x-y| = \nu_E(x)$.



Quantitative Stability: Oscillation Index vs H^1

We suppose here that E is *nearly spherical*, i.e. $\partial E = \{x + u(x)x \mid x \in \partial B_1\}$ with $u \in C^1(\partial B_1)$ and $\|u\|_{W^{1,\infty}(\partial B_1)}$ small. In this case can parametrize ∂E in terms of ∂B_1 and compute

$$\beta(E)^2 \lesssim \|u\|^2_{H^1(\partial B_1)}.$$

On the other hand, Fuglede and Fusco-Julin show, respectively, that if E is nearly spherical then

$$\frac{1}{10}\|u\|^2_{H^1(\partial B_1)} \leq \delta(E).$$

and there exists $C(n) > 0$ such that (for any set of finite perimeter)

$$\alpha(E) + \delta(E)^{1/2} \leq C(n)\beta(E).$$

In particular since $\alpha(E) > 0$, $\delta(E) \leq C(n)\beta(E)^2$, we also have $\|u\|^2_{H^1(\partial B_1)} \lesssim \beta(E)^2$.

Anisotropy: Deficit, Asymmetry, and Oscillation

We define the anisotropic deficit, Fraenkel asymmetry, and oscillation index as

$$\delta_\Phi(E) := \frac{\Phi(E)}{n|K|^{1/n}|E|^{(n-1)/n}} - 1$$

$$\alpha_\Phi(E) := \inf_{x_0 \in \mathbb{R}^n} \left\{ \frac{|E \Delta (rK + x_0)|}{|E|} \,\middle|\, |rK| = |E| \right\}$$

$$\beta_\Phi(E) := \inf_{y \in \mathbb{R}^n} \left\{ c_\Phi(n, E) \int_{\partial^* E} \left[f(\nu_E(x)) - \frac{\langle x - y, \nu_E(x) \rangle}{f_*(x-y)} \right] d\mathcal{H}^{n-1}(x) \right\}^{1/2}$$

where $c_\Phi(n, K) = 1/(n|K|^{1/n}|E|^{(n-1)/n})$. Notice the integrand for β_Φ comes from the Fenchel inequality

$$\langle x - y, \nu \rangle \leq f(\nu) f_*(x - y)$$

where equality occurs if and only if $\{\langle x - y, \nu \rangle = f(\nu)\}$ is a supporting hyperplane for K at $x - y$.

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Anisotropy: Previous Results

Theorem (Figalli-Maggi-Pratelli, '12)

There exists $C(n) > 0$ such that for any set of finite perimeter with $0 < |E| < \infty$,

$$\alpha_\Phi(E)^2 \leq C(n)\delta_\Phi(E).$$

The power of 2 is sharp.

Theorem (Neumayer '16)

- *If K is uniformly convex there exists $C(n, ..., \|\nabla^2 f\|_{C^0(\partial K)}) > 0$ such that*

$$\alpha_\Phi(E)^2 + \beta_\Phi(E)^2 \leq C\delta_\Phi(E).$$

- *If instead $n = 2$ and K is a polygon (a crystalline case), there exists $C(K) > 0$ such that the above holds.*

Main Result

The following is the main result, a direct generalization of Neumayer's result in the crystalline $n = 2$ setting.

Theorem (D. '24)

Let K be a polytope. There exists $C(n, K) > 0$ such that for any set of finite perimeter $E \subset \mathbb{R}^n$ with $0 < |E| < \infty$,

$$\alpha_\Phi(E)^2 + \beta_\Phi(E)^2 \leq C(n, K)\delta_\Phi(E).$$

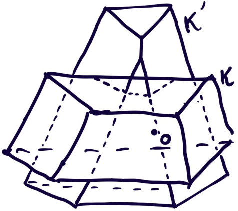
Comparison to Isotropic Case

Remark

- In the anisotropic setting we lack symmetry, so in particular we cannot appeal to symmetrization techniques as in the isotropic setting.
- The Figalli-Maggi-Pratelli result uses optimal transport methods, Neumayer uses the selection principle
- Only weak regularity theory is available. For a generic Wulff shape K can only conclude almost minimizers satisfy uniform density estimates, not (Λ, r_0)-minimizer.
- Need to pair uniform density estimates with L^1-closeness (by FMP) to get Hausdorff closeness.
- Further, in the crystalline setting $\nabla^2 f \equiv 0$ making the problem degenerate elliptic.

Overview of Argument: Parallel Polytopes

Step 1: Prove the result for *parallel* polytopes.



We say that K' is parallel to K if they share the same set of unit normals, and hence have the same amount of sides.

Overview of Argument: The Function γ_Φ

Recall that $\beta_\Phi(E)$ is defined as

$$\beta_\Phi(E) := \inf_{y \in \mathbb{R}^n} \left\{ c_\Phi(n,E) \int_{\partial^* E} \left[f(\nu_E(x)) - \frac{\langle x-y, \nu_E(x) \rangle}{f_*(x-y)} \right] d\mathcal{H}^{n-1}(x) \right\}^{1/2}$$

where $c_\Phi(n,E) = 1/(n|K|^{1/n}|E|^{(n-1)/n})$. In practice, it is much more useful to rewrite this using the divergence theorem as

$$\beta_\Phi(E)^2 = \frac{\Phi(E) - (n-1)\gamma_\Phi(E)}{n|K|^{1/n}|E|^{(n-1)/n}}$$

where

$$\gamma_\Phi(E) := \sup_{y \in \mathbb{R}^n} \int_E \frac{1}{f_*(x-y)}\, dx.$$

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Overview of Argument: A Preliminary Estimate

In particular, it can be shown that

$$\Phi(K) = (n-1)\gamma_\Phi(K) = (n-1)\int_K \frac{1}{f_*(x)}\, dx.$$

Furthermore, by the Wulff inequality $\Phi(K) = n|K|$. Accordingly, if $|E| = |K|$ then by testing $\gamma_\Phi(E)$ at the origin,

$$\begin{aligned}
\beta_\Phi(E)^2 &\leq \frac{\Phi(E)}{n|K|} - \frac{(n-1)}{n|K|}\int_E \frac{1}{f_*(x)}\, dx \\
&= \delta_\Phi(E) + \frac{(n-1)}{n|K|}\left[\int_K \frac{1}{f_*(x)}\, dx - \int_E \frac{1}{f_*(x)}\, dx\right] \\
&= \delta_\Phi(E) + \frac{(n-1)}{n|K|}\left[\int_{K\setminus E} \frac{1}{f_*(x)}\, dx - \int_{E\setminus K} \frac{1}{f_*(x)}\, dx\right].
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Overview of Argument: The Anisotropic Co-Area Formula

For any Borel $g : \mathbb{R} \to [0, \infty)$, Lipschitz $u : \mathbb{R}^n \to \mathbb{R}$, and open $\Omega \subset \mathbb{R}^n$ the (weighted) Anisotropic co-area formula states that

$$\int_\Omega f(\nabla u(x))\, g(f_*(x))\; dx = \int_0^\infty \Phi(\{u < r\}; \Omega)\, g(r)\; dr.$$

With $u(x) = f_*(x)$ and $g(r) = 1/r$ this reads

$$\int_\Omega \frac{1}{f_*(x)}\; dx = \int_0^\infty \frac{1}{r}\Phi(\{f_* < r\}; \Omega)\; dr = \int_0^\infty \frac{1}{r}\Phi(rK; \Omega)\; dr$$

since $f(\nabla f_*(x)) = 1$ for a.e. $x \in \mathbb{R}^n$ by duality.

Overview of Argument: Parallel Polytopes

Recall that $\Phi(K)$ takes the nice form

$$\Phi(K) = \sum_{i=1}^{N} f(\nu_i)\mathcal{H}^{n-1}(F_i).$$

So the computation simply involves bounding

$$\begin{aligned}
\int_{K\setminus K'} \frac{1}{f_*(x)}\,dx &= \int_0^\infty \frac{1}{r}\Phi(rK; K\setminus K')\,dr \\
&= \sum_{i=1}^{N}\int_0^1 \frac{f(\nu_i)}{r}\mathcal{H}^{n-1}(rF_i\cap(K\setminus K'))\,dr, \\
\int_{K'\setminus K} \frac{1}{f_*(x)}\,dx &= \int_0^\infty \frac{1}{r}\Phi(rK; K'\setminus K)\,dr \\
&= \sum_{i=1}^{N}\int_1^\infty \frac{f(\nu_i)}{r}\mathcal{H}^{n-1}(rF_i\cap(K'\setminus K))\,dr.
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Overview of Argument: Uniform Density Estimates

Step 2: Prove the result for E satisfying uniform density estimates. Allows to upgrade L^1 control to Hausdorff.

Need to use the following projection theorem

Theorem (Figalli-Zhang '22)

There exists $\sigma(n, K) > 0$ and $\gamma(n, K) > 0$ such that for any set of finite perimeter $E \subset \mathbb{R}^n$ with $|E| = |K|$ and $|E \Delta K| \leq \sigma$, there exists a parallel polytope K' such that $|K'| = |K|$ and

$$\Phi(E) - \Phi(K') \geq \gamma |E \Delta K'|$$

We'll call the polytope obtained from this theorem K^*.

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Overview of Argument: Uniform Density Estimates

Observe that since $\delta_\Phi(E) = \Phi(E)/(n|K|) - 1$ (and similarly for K^*), we have that

$$\delta_\Phi(E) - \delta_\Phi(K^*) = \frac{\Phi(E)}{n|K|} - \frac{\Phi(K^*)}{n|K|} \geq \frac{\gamma}{n|K|}|E\Delta K'|.$$

In particular, this implies that $\delta_\Phi(K^*) \leq \delta_\Phi(E)$.

Use Hausdorff control to show that

$$|\gamma_\Phi(E) - \gamma_\Phi(K^*)| \leq \frac{1}{2}|E\Delta K^*|.$$

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In particular, this implies that $\delta_\Phi(K^*) \leq \delta_\Phi(E)$.

Use Hausdorff control to show that

$$|\gamma_\Phi(E) - \gamma_\Phi(K^*)| \leq \frac{1}{2}|E \Delta K^*|.$$

Overview of Argument: Uniform Density Estimates

Combining these with the identity

$$\beta_\Phi(E)^2 = \frac{\Phi(E) - (n-1)\gamma_\Phi(E)}{n|K|}$$

yields

$$\begin{aligned}
\beta_\Phi(E)^2 \leq & \frac{\Phi(E)}{n|K|} - \frac{(n-1)\gamma_\Phi(K^*)}{n|K|} + C|E\Delta K^*| \\
&= \frac{1}{n|K|}\left[\Phi(E) - \Phi(K^*)\right] + \beta_\Phi(K^*)^2 + C|E\Delta K^*| \\
&\leq C[\delta_\Phi(E) - \delta_\Phi(K^*)] + \beta_\Phi(K^*)^2.
\end{aligned}$$

Overview of Argument

Step 3: Selection Principle

- Aiming for a contradiction, generate a sequence $\{E_j\}_{j=1}^{\infty}$.
- Choose

$$F_j \in \arg\min\{\Phi(F) + C_1|\beta_\Phi(F)^2 - \beta_\Phi(E_j)^2| + C_2\big||F| - |K|\big|\}$$

  These are almost minimizers in the sense they minimize a perturbed volume-constrained problem.

- Need to show F_j satisfies same properties as E_j and control $\beta_\Phi(F_j)$ and $\Phi(F_j)$. Replace $\{E_j\}_{j=1}^{\infty}$ with almost minimizers $\{F_j\}_{j=1}^{\infty}$.
- Appeal to regularity theory of almost minimizers (uniform density estimates) to apply Step 2.

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Thanks for coming!